ROSENBACH · WHITMAN
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ESSENTIALS OF PLANE TRIGONOMETRY

With Tables

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WITH TABLES

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THE

DESCRIPTION

AND

U S E

OF THE

GLOBES,

Celeſtial and *Terreſtrial;*

With

VARIETY of *EXAMPLES*

For the LEARNER'S EXERCISE:

Intended for the USE of ſuch Perſons as would attain to the KNOWLEDGE of thoſe *INSTRUMENTS;*

BUT

Chiefly deſigned for the INSTRUCTION *of the young* GENTLEMEN *at the* ACADEMY *in* Philadelphia.

To which is added

RULES for working all the CASES in Plain and Spherical TRIANGLES without a SCHEME.

By THEOPHILUS GREW
Mathematical Profeſſor.

GERMANTOWN,
Printed by CHRISTOPHER SOWER, 1753.

Trigonometry in the New World

A reproduction of the title page from the first American trigonometry. The author, Theophilus Grew, was Professor of Mathematics at the Academy in Philadelphia, which is now the University of Pennsylvania. Another page of this early text is shown on page 2

Preface

In the preparation of this textbook the authors have preserved the fundamental features of their earlier textbook in trigonometry. Suggestions offered by teachers who used the earlier book have been of great assistance in the writing of this new one.

The chief aim of the authors has been to write a textbook on trigonometry which is thoroughly teachable and in all respects distinctly modern. Every effort has been made to present material that is clear and simple, yet rigorous and stimulating. Emphasis falls on those topics which are generally recognized as essential whatever the aims of the student or the objectives of the course. In the treatment of each new principle or process, the object has been first to present carefully and completely the necessary definitions, theorems, and proofs, and then to give illustrations, illustrative examples, and problems by which the student can test his mastery of the new material.

Since this textbook is intended for students of early college or engineering-school level, the trigonometric functions are first defined for angles of any magnitude. Attention is called immediately, however, to the fact that the so-called right-triangle definitions, now commonly a part of high-school algebra and trigonometry, are merely special cases of these definitions. Radian measure is introduced early and used along with degree measure throughout the book. Likewise, the inverse-function notation is introduced when the use of tables is discussed, although this topic is treated more completely in a later chapter.

The two main parts of trigonometry, namely, the solution of triangles and the development of the interrelationships between the trigonometric functions with their applications to equations and identities, are completed in the first five chapters. Additional topics, such as line values, graphs, graphical solution of equations, inverse functions, and general values, are presented in later chapters. They may be omitted or assigned independently one of another. Many instructors find them an interesting and valuable part of a course in trigonometry. For students whose previous preparation in logarithms is not adequate, a separate chapter is devoted to the theory of logarithms and their use in computations.

The illustrations and illustrative examples are carefully selected to anticipate the difficulties of the student and at the same time to set before him applications of basic principles and well-ordered solutions of problems. The abundance of carefully graded problems affords the instructor a different selection each year for several years. In each list the first problems generally will require only the application of the principles discussed in the article or articles directly preceding, while problems at the end of the list are more difficult and will challenge even the better students. In addition, a list of general exercises arranged by chapters appears at the back of the book; this set of problems will be found useful for review.

Answers are provided for the odd-numbered problems. Thus the student is enabled to check his work in some of the problems, but is placed on his own resources in others. Answers to the even-numbered problems are available in a separate pamphlet, but are furnished to students only at the request of the instructor.

The tables are more complete than is usual. In addition to five-place logarithmic and trigonometric tables, the following tables are also included: trigonometric functions of some particular angles, conversion tables from degree measure to radian measure and vice versa, trigonometric functions of angles in radians, and important constants and their logarithms. A compact table of proportional parts for differences from 1 to 105 and a table of angles corresponding to certain rational values of the trigonometric functions appear on a card enclosed in a pocket inside the back cover of this book.

The authors gratefully acknowledge their indebtedness to their colleagues at Carnegie Institute of Technology for criticisms and suggestions. They also wish to thank the members of the Editorial Staff of Ginn and Company and the Staff of the Athenaeum Press for many courtesies.

