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Edited by J. Ehlers, München, K. Hepp, Zürich
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Complex Analysis, Microlocal Calculus and Relativistic Quantum Theory

Proceedings, Les Houches 1979

Edited by D. Iagolnitzer

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COMPLEX ANALYSIS, MICROLOCAL CALCULUS
AND RELATIVISTIC QUANTUM THEORY

Les Houches, Sept.3-13, 1979

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FOREWORD

This volume presents the Proceedings of the Colloquium on "Complex Analysis, Microlocal Calculus and Relativistic Quantum Theory", held at the Centre de Physique des Houches in September 1979.

This Colloquium originated in the contacts developed during the seventies between two groups of (French and Japanese) mathematicians and a group of theoretical physicists who, with different motivations and approaches, were led to related, or even common, problems in the study of the singularity structure of functions (or distributions, hyperfunctions, microfunctions,...) of interest either in mathematics, in complex analysis and differential and microdifferential calculus, or in physics, in relativistic quantum theory. These contacts, including in particular previous meetings organized by F. Pham in Nice in 1973 and by M. Sato and collaborators in Kyoto in 1976, had proved to be useful. The present Colloquium, which was extended to related domains of common interest, has allowed the presentation of the important new developments and the exchanges that had appeared to be desirable or needed. Let us note, in this connection, that the separation between mathematicians and physicists is not always very neat: as will appear in these Proceedings, several of them in both groups have been led recently to contributions in either domain, the differences appearing mainly in the emphasis and in the main motivations and character of the various contributions.

The topics treated have been classified in four parts. The two first parts are mainly mathematical, while the last two are more oriented towards physics. Some indications on the connections between various parts are given later.

Part I presents the recent developments of microfunction theory and of the microlocal, or microdifferential, calculus (holonomic systems, second microlocalization) and related topics (essential support theory, ...). Other miscellaneous mathematical developments, on singularities of solutions of partial differential equations, pseudodifferential operators and generalizations, spectrum of operators, asymptotic expansions, monodromy, ... will be found in Part II.

Part III is mainly devoted to the rigorous study of the general analytic and microanalytic structure of Green functions and of the S-matrix (i.e., of collision amplitudes), in axiomatic quantum field theory and in S-matrix theory, a domain in which appreciable progress has been made recently for multiparticle processes. Recent developments in the related study of Feynman integrals,and a few other topics, are also included.

Finally, an important part of these Proceedings (Part IV) is devoted to the explicit determination of the S-matrix and, in some cases, of Green functions for various models of field theory in two space-time dimensions, and to related physical and mathematical developments with emphasis on those aspects that have, or should prove to have, a general character and give hopes of further developments. Approaches based on general principles and applying to theories "with soliton behaviour" are first presented (Sect.A). The more direct approach developed recently for quantization and solution of completely integrable systems is then introduced in Sect.B, where a general analysis of such systems, in connection with the isospectral deformation, is presented on the other hand. Finally, the recent developments on holonomic quantum fields, in connection with the isomonodromic deformation, and the corresponding solution of the Ising and other models by Sato-Miwa-Jimbo is presented in Sect.C. Complements on the Ising model are also included.

While each part has its own unity, we emphasize however that the above division is to some extent arbitrary, both because the distinction between physics and mathematics is not always very neat and in view of the connections that will appear between the various parts. For instance, some texts of Part I (Iagolnitzer, Kashiwara-Kawaï, Van Den Essen) are directly relevant to the study of the S-matrix and of Feynman integrals. As another example, the study of the general structure of the S-matrix in Part III gives insight into some aspects of the two-dimensional models considered in Part IV. (Note, however, that possible extensions of specific aspects of these models to more dimensions seem to lead to structures that differ from the usual ones). Finally, a number of connections, not described here in detail, will appear between various texts of Parts I,II and IV.

