

# DYNAMICAL CHAOS

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## Chairman's introduction

BY E. C. ZEEMAN, F.R.S.

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The understanding of chaos and strange attractors is one of the most exciting areas of mathematics today. It is the question of how the asymptotic behaviour of deterministic systems can exhibit unpredictability and apparent chaos, due to sensitive dependence upon initial conditions, and yet at the same time preserve a coherent global structure. The field represents a remarkable confluence of several different strands of thought.

1. Firstly came the influence of differential topology, giving global geometric insight and emphasis on qualitative properties. By qualitative properties I mean invariants under differentiable changes of coordinates, as opposed to quantitative properties which are invariant only under linear changes of coordinates. To give an example of this influence, I recall a year-long symposium at Warwick in 1979/80, which involved sustained interaction between pure mathematicians and experimentalists, and one of the most striking consequences of that interaction was a transformation in the way that experimentalists now present their data. It is generally in a much more translucent form: instead of merely plotting a frequency spectrum and calling the incomprehensible part 'noise', they began to draw computer pictures of underlying three-dimensional strange attractors.

2. Secondly, classical differential equations remain as important as ever. Although the general theory has seen major advances in the last two decades, every now and then it runs into a brick wall. For example after Thom's spectacular success in the 1960's in proving the density of stable functions and classifying elementary catastrophes, a similar programme was attempted for dynamical systems. Structurally stable systems, however, turned out to be neither dense nor classifiable. Attention has consequently now switched back to examining classical examples with the advantage of new insight. Meanwhile there is a vast mountain of unsolved problems for the mathematician to work on. Even the notion of strange attractors is not, as yet, satisfactorily defined, and will remain so until there is enough theory built upon it to give an appropriate definition due weight.

3. Thirdly, comes the influence of the ergodic theoretical approach, bringing ideas of entropy and averaging to bear upon differential equations. Previously these two fields were studied separately: a system was thought to be either deterministic or ergodic. But now we are familiar with many examples in which the regions of predictability and chaos are closely interwoven, and the transitions between the two are of paramount importance.

4. Fourthly was the advent of fast interactive graphics in computers, which enabled one to perceive patterns within complex systems that might otherwise never have been suspected to be there, and to formulate conjectures that can then be proved by traditional methods. A beautiful example of this has been the use of

renormalization techniques to study the breakdown of invariant tori and the onset of chaos.

5. Fifthly is the development of precision experiments in chaotic physical systems. For example there have been major advances in the depth of understanding of the onset of turbulence. And the confidence that this has given to physicists and astronomers has opened their eyes to many examples, now well documented, of deterministic chaos amongst natural phenomena.

6. Sixthly, and finally, has come a new understanding of chaotic biological systems. Mathematical modelling in biology tends to be either very simple or very sophisticated. For most research in biology one needs little more mathematical equipment than the integers, but at the other end of the spectrum one needs very sophisticated mathematics, because biology is in principle far more complicated than physics. We shall see in the next few decades a new generation of mathematical biologists beginning to tackle problems in which the complexity is fundamental.

Summarizing: it is the confluence of these ideas makes the subject so rich, and promises us a fascinating meeting.

## Diagnosis of dynamical systems with fluctuating parameters

By D. RUELLÉ

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Many time evolutions occurring in Nature may be considered as non-autonomous, but dependent on parameters that vary slowly with time. It is argued here that some, but not all, of the tools used to understand chaotic dynamics remain useful in this situation.

In recent years it has been ascertained that many time evolutions observed in Nature exhibit the features of *chaos*. This means that they are deterministic time evolutions involving only a finite number of degrees of freedom, but that a complicated non-periodic behaviour is observed, due to *sensitive dependence on initial condition*. Mathematically, the deterministic time evolution corresponds to an autonomous differentiable dynamical system, sensitive dependence on initial condition means that a small perturbation of the initial condition will grow exponentially with time (as long as it does not become too large). The asymptotic evolution of the system takes place on a (usually) complicated set in phase space called a *strange attractor*.