J. B. ROSENBACH
E. A. WHITMAN
DAVID MOSKOVITZ

Carnegie Institute of Technology

GREEK ALPHABET

Letters		*Names*	*Letters*		*Names*	*Letters*		*Names*
Α	α	Alpha	Ι	ι	Iota	Ρ	ρ	Rho
Β	β	Beta	Κ	κ	Kappa	Σ	σ ς	Sigma
Γ	γ	Gamma	Λ	λ	Lambda	Τ	τ	Tau
Δ	δ	Delta	Μ	μ	Mu	Υ	υ	Upsilon
Ε	ϵ	Epsilon	Ν	ν	Nu	Φ	ϕ	Phi
Ζ	ζ	Zeta	Ξ	ξ	Xi	Χ	χ	Chi
Η	η	Eta	Ο	o	Omicron	Ψ	ψ	Psi
Θ	θ	Theta	Π	π	Pi	Ω	ω	Omega

LIST OF SYMBOLS

$\neq$, read *is not equal to.*

a_n, read *a subscript n*, or *a sub n.*

$<$, read *is less than.*

$>$, read *is greater than.*

$\leqq$, read *is less than or equal to.*

$\geqq$, read *is greater than or equal to.*

(x, y), read *point whose coordinates are x and y.*

θ' and θ'', read *theta prime and theta second* respectively.

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ESSENTIALS OF
PLANE
TRIGONOMETRY

TRIGONOMETRY.

Oblique angled Plain Triangles.

CASE I.

Given the Angles and one Side, to find the other two Sides.

As the Sine of the *Angle* oppoſite to the given *Side*, is to the given *Side*, ſo is the Sine of either of the other two *Angles* to the *Side* oppoſite thereto. For in all Plain Triangles the *Sides* are in direct Proportion to the Sines of their oppoſite *Angles*, and the contrary.

CASE II.

Given two Sides, and an Angle oppoſite to one of them, to find the other two Angles and third Side.

Firſt, As the *Side* oppoſite to the given *Angle* is to the Sine of the given *Angle*, ſo is the other given *Side* to the Sine of its oppoſite *Angle*.

Then add the two known *Angles* together, and ſubſtract their Sum from 180 Degrees, and what remains will be the third *Angle*. For all three make 180 Degrees.

Secondly, As the Sine of the given *Angle* is to its oppoſite *Side*, ſo is the Sine of the third *Angle* to the *Side* required.

Note,

A Page of an Early Trigonometry Textbook

The use of diagrams and of algebraic symbolism is far more common today than in these early textbooks

CHAPTER I

The Trigonometric Functions

1 · Introduction

Trigonometry, meaning triangle measure, had its origins in early historical times. Its beginnings were a part of the attempts to describe and measure the celestial sphere in which the sun, planets, and stars were then supposed to move. Prominent among its founders were the Greek astronomers HIPPARCHUS of Nicaea (*c.* 140 B.C.) and PTOLEMY of Alexandria (*c.* 150 A.D.). Developing slowly, trigonometry remained as a part of astronomy for some fifteen or sixteen centuries but increasing in importance within that subject. Finally trigonometry emerged as a separate subject, worthy of its own book, about 1464, in *De triangulis omnimodis, libri V*, by JOHANN MÜLLER of Königsberg, more generally known as REGIOMONTANUS.

Today trigonometry is widely used in the various branches of engineering and science. Numerical trigonometry, that part of trigonometry used in the solution of triangles, is only one of its aspects. The other and equally important aspect is analytical trigonometry, that part which is concerned with the properties of and the interrelationships between certain ratios called trigonometric functions.

2 · Directed Line Segments

Let us agree that on a given line AB (Figure 1) all distances measured in one direction are *positive* and all distances measured in the opposite direction are *negative*. We then say that the line AB is a **directed line**, and any segment of it, such as CD, is a **directed line segment**. On directed lines we shall denote the positive direction by an arrowhead. If the length of the segment CD is 4 units, then we say that the directed distance CD is 4 and the directed distance DC is -4; for brevity, we write $CD = 4$ and $DC = -4$.