The Proceedings include short contributions, which are a summary or an introduction to more complete works published elsewhere, and longer contributions, which either present original work or are review works (with some original aspects) in domains in which there was a need for an up-to-date and clear presentation of recent developments.

On behalf of the Organization Committee, I would like to thank the directors of the Centre de Physique des Houches, and in particular Mrs. M.T. Beal-Monod, for their efficient help. I also wish to thank all participants and lecturers who contributed to the success of this Colloquium, and more particularly here all lecturers who, by their efforts in the preparation of their manuscripts, will contribute to the usefulness of these Proceedings. I am finally pleased to thank Mrs. E. Cotteverte for her efficient help in the final preparation of the manuscript.

D. IAGOLNITZER

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PART I

MICROFUNCTIONS, MICROLOCAL CALCULUS AND RELATED TOPICS

ESSENTIAL SUPPORT THEORY AND u=0 THEOREMS

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The standard results of essential support theory[1], or of hyperfunction theory[2], give no information on the essential support (= singular spectrum) of a product of distributions f',f" at u=0 points X (i.e. points where the essential supports of f' and f" contain opposite directions), even when the product itself is well defined in the neighborhood of X. The same remark applies also to products of bounded operators, such as the products of collision operators encountered in some applications in S-matrix theory (see the lecture of the author in Part III). The u=0 problem encountered there is crucial and has been at the origin of several works. The approach to the u=0 problem developed in the framework of the theory of holonomic microfunctions by Kashiwara-Kawai is presented in [3] and references therein. We present here u=0 results[4] obtained in the framework of essential support theory, on products of square integrable functions that satisfy a general regularity property R at X with respect to the relevant directions of their essential support. Analogous results[4a] also hold for products of bounded operators. Details in both cases will be given in [4c].

The basic facts on the essential support are recalled in Sect.1. Property R is probably linked with the second microlocalization introduced recently in microfunction theory. It is introduced in Sect.2 at the end of which a simple example is given in terms of analyticity properties. The announced u=0 theorem is then presented in Sect.3. As in [3], the result is similar to the standard u≠0 one, except that limiting procedures that may enlarge the essential support have to be introduced. The latter are however different, the respective conditions of application of the results being themselves different in general.

1. ESSENTIAL SUPPORT

Being given a tempered distribution f defined on the space $\mathbb{R}^n$ of n real variables $x=x_1,\dots,x_n$, the essential support of f is defined, at each point X, as a cone with apex at the origin in the space $\mathbb{R}^n$ of the dual variables $u=u_1,\dots,u_n$ composed of the "singular directions" along which the generalized, or localized, Fourier transform of f at X does *not* fall off exponentially in a well specified sense. Namaly, let F_γ be defined, for every $\gamma > 0$, by

$$F_\gamma(u;X) = \int f(x)\ e^{-iu.x-\gamma|u|(x-X)^2}\ dx \qquad (1)$$

or, with an auxiliary real variable v, let :

$$F(u;v,X) = \int f(x)\ e^{-iu.x-v(x-X)^2}\ dx\ . \qquad (2)$$

Then a direction $\hat{u}_o$ is by definition outside $ES_X(f)$, or $(X,\hat{u}_o) \notin ES(f)$ if there exist a neighbouring cone $V(\hat{u}_o)$ of $\hat{u}_o$ with apex at the origin, $\alpha > 0, \gamma_o > 0$, as also a polynomial P and $q \geq 0$ such that :

$$|F(u;v,X)| < [P(|u|)\ v^{-q}]\ e^{-\alpha v} \qquad (3)$$

in the region $u \in V(\hat{u}_o)$, $0 < v < \gamma_o|u|$.