A fundamental finding is that hydrodynamic turbulence is chaotic, and described by strange attractors. Many other examples of chaos have been demonstrated clearly in various areas of physics and chemistry, and less clearly in biology and economics. Investigations of experimental chaos have been mostly of either geometric or ergodic nature. (The study of chaotic power spectra, broad band spectra, has also played an important role historically, but does not at this time yield the sort of detailed information that is provided by other techniques.) The geometric approach visualizes reconstructions of attractors and of bifurcations and is limited to weakly excited systems ('onset of turbulence'). The ergodic approach determines information dimensions, characteristic (i.e. Liapunov) exponents and entropies, and is applicable to moderately excited systems.

The analysis of relatively modest data can provide a usable power spectrum, or an estimate of the information dimension by the Grassberger–Procaccia algorithm†. In general, however, the detailed diagnosis of chaotic dynamical systems requires long time series of high quality (stability of parameters of the system and precision of experimental measurements). One faces then the problem that the systems for which our techniques work best are not those in which we are mostly interested. Among the latter we may quote pulsating variable stars,

† See Grassberger & Procaccia (1983); see also Eckmann & Ruelle (1985) for a general review of the ergodic approach.

electroencephalograms and time series of economics. One can of course dismiss at least the last two examples by arguing (reasonably) that the electrical activity of the brain, and the stock market, are not autonomous dynamical systems with few degrees of freedom.

Against this reasonable view let me remark that information dimension estimates for EEGs (electroencephalograms) (see Layne *et al.* 1986; Rapp *et al.* 1987) and time series of economics (see Scheinkman & Le Baron 1986) are not at all suggestive of pure randomness. Let me talk of EEG data that I have seen (some from Rapp and some from Lehmann, analysed in Geneva in collaboration with J.-P. Eckmann and S. Kamphorst). They suggest that there are many degrees of freedom, or 'modes', with decreasing amplitudes, and that computations of information dimension yield variable results depending on which modes have amplitudes sufficiently large to be captured by a given calculation. This type of 'explanation' is, however, basically unsatisfactory because 'modes' cannot in general be separated in a truly nonlinear theory.

What then? I suggest that some interesting time evolutions occurring in Nature, those with *adiabatically fluctuating parameters* (AFPs), although not represented by an autonomous dynamical system, are accessible to analysis. I have in mind evolutions of the type

$$dx/dt = F(x, \lambda(t)) \quad (\text{continuous time})$$

or

$$x_{n+1} = f(x, \lambda(n)) \quad (\text{discrete time}),$$

where the time dependence of  $\lambda$  is assumed to be adiabatic (slow compared with the characteristic times of the autonomous systems obtained by fixing  $\lambda$ ), and not too large. Then, instead of a fixed attractor  $A$ , we have a family  $(A_\lambda)$  depending on  $\lambda = \lambda(t)$  or  $\lambda(n)$ . The evolution of  $\lambda$  might itself be determined by a dynamical system, but we consider it as arbitrarily given *a priori*. Note that a time evolution of the above type is expected both for EEGs and in economics. Note also that *noise* can be accommodated in our  $\lambda$ -dependence provided that it satisfies the requirement of adiabaticity.

A first remark is that in a system with AFPs the information dimension will be considerably messed up, because instead of looking at an attractor  $A$  we are looking at a union  $\cup A_\lambda$ . The observed information dimension will thus be the dimension of the attractors  $A_\lambda$  (supposed to be independent of  $\lambda$ ) plus the dimension of the set of  $\lambda$ s in parameter space. If we observe, for instance, an attracting periodic orbit with slowly decreasing amplitude (mechanical oscillations with friction) we shall obtain a dimension equal to  $1 + 1 = 2$ . Long-term evolution in economics would similarly increase the dimension by 1. All we can say in general is that the observed dimension is an upper bound to the dimensions of the  $A_\lambda$  (more precisely one should speak of the information dimension of invariant probability measures carried by  $A_\lambda$ ). If one suspects (as in economics) that there is a long time 'secular' evolution of the system, this can be checked by taking an *early* point  $X(t_0)$  on the reconstructed attractor and looking at the statistics of times at which the point  $X(t)$  comes back close to  $X(t_0)$ . (For a long time series these times will be predominantly at the beginning of the series if there is a secular evolution of the system.)



