Lecture Notes in Mathematics 1580

Mario Milman

Extrapolation and Optimal Decompositions

with Applications to Analysis

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- a table of contents;
- an informative introduction, perhaps with some historical remarks: it should be accessible to a reader not particularly familiar with the topic treated;
- a subject index: as a rule this is genuinely helpful for the reader.

Further remarks and relevant addresses at the back of this book.

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To Vanda

Preface

In these notes we continue the development of a theory of extrapolation spaces initiated in [57]. One of our main concerns has been to connect the fundamental processes associated with the construction of interpolation and extrapolation spaces "optimal decompositions" with a number of problems in analysis. In particular we study extrapolation of inequalities related to Sobolev embedding theorems, higher order logarithmic Sobolev inequalities, *a.e.*convergence of Fourier series, bilinear extrapolation of estimates in different settings with applications to PDE's, apriori estimates for abstract parabolic equations, commutator inequalities with applications to compensated compactness, a functional calculus associated with positive operators on a Banach space, and the iteration method of Nash/Moser to solve nonlinear equations.

Many of the results presented in these notes are new and appear here for the first time. We have also included a number of open problems throughout the text. While we hope that these features could make our work attractive to specialists in the field of function spaces it is also hoped that the central rôle that the applications play in our development could also make it of interest to classical analysts working in other areas. In order to facilitate the task of these prospective readers we have tried to provide sufficient background information with complete references and included a brief guide to the literature on interpolation theory. We have also tried to make the contents of different chapters as independent of each other as possible while at the same time avoiding too much repetition. Finally we have also included a subject index and notation index.

It is a pleasure to record here my gratitude to a number of people and institutions who have helped me to complete these notes over the years. In particular I am grateful to Björn Jawerth with whom I have spend a lot of time over the years talking about extrapolätion. My friends DMGSJ played also an important extrapolatory role, both professionally and otherwise. The first version of the book was written during a membership, partially supported by an NSF grant, at the Institute for Advanced Study. I am most grateful to the Institute and Professor L. Caffarelli, for their support and for providing such an stimulating environment for my work. The notes were further developed while I was visiting the University of Paris (Orsay), The Centre for Ricerca (Barcelona) and the University of Zurich. I am particularly grateful to Professors Herbert Amann (Zurich), Aline Bonami (Orsay), Joan Cèrda (Barcelona) for their support and interest in my work.

Contents

Chapter 1

Introduction

In the last thirty years or so interpolation theory has become an important chapter in the field of function spaces. The origins of the theory are the classical interpolation or convexity theorems of Riesz, Thorin, and Marcinkiewicz. These classical results were subsequently extended by many authors including Calderón, Cotlar, Salem, Zygmund. The foundations of the general theory were laid down in the sixties by Aronszajn, Calderón, Gagliardo, Krein, Lions, Peetre, among others. It has since been extended, perfected, and applied, by many mathematicians. We refer the reader to the section at the end of this chapter for a brief guide to the available literature.

The theory has found many applications to classical analysis. In particular it has become an important tool in theories as diverse as partial differential equations, approximation theory, harmonic analysis, numerical analysis, operator theory, etc.

In the last few years, mainly in collaboration with B. Jawerth, we have been developing a new theory of "extrapolation spaces" which somehow is the converse of interpolation theory. The classical framework of interpolation theory can be briefly described as follows. We are given a pair of compatible Banach spaces (X, Y) and we attempt to construct all the spaces with the interpolation property between them. The real and complex methods, for example, provide parameterized families of spaces $(X, Y)_{\theta,q}$, $[X, Y]_\theta$, with the interpolation property. In extrapolation we conversely ask: given a family of interpolation spaces can we reconstruct the originating pair? This question is, of course, directly related to best possible interpolation theorems. In practice, however, one is also very much interested in weaker formulations of this general question. Thus, given a family of estimates for an operator T acting on interpolation spaces, one wishes to "extrapolate" this information either as an "extrapolation theorem" (i.e. a continuity result for T, the model for which is provided by the classical extrapolation theorem of Yano) or an "extrapolation inequality" for T where the extrapolation estimate is usually based on the basic functionals of real interpolation (K, J, E functionals). In a sense one could also consider "extrapolation" as chapter of the theory of interpolation of infinitely many spaces, since one is trying to obtain information from an infinite family of spaces. The precise connection between these theories is an interesting open problem.

The usual relationship between modern analysis and classical analysis is that the former provides a framework for consolidation, extension, and simplification of results of the latter. At times, however, it is the general framework that suggests the right questions, and the techniques of modern analysis sometimes also provide the answers. It is this symbiotic state of affairs that, from our perspective, makes the general field of interpolation theory interesting.