The important factor in these bounds is the factor $e^{-\alpha v} \equiv e^{-\alpha\gamma|u|}$, which expresses exponential fall off in the direction $\hat{u}_o$ for all $\gamma > 0$ sufficiently small with a rate of fall off at least proportional to γ . (Bounds of the form (3) without this factor are always satisfied). Whereas the exponential fall-off at $\gamma=0$, i.e. of the usual Fourier transform $\tilde{f}$ of f, corresponds to analyticity properties independent of the real point X (see e.g. [1]), the above notion of essential support characterizes by duality the real points where f is analytic or is the boundary value of an analytic function, and more generally possible decompositions of f into sums of boundary values of analytic functions from specified directions (which may depend on X) : see details in [1], where the notion of essential support and the results above are extended to general distributions defined in $\mathbb{R}^n$ or on a real analytic manifold.

The characterization of analyticity properties obtained[2] in hyperfunction theory in terms of the notion of *singular spectrum* is similar to above, except that the boundary values involved in the decompositions of f may à priori be hyperfunctions, even when f itself is a distribution. It is, however, proved in [5] that the two notions do coïncide for distributions (and coïncide with Hormander's "analytic wave front set").

2. REGULARITY PROPERTY R (for square integrable functions)

The regularity property R is a condition on the way rates of exponential fall off of the generalized Fourier transform F_γ tend to zero in certain situations when directions of the essential support are approached. It asserts more precisely that certain uniform bounds are then satisfied (see below).

Being given X and $\hat{u}_o \notin ES(f)$, it follows from results of [1] that α in the bounds (3) can always be chosen arbitrarily close to

$$\mathrm{Max}_{\alpha'>0}\ \{\alpha';\hat{u}_o \notin ES_x(f)\ ,\ \forall x \text{ s.t. } (x-X)^2 < \alpha'\}\ ,$$

(with appropriate choices of $V(\hat{u}_o)$ and $\gamma_o \geq 0$).

Let us then consider a direction $\hat{u}_o$ that now belongs to the boundary of $ES_X(f)$, and moreover to the boundary of $\bigcup_{x\in N} ES_x(f)$, where N is some real neighborhood of X. The above result entails that there exists $\alpha > 0$ (depending only on N) such that bounds of the form (3) be satisfied at X with this common *uniform* α for all directions $\hat{u}$ outside $\bigcup_{x\in N} ES_x(f)$. The choice of $V(\hat{u})$, $\gamma_o > 0$, ... depends on the other hand on $\hat{u}$, and as a matter of fact $\gamma_o(\hat{u})$ necessarily tends to zero when the direction $\hat{u}_o$ is approached, since $\hat{u}_o \in ES_X(f)$. The main content of property R at $(X,\hat{u}_o)$, when it holds, is the condition that $\gamma_o(\hat{u})$ should not tend to zero faster than linearly with respect to the angle of $\hat{u}$ with the boundary of $\bigcup_{x\in N} ES_x(f)$.

In order to introduce the precise statement of property R for square integrable functions, we mention the following results that always hold in this case. First[4a], the bracket $[P(|u|)v^{-q}]$ can always be replaced in the bounds (3) by a square integrable function d_v of u whose norm is independent of v. This function may à priori depend on the direction considered outside $ES_X(f)$. On the other hand, there always exists, as easily seen, a uniform square integrable function d of u such that the bounds $|F(u;v,X)| < d(u)$, without the exponential fall-off factor $e^{-\alpha v}$, be satisfied everywhere, in the whole region $v \geq 0$.

<u>Property R</u> (for square integrable functions)
"Being given X and a direction $\hat{u}_o$ of $\partial(ES_X(f))$, property R is by definition satisfied by f at $(X,\hat{u}_o)$ if, being given any real neighborhood N of X, there exist a neighboring cone $V(\hat{u}_o)$ of $\hat{u}_o$ with apex at the origin, $\alpha > 0$, $\chi > 0$ and a square integrable function d_v of u, whose norm is independent of v, such that :

$$|F(u;v,X)| < d_v(u)\, e^{-\alpha v} \tag{4}$$

in the region $u \in V(\hat{u}_o)$, $0 \leq v < \chi$ dist. $\{u, \bigcup_{x\in N} ES_x(f)\}$" .