Contrary to the dimension, the higher characteristic exponents may not be much perturbed by the fluctuation of the parameters  $\lambda$ . In other words, it will often be the case that the higher characteristic exponents are stable *practically* under small changes of  $\lambda$ ; note however that this is not a mathematical statement of continuity. (If the time evolution of  $\lambda(t)$  is given by a dynamical system, adiabaticity will correspond to small exponents for the  $\lambda$ -evolution, and those will not interfere with the higher characteristic exponents of the global system.) Therefore the determination of the higher characteristic exponent (or exponents) is very desirable for systems with AFPS, because it (or they) can provide more unambiguous information than the information dimension.†

We turn now to the problem of short-term predictions for the time evolution of dynamical systems. Consider for simplicity a time series  $(u_i)$  corresponding to a system with discrete time, and chose an embedding dimension  $n$  such that the points  $(x_i, \dots, x_{i+n-1}) \in \mathbb{R}^n$  give a faithful representation of the dynamical system on its attractor. Then one can determine a continuous function  $\Phi$  such that

$$u_{i+n} = \Phi(u_i, u_{i+1}, \dots, u_{i+n-1})$$

(see Ruelle 1987, §3). The introduction of AFPS will not make this representation useless. Therefore short-term predictions of the evolution of a dynamical system with fluctuating parameters remain possible.

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#### Discussion

N. O. WEISS (*Department of Applied Mathematics and Theoretical Physics, University of Cambridge, U.K.*). Surely there will be difficulties if one is dealing with a physical system involving several disparate timescales. One might attempt to describe its behaviour by techniques involving separation of scales or averaging, but if one simply increases the embedding dimension there may be spurious results.

† See Eckmann *et al.* (1987) for the description of an algorithm for the determination of the characteristic exponents.

For example, solar magnetic activity shows several different timescales, all of which appear to involve chaos. The day-to-day variation (which has been studied by Spiegel and his colleagues) can be separated from the 11-year solar cycle (which is aperiodic) and from the long-term irregular modulation associated with grand minima. No doubt there are other timescales too and each of them could have its own low-dimensional attractor.

D. RUELLE. The choice of a unit of time between measurements, and of the total recording time, operate a certain choice of timescale. The effect of smaller scales may appear as *noise*, and that of longer scales as *drift*. The ideas of this paper would apply to this drifting situation.

## Nonlinear dynamics, chaos and complex cardiac arrhythmias

BY L. GLASS<sup>1</sup>, A. L. GOLDBERGER<sup>2</sup>, M. COURTEMANCHE<sup>1</sup>  
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Periodic stimulation of a nonlinear cardiac oscillator *in vitro* gives rise to complex dynamics that is well described by one-dimensional finite difference equations. As stimulation parameters are varied, a large number of different phase locked and chaotic rhythms is observed. Similar rhythms can be observed in the intact human heart when there is interaction between two pacemaker sites. Simplified models are analysed, which show some correspondence to clinical observations.

### 1. INTRODUCTION

The normal adult human heart at rest usually beats at a rate of between 50 and 100 times per minute. In many circumstances, some of which are life-threatening, but most of which are not, the normal rhythmicity is altered, resulting in abnormal rhythms called cardiac arrhythmias. The point of this paper is to show that a branch of mathematics called nonlinear dynamics may be useful in the analysis of physiological processes believed to underlie normal heart rate regulation and some cardiac arrhythmias.

