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Société Mathématique du Canada

Second Edmonton Conference on APPROXIMATION THEORY

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PREFACE

The Second Edmonton Conference on Approximation Theory was held in Edmonton, Alberta, June 7-11, 1982. The Conference was devoted to Approximation Theory and related topics, including spline approximation, computational problems, complex and rational approximation, and techniques from harmonic analysis and the theory of interpolation of operators. In conformity with the requirements of this series, this volume consists of refereed papers by some of the invited speakers.

The participants of the conference were:

J.A.H. Alkemade, W.A. Al-Salam, B.M. Baishanski, K. Bartke, R.K. Beatson, H. Becker, C. Bennett, R. Bojanic, S.S. Bonan, D. Borwein, P.B. Borwein, D. Boyd, P.L. Butzer, J.S. Byrnes, B.A. Chalmers, B.L. Chalmers, W. Cheney, C. Chui, Z. Ciesielski, C. Coatmelec, F. Deutsch, R. DeVore, Z. Ditzian, C. Dunham, N. Dyn, P. Erdös, J.J.F. Fournier, C. Frappier, W.H.J. Fuchs, A. Giroux, M. Goldstein, H.H. Gonska, T.N.T. Goodman, W. Haussmann, A. Jakimovski, R.N. Kesarwani, D. Khavinson, P.E. Koch, J. Korevaar, A.T. Lau, L. Leindler, D. Leviatan, G.G. Lorentz, W.A.J. Luxemburg, M.T. McGregor, J. Mach, M.J. Marsden, A. Meir, H.N. Mhaskar, C. Micchelli, R.J. Nessel, P. Nevai, D.J. Newman, T. Nishishiraho, D.V. Pai, J. Peetre, G. Phillips, Q.I. Rahman, S.D. Riemenschneider, E.B. Saff, R.B. Saxena, H.J. Schmid, I.J. Schoenberg, C.F. Schubert, L.L. Schumaker, A. Sharma, R.C. Sharpley, B. Shawyer, B. Shekhtman, Y.G. Shi, O. Shisha, P.C. Sikkema, S.P. Singh, R.D. Small, P.W. Smith, H.M. Srivastava, R.J. Stroeker, F. Stenger, J. Szabados, B. Thorpe, J.L. Ullman, R.S. Varga, A.K. Varma, R. Vermes, P. Vértesi, H. Wallin, K. Wiggins, C. Wlodarski, D. Wulbert.

Sponsored by the Canadian Mathematical Society, the Conference was supported by grants from the Natural Sciences and Engineering Research Council of Canada and the University of Alberta. We wish to express our

graditude for the financial support of the above mentioned institutions; also, we wish to thank the conference secretary Mrs. Donna Pawluk, and Mrs. Christine Fischer for her patient work in preparing the camera-ready copy for this volume.

Z. Ditzian

A. Meir

S. Riemenschneider

A. Sharma

LIST OF SPEAKERS AND TITLES

J.A.H. Alkemade, "On exponential operators".

B.M. Baishanski, "Polynomials of a given length".

K. Bartke, "Numerical aspects of rational approximation".

R.K. Beatson, "Constrained approximation".

C. Bennett, "Maximal functions and oscillation".

R. Bojanic, "Estimates for the rate of approximation of functions of bounded variation by Hermite-Fejér polynomials".

S.S. Bonan, "Mean convergence of Lagrange interpolation at zeros of generalized Hermite polynomials".

D. Borwein, "Transformation of certain sequences of random variables by generalized Hausdorff matrices".

P.B. Borwein, "Rational approximation with real poles to e^{-x} and x^n".

D. Boyd, "A convergence property of the rational approximants generated by Schur's algorithm".

P.L. Butzer, "Whittaker's cardinal series representation, the Poisson summation formula, and approximate integration".

J.S. Byrnes, "Piecewise polynomial approximation on $(0,\infty)$".

B.L. Chalmers, "The form of the minimal projection operator".

E.W. Cheney, "Existence, characterization, and computation of best approximations in tensor-product subspaces".