Fig. 1

On a directed line $X'X$ (Figure 2) of unlimited length, let us choose the point O, called the **zero point** or **origin**, from which to measure distances. Then with an arbitrarily chosen length as unit, lay off from O the directed distances 1, 2, 3, · · ·, and − 1, − 2, − 3, · · ·. If we then mark the terminal points of these directed distances with the corresponding numbers, we have a **number scale** on which each integer corresponds to a specific point. By subdividing the various unit lengths we can locate

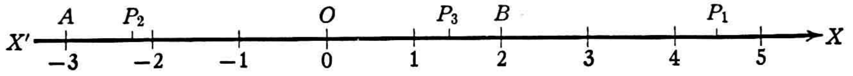

Fig. 2

points corresponding to numbers which are not integers. On such a scale there is precisely one real number which corresponds to each point. Hereafter, when the word *number* is used in this book, it will be understood to mean a real number.

ILLUSTRATIONS. On the number scale in Figure 2, A corresponds to -3; 2 corresponds to B; P_1 (read "*P sub one*") corresponds to 4.5; P_2 corresponds to $-2\frac{1}{4}$; P_3 corresponds to $\sqrt{2}$ if OP_3 is the diagonal of a square of unit length. We also express these as follows:

$$OA = -3,\ OB = 2,\ OP_1 = 4.5,\ OP_2 = -2\tfrac{1}{4},\ OP_3 = \sqrt{2}.$$

EXERCISE 1

1. Draw a number scale and locate on it the points P_1, P_2, P_3, A, B, C which correspond, respectively, to the following numbers: $3, -2, -1.5, -4\frac{2}{3}, \sqrt{5}$, and $-\sqrt{13}$.

2. From the number scale in Figure 2, give the directed distance that is equal to the line segment (*a*) AB; (*b*) BP_1; (*c*) AP_2; (*d*) BA; (*e*) BP_2; (*f*) P_1P_2; (*g*) P_3A; (*h*) P_3P_1.

3. From the number scale of Problem 1, give the directed distance that is equal to the line segment (*a*) P_2P_1; (*b*) P_1P_3; (*c*) CB; (*d*) AC; (*e*) P_3P_2; (*f*) P_3C; (*g*) BP_1; (*h*) P_2A.

3 · Rectangular Coordinates

Let $X'X$ and $Y'Y$ be two directed lines intersecting at right angles at the point O, and let a number scale be constructed on each with O as the zero point and with units as shown in Figure 3. The line $X'X$ is called the **x-axis**, the line $Y'Y$ is called the ***y*-axis**; the two axes *together* are called the **coordinate axes.** The point O is called the **origin.**

If from any point P in the plane of the coordinate axes, perpendiculars, PM and PN, are dropped to the axes, two numbers which correspond to M and N are determined. The first of these numbers is called the **abscissa** or ***x*-coordinate** of P; the second is called the **ordinate** or ***y*-coordinate** of P. If these numbers are x and y, respectively, the point P is said to be the point (x, y) or the point $P(x, y)$.

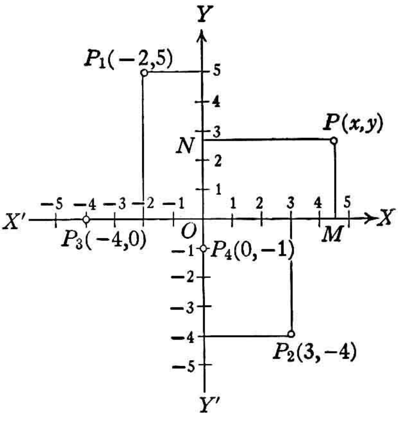

Fig. 3

Illustrations. In Figure 3, the point P_1 has the coordinates $(-2, 5)$; to the coordinates $(3, -4)$ there corresponds the point P_2; the point P_3 has the coordinates $(-4, 0)$; and the point with the coordinates $(0, -1)$ is P_4.