The idea that mathematical analysis can play a role in understanding cardiac arrhythmias is not novel. Indeed, in the 1920s it was demonstrated that as parameters in mathematical models for the heart were varied, several different rhythms that resembled clinically observed arrhythmias could be generated (Mozart 1924; van de Pol & van der Mark 1928). In nonlinear mathematics, these changes in the qualitative features of the rhythms that are observed as parameters vary are called bifurcations. Thus the problem of understanding cardiac arrhythmias in the human heart is identified with understanding the bifurcations and complex dynamics in mathematical models of the human heart.

One type of dynamic behaviour that is the object of intensive analysis in mathematics is chaos. Loosely, chaos is defined as aperiodic dynamics in deterministic systems in which there is sensitive dependence to the initial conditions. This means that although in principle one could determine precisely the future evolution of the system starting from some initial condition, for chaotic dynamics any difference in the initial condition, no matter how small, will eventually lead to marked differences in the future evolution of the system. Although the existence of chaos was known to Poincaré and others since the end of the last century, in the

past decade there has been a recognition of the potential significance of chaos in understanding the genesis of aperiodic dynamics experimentally observed in the natural sciences (Cvitanovic 1984). Unfortunately, there is in our view not yet an adequate operational definition for chaos in experimental or naturally occurring systems, but see Mayer-Kress (1986) for recent advances. The concept of chaos excludes non-deterministic stochastic processes, such as the Poisson process or random walk. It is not yet known how to measure the relative contribution of chaos as opposed to non-deterministic stochastic processes in experimental data.

Normal individuals show marked fluctuations in heart rate (Kitney & Rompelman 1980; Kobayashi & Musha 1982; Pomeranz *et al.* 1985; De Boer *et al.* 1985). In addition, cardiac arrhythmias are often extremely irregular and unstable (Pick & Langendorf 1979; Schamroth 1980). The adjective 'chaotic' is sometimes used to characterize cardiac arrhythmias that are believed to arise when there are several pacemaker sites competing for control of the myocardium (Katz 1946; Phillips *et al.* 1969; Chung 1977). It has been proposed that chaotic dynamics, in the mathematical sense, may underlie normal heart-rate variability (Goldberger *et al.* 1984; Goldberger & West 1987) as well as certain cardiac arrhythmias in humans (Guevara & Glass 1982; Smith & Cohen 1984; Glass *et al.* 1986*b*). The absence of a clear definition for chaos in experimental data has led to controversy. For example, ventricular fibrillation, an arrhythmia that leads to rapid death, is frequently called chaotic by clinicians, and it has been proposed that it may be associated with chaos in deterministic systems (Smith & Cohen 1984). However, there are marked periodicities during ventricular fibrillation, and the presence of deterministic chaos in this arrhythmia has been questioned (Goldberger *et al.* 1985, 1986).

In humans it is frequently difficult to analyse the mechanism underlying an arrhythmia, and systematic experimental studies are usually not feasible. One means of analysis is from the electrocardiogram (ECG), a record of electrical potential differences on the surface of the body that reflects the electrical activity associated with the heartbeat. Because the ECG can be obtained with lightweight monitors, it can be readily recorded over long time intervals. The ambulatory (Holter) ECG is an important means for evaluating patients. Holter recordings for as long as 24 h can be readily obtained, but conventional analysis of such records is limited. The great wealth of data about the dynamics of the heart that is contained in such records is generally distilled to characterize the mean heart rate and range. The presence and frequency of abnormal electrocardiographic complexes, which reflect abnormalities in cardiac impulse formation and propagation, are also determined. However, the analysis of long-term fluctuations in the Holter ECG is largely ignored.