<u>Example</u>.

Let f be, in the neighborhood of X, the boundary value of an analytic function <u>f</u> from the directions of an open (convex) cone Γ in Imz-space (where z=x+iy is the complexified variable of x), in which case $ES_X(f) \subset C$, where C is the closed dual cone of Γ. The above analyticity property means that there exist a neighborhood N of X and an open set B with profile Γ at the origin in y-space, such that <u>f</u> is analytic in $\{z=x+iy : x\in N, y\in B\}$. B has Γ as its profile at the origin if, being given any open cone Γ' with apex at the origin whose closure is contained (apart from the origin) in Γ, there exists $\rho > 0$ such that $\Gamma'\cap\{|y| < \rho\} \subset B$; ρ may, however, shrink to zero in general when Γ' expands to Γ.

Let $\hat{e}$ be a direction of the boundary $\partial\Gamma$ of Γ , if it exists, for which this is not the case, more precisely for which there exist $\rho(\hat{e}) > 0$ and a neighbouring cone $V(\hat{e})$ of $\hat{e}$ with apex at the origin such that the set $\{y; |y|<\rho(\hat{e}), y\in V(\hat{e})\cap\partial\Gamma\}$ belongs to the closure of B. If, moreover, f is bounded in its analyticity domain when this set is approached, then property R is satisfied at $(X,\hat{u}_o)$ for any $\hat{u}_o$ of ∂C such that $\hat{u}_o.\hat{e}=0$. The converse is also essentially true.

3. U=0 THEOREM

Theorem - "Let f', f" be square integrable functions such that property R is satisfied by f' and f" at $(X,\hat{u}_o)$ and $(X,-\hat{u}_o)$ respectively, for any direction $\hat{u}_o$ such that $\hat{u}_o \in ES_X(f')$, $-\hat{u}_o \in ES_X(f'')$. Then :

$$ES_X(f'f'') \subset \{u; \exists\, u'_n, u''_n, X'_n, X''_n, u'_n \in ES_{X'_n}(f'), u''_n \in ES_{X''_n}(f'') \quad ,$$

$$X'_n \to X, X''_n \to X, u'_n + u''_n \to u \text{ when } n \to \infty\} \; . \qquad (5)''$$

The (easy) proof is based on the formula :

$$F(u;v,X) = \int F'(u'; \frac{v}{2},X)\, F''(u-u'; \frac{v}{2},X)\, du' \qquad (6)$$

where f=f'f". The result is a generalization of the particular case obtained when f',f" satisfy the properties of the example of Sect.2 : if $\Gamma' \cap \Gamma''$ is empty (in which case X is a u=0 point), but if $\partial\Gamma' \cap \partial\Gamma''$ contains a direction $\hat{e}$ (or a set of such directions) such that the properties described at the end are satisfied by f' and f", then $ES_X(f)$ is contained in its dual half-space (or in their intersection).

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THE THEORY OF HOLONOMIC SYSTEMS WITH REGULAR SINGULARITIES AND ITS RELEVANCE TO PHYSICAL PROBLEMS

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The purpose of this report is to give a partial resume of our articles [11], [12] and indicate their relevance to physical problems. So we have tried to make this report easy to understand, especially for physicists, by sometimes sacrificing the generality of the statement. We have also concentrated our attention on topics which, we hope, relate to physical problems. In this report we use the same notations as in [12].

In the theory of linear ordinary differential equations and its applications, equations with regular singularities play a central role. Hence one might naturally wish to extend the theory to several variables case. Then one natural choice would be to concentrate our attention on holonomic systems, partly because they share the basic finiteness theorem with ordinary differential equations ([4], [8]) and partly because, not general systems, but holonomic systems are particularly important in applications (e.g. [9], [13]. [15], [20],...).