C. Chui, "On spaces of piecewise polynomials with boundary conditions II. Type I triangultions".

Z. Ciesielski, "Spline bases in Hardy spaces".

C. Coatmelec, Prolongement explicite des elements de $L^p(\Omega)$ et majoration du module de regularite du prolongement".

F. Deutsch, "Continuous selections and linear selections for metric projections".

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M. Goldstein, "Weak Arakelian sets".

H.H. Gonska, "Some remarks on Hermite-Fejér-type interpolation".

T.N.T. Goodman, "Solvability of cardinal spline interpolation problems".

W.H. Haussmann, "Best L^1-approximation by blending functions".

A. Jakimovski, "Some remarks on an interpolation problem".

D. Khavinson, "L^1-approximation of continuous functions by elements of a uniform algebra".

P.E. Koch, "Jackson theorems for generalized polynomials with special applications to trigonometric and hyperbolic functions".

J. Korevaar, "Müntz-type theorems for arcs and for $\mathbf{R}^n$".

L. Leindler, "On relations of coefficient conditions".

D. Leviatan, "Comonotone approximation by polynomials and splines".

G.G. Lorentz, "Properties of trigonometric polynomials".

W.A.J. Luxemburg, "On a conjecture of Whitham and a theorem of Vekua".

J. Mach, "Best n-nets and best n-dimensional extremal subspaces".

M.J. Marsden, "Some open problems on variation diminishing splines".

A. Meir, "An upper bound for the measure of convexity of L^p spaces".

H.N. Mhaskar, "On the domain of the convergence of expansions in polynomials orthogonal on the whole real line with respect to general weight functions".

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D.V. Pai, "On set valued f-projections and f-farthest point mappings".

J. Peetre, "Hankel operators, rational approximation and allied questions in analysis".

G. Phillips, "Polynomial approximation using equioscillation on the extreme points of Chebyshev polynomials".

Q.I. Rahman, "On zeros of rational functions arising from certain determinants".

S.D. Riemenschneider, "Interpolation with box splines on $\mathbf{R}^2$.

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H.J. Schmid, "Symmetric cubature formulas for the square".

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BEST n-NETS IN NORMED SPACES

Dan Amir and Jaroslav Mach

ABSTRACT. We show that a sufficient condition of Keener is not necessary for the existence of best n-nets for bounded sets, $n > 1$. Furthermore, it is shown that for any $n > 1$, $C(S)$, S infinite compact metric, contains a bounded set without a best n-net.

In 1964, Garkavi [2] introduced the concepts of Chebyshev center, best n-net and best n-dimensional cross-section. He used compactness and contraction arguments to prove their existence. These arguments apply to most classical spaces except, unfortunately, the spaces of continuous functions $C(\Omega)$. Kadec and Zamyatin [6] proved that any bounded set in $C(\Omega)$ has a Chebyshev center. Brown [5] proved that even compact sets in $C(\Omega)$ do not need to have a best n-dimensional cross-section. The question whether any bounded set in $C(\Omega)$ has a best n-net for $n > 1$ has remained open. It was proved in [1] that the answer to this question is positive for compact sets. We show here that in general the answer is negative (Theorem 2). Keener [3] gave a sufficient condition for the existence of best n-nets in normed spaces. Since this condition is also necessary for $n = 1$ (the Chebyshev center case) he asked whether this is true for $n > 1$. We show here that this is not the case (Theorem 1).

Let X be a normed space, A a bounded subset of X, $n \in \mathbf{N}$. A set $\{y_1,\ldots,y_n\}$ of n points in X is called a *best n-net* of A if there is a number $r > 0$ such that A is in the union of n closed balls with centers in $y_1,\ldots,y_n$ and radius r, and there is no collection of n closed balls whose common radius is less than r and whose union contains A.