In these notes we study the connection between optimal decompositions, extrapolation, and its applications to other areas of analysis. In particular we also explore the role that cancellation plays in interpolation/extrapolation theory. We have arranged the development of the theory vis a vis concrete applications to classical analysis in order to emphasize our point of view. Thus the development of theory in this book is not "linear." A prospective reader interested mainly in the abstract theory could certainly skip the applications developed in each of the chapters. On the other hand we hope that the specific applications will be of interest to analysts working in different fields. As a consequence we have tried to make the reading of each of these sections as independent as possible from the others while at the same time trying to avoid too much repetition.

Most of the results presented in these notes are new or have not appeared in book form before. It is hoped that they will serve as the basis of a larger, more detailed, and formal book. The author would therefore welcome suggestions, remarks and corrections.

The notes are organized as follows. In Chapter **2** we provide a detailed introduction to extrapolation theory (the reader is referred to [57] and [58] for further information and other specific applications). In this chapter we also provide detailed computations of extrapolation spaces for different scales. In Chapter **3** we discuss K/J inequalities in the context of limiting embedding theorems for interpolation scales with specific applications to Sobolev embeddings, in particular we relate these results to limiting inequalities by Kato and Ponce [63] , and Beale, Kato and Majda [5] concerning singular integrals. In Chapter **4** we study the Δ method of extrapolation, develop new tools to compute it, and apply these results to sharpen recent estimates by Müller [82] and Iwaniec and Sbordone [54] on the integrability of the Jacobian of orientation preserving maps. We also give an application to Sobolev imbedding theorems extending recent work by Fusco, P. L. Lions and Sbordone [42]. In Chapter **5** we study bilinear extrapolation and prove general bilinear extrapolation theorems of Yano type. As an application, involving the Schatten classes, we derive an end point inequality, due to Constantin [22], of a well known theorem of Cwikel [28] concerning the singular values of Schrodinger operators (which leads to Constantin's [22] limiting version of the collective Sobolev estimates of Lieb [70]). In Chapter **6** we give a number of new reiteration theorems for extrapolation spaces and apply our results to give a new approach to end point estimates for the maximal operator of Fourier partial sums due to Sjölin [95] and Soria [97]. In this chapter we also provide a new approach to the higher order logarithmic Sobolev estimates of Gross and Feissner (cf. [46], [41]). In Chapter **7** we study the role of cancellation in interpolation/extrapolation theory: we prove new commutator theorems and establish the relationship between commutator theorems and a functional calculus for positive operators on a Banach space. We believe that this connection could lead to applications in the theory of parabolic equations. In this chapter we also develop in detail an application to the theory of compensated compactness of Murat and Tartar, and, in particular, we indicate how commutator theorems can be used to obtain sharp integrability theorems for Jacobians of orientation preserving maps. In the short Chapter **8** we consider families of operators an establish an abstract version of a Sobolev embedding theorem due to Varopoulos [102]. In

Chapter **9** we indicate rather briefly the role of extrapolation spaces in the theory of abstract parabolic equations in Banach space and establish a number of new end point estimates. Finally, in Chapter **10**, we discuss the iteration process of Nash/Moser in the setting of scales with smoothing and interpolation /extrapolation spaces, in particular we establish a precise relationship between the theory of scales of spaces with smoothing and interpolation scales as well as provide an interpretation of the paracommutators of Hörmander [50] in terms of optimal decompositions.

1.1 A Very Brief Guide To The Literature On Interpolation

This guide is intended to provide a rather incomplete and super brief guide to background literature on interpolation theory relevant to the developments in these notes. It is intended to help a newcomer to the field to begin to study the literature. We apologize in advance if your favorite book or paper is not quoted here. The classical reference books on interpolation theory include [13], [8], [101], and [67]. These books also develop many of the early applications of the theory to harmonic analysis, approximation theory, semigroups and partial differential equations. The book [65] also presents a detailed account of interpolation and its applications to operator ideals. More recent contributions are the research monograph [85] which develops a new approach to interpolation theory, the book [7] which presents a detailed study of rearrangement invariant spaces and the computation of K functionals, and the ground breaking treatise [12] presenting new important theoretical developments in the field.

The monograph [57] presents the first detailed account of extrapolation theory and contains a bibliography of 60 items, as well as brief historical survey. Further results and applications of extrapolation are contained in [58] (which also contains an extensive bibliography) and the forthcoming paper [27] which deals with complex extrapolation.

The theory of second order and commutator estimates in interpolation theory and their applications was developed in, among other papers, [89], [60], [25], [62], [61]. Finally we should mention that the recent book [26] that contains an extensive list of unsolved problems in the area.