Now, let $\mathcal{M}$ be a holonomic $\mathcal{E}_X$-Module on $\Omega \subset T^*X - T^*_X X$. Let Λ be its characteristic variety. (Hence Λ is a Lagrangian variety possibly with singularities.) For each point p in the non-singular locus Λ_{reg} of Λ, we can find a suitable contact transformation φ which is defined in a neighborhood U of p and brings $\Lambda_{reg} \cap U$ to the form $T^*_Y X \cap \varphi(U)$ for a non-singular hypersurface Y in X. We choose a coordinate system $x = (x_1,\dots,x_n)$ near $\pi(\varphi(p))$(*) so that $Y = \{x_1 = 0\}$ and $\varphi(p) = (0;dx_1\infty)$. Then by definition, $\mathcal{M}$ is with regular singularities along $T^*_Y X$ at $\varphi(p)$ if and only if $\mathcal{M}$ is isomorphic to the following $\mathcal{E}_X$-Module $\mathcal{N}$ in a neighborhood of $\varphi(p)$:

$$\mathcal{N} \ : \ (p_j I - A_j)U = 0 \qquad (j=1,\dots,n),$$

where U is a column vector with N unknown functions, I is the identity matrix of size $N \times N$, $p_1 = x_1 D_1$, $p_j = D_j$ $(j=2,\dots,n)$ and A_j $(j=1,\dots,n)$ is an $N \times N$ matrix of micro-differential operators which satisfies the following condition:

(1) There exists an integer m such that the order of each component of each power A_j^{ℓ} $(\ell \geqq 0)$ is less than m.

(When a matrix A of micro-differential operators satisfies the condition (1), A is said to be of order at most 0.)

We define the notion of a holonomic $\mathcal{E}_X$-Module with R.S. (= the abbreviation of regular singularities) by requiring that it is with regular singularities along Λ at each point in Λ_{reg}. A holonomic $\mathcal{D}_X$-Module $\mathcal{L}$ is said to be with R.S. if and only if so is $(\mathcal{E}_X \underset{\mathcal{D}_X}{\otimes} \mathcal{L})|_{T^*X - T^*_X X}$.

See [12] Chap. I and Chap. V for more intrinsic definition of systems with R.S. and results which show the naturality of the above definition.

An important class of holonomic $\mathcal{D}_X$-Modules is that of the D-type equations. A D-type equation corresponds to the sheaf of meromorphic sections of a regular integrable connection in the sense of Deligne [2], namely, the sheaf of Nilsson class functions. In terms of $\mathcal{D}_X$-Modules, it is characterized as follows:

Definition 1. A holonomic $\mathcal{D}_X$-Module $\mathcal{L}$ is said to be of D-type (=Deligne type) with singularities along a hypersurface $Y = f^{-1}(0)$, if and only if it satisfies the following three conditions:

(2) $SS(\mathcal{L})$(**) $\subset \pi^{-1}(Y) \cup T^*_X X$

(3) $\mathcal{L}$ is with regular singularities along $T^*_{Y_{reg}} X$.

(4) $f: \mathcal{L} \longrightarrow \mathcal{L}$ is bijective.

(*) Here and in what follows, π denotes the canonical projection from T^*X to X.

(**) $SS(\mathcal{L})$ denotes the characteristic variety of the $\mathcal{D}_X$-Module $\mathcal{L}$.

Remark 1. Since there may be some irreducible components other than $T^*_Y X$ in $SS(\mathcal{L})$, the condition (3) does not imply immediately that $\mathcal{L}$ is with R.S. The proof for this fact requires (at least, at present) the use of $\mathcal{D}^\infty$, the sheaf of linear differential operators of infinite order. ([12] Chap. V, Theorem 5.2.3.)

Example 1. Let f be a holomorphic function defined on X. Suppose that f is not identically zero on any connected components of X. Then, for generic α, $\mathcal{D}_X f^\alpha$ is a holonomic system of D-type. ([11] Remark 1.2. The precise condition which guarantees α to be generic is also given there.)