The problem of finding a best n-net ($n>1$) seems to be considerably more difficult than that of the Chebyshev center case, even in the Euclidean case (e.g. the well known problem of distributing most efficiently n points on the sphere). The troubling features of the n-net problem can be exhibited in

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the simplest 2-dimensional example. The best 2-nets for the equilateral triangle $\{x,y,z\}$ are $\{\frac{x+y}{2},w\}$, $\{\frac{x+z}{2},u\}$, $\{\frac{x+z}{2},v\}$, where $d(x,u)$, $d(y,v)$, $d(z,w) \leq \frac{1}{2}d(x,y)$. But a small change of the point x so that $d(x,y) > d(x,z) = d(y,z)$ reduces the best 2-net set to $\{\frac{x+z}{2},[\frac{y+z}{2},y]\}$. This shows the instability and discontinuity of the best 2-net operator. This is why, when compactness or contraction arguments do not work, the existence of a best 2-net is not easy to establish.

From the classical spaces, this leaves open the problem whether best n-nets (n>1) must exist for bounded sets in $C(\Omega)$-spaces. Garkavi [2] proved the existence in c_0. His proof is easily modified to yield:

PROPOSITION. For every Γ, every bounded $A \subset \ell_\infty(\Gamma)$ has best n-nets in $c_0(\Gamma)$ for all $n \in \mathbf{N}$.

PROOF. Let r_n be the relative n-net radius of A in $c_0(\Gamma)$ (i.e., the number $\inf \sup_{x\in A} d(x,Y)$, where the infimum is taken over all $Y \equiv \{y_1,\ldots,y_n\} \subset c_0(\Gamma)$). Let $Y^m \equiv \{y_1^m,\ldots,y_n^m\}$, $y_i^m \in c_0(\Gamma)$, satisfy $\sup_{x\in A} d(x,Y^m) < r_n + \frac{1}{m}$. Since $\Gamma_0 \equiv \bigcup_m \operatorname{spt} Y^m$ is countable, and since $|a(\gamma)| \leq r_n$ for every $a \in A$ and $\gamma \notin \Gamma_0$, it suffices to take a best n-net for $A|_{\Gamma_0}$ in $c_0(\Gamma_0)$ and extend it by zeros on $\Gamma - \Gamma_0$. Thus the problem is reduced to relative best n-nets in c_0 for bounded sets in ℓ_∞. But the fact that $A \subset c_0$ plays no role in Garkavi's proof. □

Keener [3] defined, for a bounded $A \subset E$, the $A(n)$-*topology* of E as the one in which the closed balls $\{B(x,r),\ x \in A,\ r \geq r_n(A)\}$ (where $r_n(A)$ is the n-net radius of A) is a subbasis for the closed sets, and called E an $M(n)$-*space* if $(E,A(n))$ is compact for every bounded $A \subset E$. He showed that if E is an $M(n)$-space then every bounded subset of E has a best n-net, and that for $n = 1$ the converse is also true. Therefore he asked whether the converse holds also for $n > 1$.

Keener observed that dual spaces are $M(n)$ (since every $A(n)$ is weaker than the w*-topology). It is also immediate to check that if F is an $M(n)$-space and E is the range of a contractive projection from F, then E is an $M(n)$-space, too. Keener showed that c_0 is $M(n)$ for every n. (His proof, however, uses the existence of a best n-net, so that it does not supply an alternative to Garkavi's result.) The following example shows that the answer to Keener's question is negative.

THEOREM 1. There exists a Banach space E in which every bounded set has a best n-net for every $n \in \mathbf{N}$, yet the $A(n)$ topologies are not compact for all $n \geq 2$ for some bounded $A \subset E$.

PROOF. Let Γ be an uncountable set, $m(\Gamma)$ the space of bounded functions on Γ equipped with the sup norm, $E = \{x \in m(\Gamma);\ x$ is constant except on a countable subset Γ_x of $\Gamma\}$. E is a closed subspace of $m(\Gamma)$, hence a Banach space. If $A \subset m(\Gamma)$ is bounded, let $x_1^j,\dots,x_n^j \in E$, $j \in \mathbf{N}$, be such that