Note. It is important to observe that the line segments whose directed distances are abscissas or ordinates are always directed *from* the coordinate axes. Thus, in each of the diagrams of Figure 4,

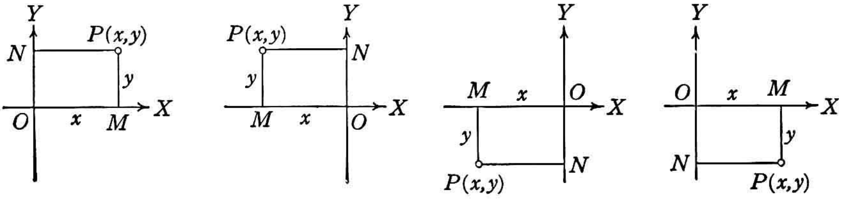
Fig. 4

$$x = OM \text{ but } x \neq MO;\ MO = -x;$$
$$y = MP \text{ but } y \neq PM,\ PM = -y.$$

The coordinate axes divide their plane into four parts called **quadrants**, which are conveniently designated as shown in Figure 5 (Q I is read "*quadrant one*"). Also shown in this figure are the signs of the coordinates in each of the quadrants. A point on either of the coordinate axes is considered not to be in any quadrant. The abscissa of every point on the $Y'Y$ axis is zero; the ordinate of every point on the $X'X$ axis is zero.

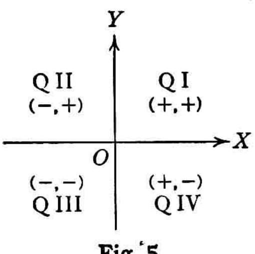

Fig. 5

HISTORICAL NOTE. Rectangular coordinates are also called Cartesian coordinates. They are so named after RENÉ DESCARTES, a distinguished French mathematician and philosopher, in whose *La Géométrie* (1637) the use of coordinates first appeared in print.

4 · Radius Vector

The *length** of the line segment which joins the origin O to any point P is called the **radius vector** of P and will be denoted by r. **The radius vector of the origin is defined as zero; the radius vector of any other point is always taken as positive.**

The relation
$$r^2 = x^2 + y^2 \tag{1}$$
is satisfied by the coordinates (x, y) and the radius vector r of any point P in the plane (Figure 6). When two of the numbers x, y, and r are given, the third can be determined by use of equation (1).

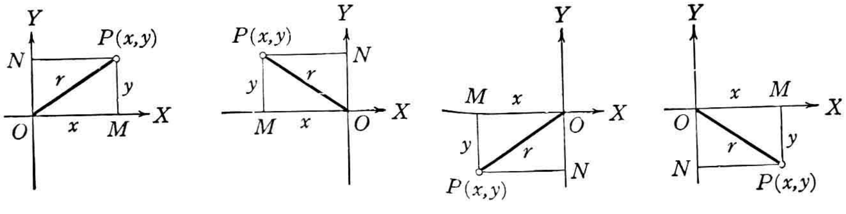

Fig. 6

ILLUSTRATION 1. A given point has $r = 5$ and $y = -4$; then $x^2 = 25 - 16 = 9$; $x = \pm 3$. Hence the point may be either $(3, -4)$ or $(-3, -4)$.

ILLUSTRATION 2. A given point has $x = -15$, $y = 8$; then $r^2 = 225 + 64 = 289$; $r = 17$.

EXERCISE 2

Plot each of the following points, state the quadrant, if any, in which it lies, and find the length of its radius vector:

1. $(3, 4)$.
2. $(12, -5)$.
3. $(-8, -15)$.
4. $(-24, 7)$.
5. $(4, -6)$.
6. $(-5, -3)$.
7. $(0, -2)$.
8. $(-1, 0)$.
9. $(0, 3)$.
10. $(0, 0)$.
11. $(-\sqrt{3}, 1)$.
12. $(1, -1)$.

---

*The word *length* will always denote a positive number in contradistinction to a *directed distance* which may be positive or negative.