Example 2. Let f be the same as in Example 1. Denote $f^{-1}(0)$ by Y. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-Module which is with regular singularities along $T^*_{Y_{reg}} X$ at each point x in Y_{reg}. Then the localization $\mathcal{M}_f^{(*)}$ of $\mathcal{M}$ along Y is a holonomic system of D-type along Y. ([12] Chap. II, §3, Proposition 2.3.4.)

The importance of the D-type equation lies in the fact that it is "generic" in the sense of Theorem 1 below.

Definition 2. A Lagrangian variety Λ in $T^*X-T^*_X X$ is said to be in a generic position at p in Λ if and only if $\Lambda \cap \pi^{-1}(\pi(p)) = \mathbb{C}^\times p$ holds in a neighborhood of p.

Theorem 1. Let $\mathcal{M}$ be a holonomic $\mathcal{E}_X$-Module defined in a neighborhood of p in $T^*X-T^*_X X$. Suppose that $\Lambda \underset{def}{=} \mathrm{Supp}\,\mathcal{M}$ is in a generic position at p in Λ. Then there exist a holonomic system $\mathcal{L}$ of D-type along $\pi(\Lambda)$, a holonomic $\mathcal{D}$-sub-Module $\mathcal{P}$ of $\mathcal{L}$ with $SS(\mathcal{P}) \subset T^*_X X^{(**)}$ and an injective $\mathcal{D}_{X,\pi(p)}$-linear homomorphism $\phi: \mathcal{M}_p \longrightarrow (\mathcal{L}/\mathcal{P})_{\pi(p)}$.

If, in addition, any order of any sections of $\mathcal{M}$ is not an half integer, then we do not need to divide $\mathcal{L}$ by a sub-Module $\mathcal{P}$, namely, $\mathcal{M}_p$ can be imbedded into $\mathcal{L}_{\pi(p)}$.

(*) Here $\mathcal{M}_f$ is equal to $\bigcup_{j=0}^{\infty} f^{-j}\mathcal{M}$, and it is known to be holonomic. ([5], Theorem 3.1.)

(**) Hence $\mathcal{P}$ is isomorphic to $\mathcal{O}_X^r$ for some integer r.

Remark 2. The assumption that Λ is in a generic position at p is not too restrictive, because a suitable contact transformation brings Λ to this form.

In the course of the proof of Theorem 1, we also obtain the following

Theorem 2. Assume the same conditions on $\mathcal{M}$ and Λ as in Theorem 1. Then $\mathcal{M}_p$ is a finitely generated $\mathcal{D}_{X,\pi(p)}$-Module. Furthermore we have

$$\mathcal{E}_{X,p'} \underset{\pi^{-1}\mathcal{D}_{X,\pi(p)}}{\otimes} \mathcal{M}_p = \begin{cases} \mathcal{M}_p & \text{if } p' = p \\ 0 & \text{if } p' \text{ is in } \pi^{-1}\pi(p) - T^*_X X - \mathbb{C}^{\times} p. \end{cases}$$

This theorem is useful in investigating the analytic structure of Feynman amplitudes at cuspidal points of Landau-Nakanishi surfaces, which are far from the physical region. (Cf. [13] §3.)

Remark 3. As a matter of fact, a much more striking result is proved in [12] (Chap. IV, Theorem 4.1.1): Theorem 1 holds for an arbitrary holonomic $\mathcal{E}_X$-Module if we use micro-differential operators of infinite order, namely, there exist a holonomic system $\mathcal{L}$ of D-type, its $\mathcal{D}$-sub-Module $\mathcal{P}$ with $SS(\mathcal{P}) \subset T^*_X X$ and a $\mathcal{D}^{\infty}_{X.\pi(p)}$-linear homomorphism