$$\sup_{x\in A} d(x,\{x_1^j,\dots,x_n^j\}) \leq r_n(A) + \frac{1}{j}$$

where $r_n(A)$ is the relative n-net radius of A with respect to E. Let $\Gamma_0 = \bigcup_{j=1}^{\infty} \bigcup_{i=1}^{n} \Gamma_{x_i^j}$ (which is countable), $F = \{x \in E;\ x$ is constant off $\Gamma_0\}$. Clearly $F \cong m(\Gamma_0{}^*)$, hence F has best n-nets for bounded sets in $m(\Gamma)$ and $\bar{r}_n(A) = r_n(A)$ (where $\bar{r}_n(A)$ is the relative n-net radius with respect to F) is attained. On the other hand, let $\Gamma = \Gamma_1 \cup \Gamma_2$, Γ_i disjoint uncountable. Consider $A = \{1 + 2\chi_{\Delta_1};\ \Delta_1 \subset \Gamma_1$ countable$\} \cup \{-1 - 2\chi_{\Delta_2},\ \Delta_2 \subset \Gamma_2$ countable$\}$. Clearly, $\{-2,2\}$ is a best 2-net for A with radius 1. But the balls $B(x,2)$, $x \in A$, have nonempty finite intersections, while $\bigcap_{z\in A} B(z,2)$ is empty in E. □

REMARK. Since the $A(n+1)$ topology is stronger than the $A(n)$ topology, $C(\Omega)$ is not $M(n)$ for any $n > 1$, unless Ω is extremally disconnected (in which case the existence of best n-nets is guaranteed since $C(\Omega)$ admits a projection of norm one from some $C(\beta D) = \ell_1(D)^*$).

THEOREM 2. If some $\omega \in \Omega$, Ω compact Hausdorff, is an accumulation point of two disjoint sequences (s_n) and (t_n), then there is a bounded sequence A in $C(\Omega)$ with no best 2-net.

PROOF. We may pass to subsequences and assume that $(s_n) \cup (t_n)$ is a relatively discrete set, and find disjointly supported Urysohn functions (u_n), (v_n) satisfying $0 \leq u_n \leq u_n(s_n) = 1$, $0 \leq v_n \leq v_n(t_n) = 1$. Let

$$g_n = 4 + u_n,\quad h_n = 1 + \frac{n+3}{n+1} \sum_{i=1}^{n}(u_i+v_i) - \frac{2n+4}{n+1} v_n,$$

$$A = \{-2\} \cup \{g_n;\ n \in \mathbf{N}\} \cup \{h_n;\ n \in \mathbf{N}\} .$$

Let $k_m = 3 - \sum_{i=1}^{m} v_i$. Then $\|h_n - k_m\|$ equals 3 if $n > m$ or equals 2 if

$n \leq m$, and $\|h_n\| \leq \|h_{m+1}\| = (2m+6)/(m+2)$ for $n > m$, hence $\sup_{x\in A} d(x, \{0,k_m\}) = (2m+6)/(m+2) \to 2$. However, if we had $\bar{x},\bar{y} \in C(\Omega)$ with $\sup_{x\in A} d(x,\{\bar{x},\bar{y}\}) = 2$, we might assume $\bar{x} \leq 0$ (to get -2), but then we must have $\|\bar{y} - g_n\| \leq 1$ and $\|\bar{y} - h_n\| \leq 1$. In particular, since $g_n(s_n) = 5$ and $h_n(t_n) = 0$, we must have both $\bar{y}(\omega) \geq 3$ and $\bar{y}(\omega) \leq 2$ which is impossible.□

In particular, the spaces $C(\Omega)$ do not admit best n-nets for bounded sets in each of the following cases:

(i) Ω is compact metric.

(ii) Ω is a infinite ordinal.

(iii) $\Omega = D^*$, the one point compactification of a discrete infinite D.

REMARK. Keener notes that c_0 is not compact in the topology generated by the complements of all closed balls, and says that "it is apparently unknown if there are non-conjugate spaces with this property" -- however, by a result of Lindenstrauss [4], this property is shared by all spaces E admitting a projection of norm one from E^{**}, e.g. $L_1(\mu)$ spaces and P_1 spaces (which need not be dual spaces).