13. Determine the quadrant, or quadrants, in which a point must lie if (*a*) its abscissa is positive and its ordinate is negative; (*b*) its abscissa is negative and its ordinate is positive; (*c*) its abscissa and ordinate are both negative; (*d*) its abscissa and ordinate are both positive; (*e*) its abscissa and ordinate are alike in sign; (*f*) its abscissa and ordinate are unlike in sign.

In Problems 14 to 19, two of the three numbers x, y, and r associated with a point are given. Find the third number and plot the point.

14. $x = -8$, $y = -6$. **16.** $x = 1$, $r = 2$. **18.** $y = 4$, $r = 10$.

15. $y = -7$, $r = 25$. **17.** $x = -3$, $y = -9$. **19.** $x = -1$, $r = \sqrt{5}$.

20. Where are all the points for which (*a*) $x = 1$? (*b*) $y = -2$? (*c*) $x = -3$? (*d*) $y = 4$? (*e*) $r = 5$? (*f*) $x = 0$? (*g*) $y = 0$? (*h*) $r = 0$?

21. Determine the quadrant, or quadrants, in which $P(x, y)$ will lie when (*a*) $y/x > 0$; (*b*) $x/r < 0$; (*c*) $r/y > 0$; (*d*) $x/y = -1$; (*e*) $r/x = 2$; (*f*) $y/r = -\frac{3}{4}$; (*g*) $x/r = 0$; (*h*) $x = y$; (*i*) $y = r$; (*j*) $r = -x$.

The point (a, b) is in (*a*) *Q I,* (*b*) *Q II,* (*c*) *Q III,* (*d*) *Q IV. Locate each of the following points as to quadrant and find its radius vector.*

22. $(-a, b)$. **23.** $(-a, -b)$. **24.** $(b, -a)$. **25.** $(-a, a)$. **26.** $(-b, -b)$.

27. If a given point has a as its abscissa and c as its radius vector, find its ordinate when the point is in (*a*) Q I; (*b*) Q II; (*c*) Q III; (*d*) Q IV.

28. If a given point has b as its ordinate and c as its radius vector, find its abscissa when the point is in (*a*) Q I; (*b*) Q II; (*c*) Q III; (*d*) Q IV.

29. Find the length of the line segment AB if the coordinates of A and B are respectively: (*a*) $(-1, 2)$, $(3, 5)$; (*b*) $(4, -6)$, $(-8, -1)$; (*c*) (a, c), (b, d).

5 · Directed Angles

The definition of an angle as the figure formed by two lines drawn from the same point is not sufficiently general for the purposes of trigonometry where we make use of angles of any magnitude, positive, negative, or zero. A more general concept of angles may be formed as follows:

An angle is considered as generated by the rotation of a line, in a fixed plane, about one of its points from an original position, called the **initial side** *of the angle, to a final position, called the* **terminal side** *of the angle. The point about which the rotation takes place is called the* **vertex** *of the angle.*

The **size** or **value** of the angle is determined by the *amount of rotation*, and the angle is said to be **positive** or **negative** according as the rotation is *counterclockwise* or *clockwise*. An **arc** and an **arrowhead** are used to indicate the *amount* and the *direction* of the rotation.

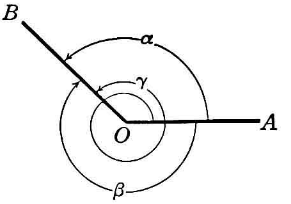

Fig. 7

ILLUSTRATION 1. In Figure 7, the angle α is 135°, the angle β is $-225°$, and the angle γ is 495°.

An angle is said to be in standard position *if its vertex is at the origin and its initial side coincides with the positive x-axis.* **An angle is said to be in a certain quadrant** *if its terminal side lies in that quadrant when the angle is placed in standard position; if the terminal side lies along one of the coordinate axes, the angle is called a* **quadrantal angle.**

ILLUSTRATION 2. 246° is in Q III, $-345°$ is in Q I, and 180° is quadrantal.