$$\phi: (\mathcal{E}^{\infty}_X \underset{\mathcal{E}_X}{\otimes} \mathcal{M})_p \longrightarrow (\mathcal{D}^{\infty}_X \underset{\mathcal{D}_X}{\otimes} (\mathcal{L}/\mathcal{P}))_{\pi(p)}$$

such that the associated homomorphism

$$\tilde{\phi} \underset{\text{def}}{=} 1 \otimes \phi: (\mathcal{E}^{\infty}_X \underset{\mathcal{E}_X}{\otimes} \mathcal{M})_p \to \mathcal{E}^{\infty}_{X,p} \underset{\mathcal{D}^{\infty}_{X,\pi(p)}}{\otimes} (\mathcal{L}/\mathcal{P})_{\pi(p)}$$

is an injective $\mathcal{E}^{\infty}_p$-linear homomorphism.

In fact, this imbedding theorem is the most important result in [12]. Most of the main results of [12] follow from this. So, here we will sketch some basic ideas of the proof. We like to mention that the method used in [9] to discuss the hierarchical principle for Feynman amplitudes is one of the essential ingredients of our proof.

We first introduce a vector space C, which is an analogue of the space of germs of microfunctions. It is the space of holomorphic functions defined on a cone with its apex at the origin, modulo the space of holomorphic functions defined on a neighborhood of the origin. (See (4.5.1) of [12] Chap. IV, §5 for the rigorous definition.) This space is introduced to make clear the action of micro-differential operators on holomorphic functions. (Cf. [6], [1] and [9].) We can then prove that $V = \mathrm{Hom}_{\mathcal{E}_p^\infty}(\mathcal{M}_p^\infty, C)$ is finite-dimensional and that $\mathcal{M}_p^\infty$ is imbedded into $\mathrm{Hom}_{\mathbb{C}}(V, C)$. Let $\{s_j\}_{j=1}^N$ be a system of generators of $\mathcal{M}$, let $\{\phi_\nu\}_{\nu=1}^{\ell}$ be a base of V and let $\varphi_{j,\nu}$ be a holomorphic function whose modulo class in C is $\phi_\nu(s_j)$. Then we can prove that $\varphi_{j,\nu}$ can be extended to a multi-valued holomorphic functions on $X-\pi(\Lambda)$. Furthermore, thus extended functions are with finite determination. Since $\pi(\Lambda)$ is a hypersurface, we can choose a holomorphic function f on X such that $\pi(\Lambda) = f^{-1}(0)$. In this case, the following result is proved by Deligne [2].

Let φ denote one of $\varphi_{j,\nu}$. Then there exists Nilsson class functions ψ_k and single-valued holomorphic fucntion a_k defined on $X-f^{-1}(0)$ such that $\varphi = \sum_{k=1}^{m} a_k\psi_k$ holds.

Using this result we can obtain a much sharper result to the effect that φ has the form $\sum_{k=1}^{m} Q_k\psi_k$ with linear differential operators Q_k (of course, possibly with infinite order). See [12] Chap. II, §2 Theorem 2.2.4 for the precise statement and the proof. Here, instead of repeating the proof, we present a heuristic, but instructive argument which, we hope, convince the reader that there must exist such an operator Q_k.

We know ([5], Theorem 2.7) that there exists a non-zero polynomial $b(s)$ which satisfies

$$(5) \qquad b(s)f^s\varphi_1 = P(s,x,D_x)(f^{s+1}\varphi_1)$$

for some $P(s) \underset{\mathrm{def}}{=} P(s,x,D_x)$ in $\mathcal{D}_X[s]$, i.e., for some linear differential operator which is a polynomial in s. Then, a superficial application of (5) would entail

$$(6) \qquad f^{-p}\varphi_1 = \frac{P(-p)\cdots P(-1)}{\prod_{s=1}^{p} b(-s)}\varphi_1 .$$

Since a_1 is single-valued on $X-f^{-1}(0)$, it has the form