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ESTIMATES FOR THE RATE OF APPROXIMATION OF FUNCTIONS OF BOUNDED VARIATION BY HERMITE-FEJÉR POLYNOMIALS

R.Bojanic and F.H. Cheng

1. INTRODUCTION. Let f be a real-valued function defined on $[-1,1]$. The polynomial $H_n(f,x)$ of Hermite-Fejér interpolation based on the zeros $x_{kn} = \cos(\frac{2k-1}{2n}\pi)$, $k = 1,\ldots,n$, of the Chebyshev polynomial $T_n(x) = \cos(n \arccos x)$ is defined by

$$(1.1) \qquad H_n(f,x) = \sum_{k=1}^{n} f(x_{kn})(1-xx_{kn})\left(\frac{T_n(x)}{n(x-x_{kn})}\right)^2 .$$

It was proved by L. Fejér [1] that $H_n(f,x)$ converges to $f(x)$ uniformly on $[-1,1]$ whenever f is a continuous function on $[-1,1]$. Since then, this subject has been studied extensively and various quantitative versions of Fejér's result are known (see [2]-[9]).

In this paper we shall study the behavior of Hermite-Fejér polynomials for functions of bounded variation. We shall give an estimate for the rate of convergence of $H_n(f,x)$ for functions of bounded variation at points of continuity. In addition, we shall prove that at points of discontinuity where $f(x+0) \neq f(x-0)$ the Hermite-Fejér polynomials $H_n(f,x)$ do not converge.

2. RESULTS. Our first result can be stated as follows:

THEOREM 1. Let f be a function of bounded variation on $[-1,1]$ and continuous at $x \in (-1,1)$. Then for all n sufficiently large

$$(2.1) \qquad |H_n(f,x) - f(x)| \le \frac{64\, T_n^2(x)}{n} \sum_{k=1}^{n} V_f\left[x - \frac{\pi}{k}, x + \frac{\pi}{k}\right] + 2V_f\left[x - \frac{\pi|T_n(x)|}{2n}, x + \frac{\pi|T_n(x)|}{2n}\right] .$$

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Here, $V_f[a,b]$ is the total variation of f on $[a,b]$, usually denoted by $V_a^b(f)$*.

As far as the precision of estimate (2.1) is concerned, consider the Hermite-Fejér polynomial of the function $f(x) = x^2$ at $x = 0$, for even n. Since $T_n(0) = 1$ if n is an even integer, we have

$$H_n(f,0) - f(0) = \sum_{k=1}^{n} \frac{T_n^2(0)}{n^2} = \frac{1}{n} .$$

On the other hand, from (2.1) it follows that

$$|H_n(f,0)-f(0)| \leq \frac{64}{n} \sum_{k=1}^{n} V_f\left[-\frac{\pi}{k},\frac{\pi}{k}\right] + 2V_f\left[-\frac{\pi}{2n}, \frac{\pi}{2n}\right]$$

$$\leq \frac{128}{n} \sum_{k=1}^{n} V_f\left[0, \frac{\pi}{k}\right] + 4V_f\left[0, \frac{\pi}{2n}\right].$$

Since $V_f[0,\delta] = \delta^2$, we have

$$|H_n(f,0) - f(0)| \leq \frac{128\pi^2}{n} \sum_{k=1}^{n} \frac{1}{k^2} + \frac{\pi^2}{n^2} < \frac{C}{n}$$

for some $C > 1$. Hence for the function $f(x) = x^2$ when n is even integer we have

$$\frac{1}{n} \leq |H_n(f,0) - f(0)| \leq \frac{C}{n}$$

for some positive constant $C > 1$. Therefore (2.1) cannot be improved asymptotically.

However, if x is a point of discontinuity of f where $f(x+0) \neq f(x-0)$, the sequence $(H_n(f,x))$ is no longer convergent. This is the content of the following theorem.

THEOREM 2. If f is a function of bounded variation on $[-1,1]$ and $x \in (-1,1)$ then

$$\lim{}^{\sup}_{\inf} H_n(f,x) = \frac{1}{2}\left(f(x+0)+f(x-0)\right) \pm \frac{1}{2}|f(x+0) - f(x-0)|\,\beta(x)$$

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*We assume here and in the rest of the paper that f is extended to the entire real line by $f(x) = f(1)$ for $x > 1$ and $f(x) = f(-1)$ for $x < -1$.