Angles whose terminal sides coincide when placed in standard position are called **coterminal angles.**

Since two angles are coterminal if they are equal or if they differ from each other by a positive or negative integral multiple of 360°, the number of angles coterminal with any given angle is unlimited. We shall call the angle zero and the positive angles less than 360° **primary angles.**

ILLUSTRATION 3. In Figure 7, the angles α, β, and γ are coterminal, and α is a primary angle. The primary angle coterminal with β and γ is α. The angles coterminal with 100° are $100° + 1 \cdot 360°$ or 460°, $100° + 2 \cdot 360°$ or 820°, $\cdots$; $100° - 1 \cdot 360°$ or $-260°$, $100° - 2 \cdot 360°$ or $-620°$, $\cdots$.

EXERCISE 3

*(**a**) Place the following angles in standard position and indicate each angle by arc and arrowhead. (**b**) Find the values of two coterminal angles, one positive and one negative, and indicate them by arcs and arrowheads in the same figure.*

1. 60°.	**4.** 310°.	**7.** $-200°$.	**10.** 270°.	**13.** 495°.
2. 150°.	**5.** $-40°$.	**8.** $-345°$.	**11.** $-90°$.	**14.** 600°.
3. 225°.	**6.** $-110°$.	**9.** 180°.	**12.** $-180°$.	**15.** $-690°$.

Each of the following points is on the terminal side of an angle in standard position. Draw the terminal side and indicate by arcs and arrowheads the primary angle and one negative angle for which it is the terminal side.

16. $(4, 5)$.	**18.** $(-7, -1)$.	**20.** $(2, 0)$.	**22.** $(0, -5)$.	**24.** $(-1, \sqrt{3})$.
17. $(-3, 2)$.	**19.** $(6, -4)$.	**21.** $(0, 3)$.	**23.** $(-4, 0)$.	**25.** $(-\sqrt{2}, -\sqrt{2})$.

6 · Angular Measure

There are two systems in common use for the measuring of angles, the **degree-measure** or **sexagesimal** system and the **radian-measure** or **circular** system.

We assume that the student is already familiar with the sexagesimal system in which the unit of measure is the **degree**. *A* ***degree*** *is an angle which, if its vertex is placed at the center of a circle, intercepts an arc that is* $\frac{1}{360}$ *of the circumference.* A degree is subdivided into 60 **minutes**, and a minute into 60 **seconds**.

The circular system is used extensively in higher mathematics and in the various branches of engineering and science. The unit of measure is the **radian**.

A radian is an angle which, if its vertex is placed at the center of a circle, intercepts an arc on the circumference equal in length to the radius of the circle.

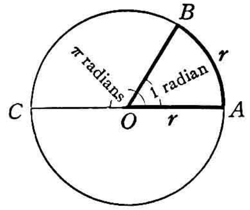

Fig. 8

ILLUSTRATION. In Figure 8, the central angle AOB (written $\angle AOB$) is one radian, since the length of the intercepted arc AB is r, the radius of the circle.

The relation between the measure of an angle in degrees and in radians follows as an immediate consequence of the definition of a radian and the theorem in plane geometry which states that angles at the center of a circle are proportional to their intercepted arcs. Thus, from Figure 8, we have

$$\frac{\angle AOC}{\angle AOB} = \frac{\text{arc } ABC}{\text{arc } AB}, \quad \text{or} \quad \frac{\angle AOC}{1 \text{ radian}} = \frac{\pi r}{r}.$$

Therefore $\angle AOC = \pi$ radians. Since $\angle AOC = 180°$, we obtain

$$180° = \pi \text{ radians} = 3.141593^- \text{ radians.} \tag{2}$$

From (2) we obtain the relations

$$1° = \frac{\pi}{180} \text{ radian} = 0.0174533^- \text{ radian,} \tag{3}$$

and

$$1 \text{ radian} = \frac{180°}{\pi} = 57.29578°^- = 57° \, 17.75'^-. \tag{4}$$

Equations (3) and (4) give the **conversion factors** by which we are able to convert from degrees to radians and from radians to degrees. To