One class of arrhythmias that has recently been the subject of much attention results from the presence of two pacemakers: the normal (sinus) pacemaker and a pacemaker at an ectopic (non-sinus) location. Such rhythms, whose existence has been recognized since the start of this century (Fleming 1912; Kaufmann & Rothberger 1917) are now called parasystolic rhythms. The possibility for interactions between the sinus rhythm and the ectopic rhythm often complicates

interpretation of such rhythms. However, recent workers have made great progress in developing both experimental (Jalife & Moe 1976; Jalife & Michaels 1985) and theoretical (Moe *et al.* 1977; Swenne *et al.* 1981; Ikeda *et al.* 1983) models for parasystole. Interpretation of ECG records has led to the recognition of the importance of parasystolic mechanisms (Jalife *et al.* 1982; Nau *et al.* 1982; Castellanos *et al.* 1984).

Here we consider the interaction between a fixed periodic stimulus and a cardiac oscillator. Such a problem is of interest because it is amenable to experimental and theoretical analysis and because of its relevance to the interpretation of parasystolic rhythms. In §2 we consider the effects of periodic stimulation of spontaneously beating aggregates of cells from embryonic chick heart (Guevara *et al.* 1981; Glass *et al.* 1983, 1984, 1986*b*). Theoretical analysis of this system shows that periodic dynamics are expected at some stimulation frequencies and amplitudes, whereas chaotic dynamics are expected for other stimulation parameters. Experiments are in close agreement with the theory. In §3 we develop a theoretical model for parasystole. The model extends previous theoretical models of parasystole (Moe *et al.* 1977; Swenne *et al.* 1981; Ikeda *et al.* 1983; Glass *et al.* 1986*a*). We describe the bifurcations in the theoretical model and show that chaotic dynamics is expected over some regions of parameter space. In §4 we discuss Holter ECG records from ambulatory patients who display frequent ectopic beats. These records may show extremely irregular dynamics which we discuss in the context of chaotic dynamics and modulated parasystole. Finally, the significance of this approach to the analysis of cardiac dynamics is discussed.

## 2. PERIODIC STIMULATION OF A CARDIAC OSCILLATOR

In this section we describe the effects of single and periodic stimulation of an aggregate of spontaneously beating cells from embryonic chick heart. As this work has been described in several recent publications, we briefly summarize the main results and refer the reader elsewhere for more details (Guevara *et al.* 1981; Glass *et al.* 1983; Glass *et al.* 1984; Glass *et al.* 1986*b*; Guevara *et al.* 1986).

Spontaneously beating aggregates of ventricular heart cells are formed by dissociating the ventricles of seven-day embryonic chicks and allowing the cells to reaggregate in tissue culture medium. The resulting aggregates are approximately 100–200  $\mu\text{m}$  in diameter and each beats with its own intrinsic frequency, which lies in a range of about 60–120 times per minute (DeHaan & Fozzard 1975). A glass microelectrode is inserted intracellularly and can be used to inject single and periodic current pulses into the aggregate. In the present context, the electrical stimulator is analogous to the sinus rhythm, and the aggregate is analogous to an ectopic focus. Clearly, this represents a gross oversimplification of the anatomically complex heart, as it in no sense takes into account the spatial heterogeneity of cardiac tissue nor the various feedback mechanisms that act to modulate cardiac activity *in vivo*. Nevertheless, as stimulation parameters are varied, this model system generates a great variety of rhythms that resemble clinically observed arrhythmias. Some of these rhythms are periodic with  $N$  cycles of the periodic stimulation for each  $M$  cycles of the cardiac oscillation ( $N:M$  phase locking).

Other rhythms are aperiodic (figure 1). The dynamics of this system can only be understood by using techniques in nonlinear dynamics. Thus, this model system is useful to fix ideas and to form a foundation for the analysis of more complex situations.

In response to a single pulse of electrical current, the phase of the oscillation is usually reset. The magnitude of the resetting is proportional to the amplitude and the phase of the current pulse. Generally within a few cycles, the rhythm is re-established at the same frequency as before but with a permanent shift of phase. The re-establishment of the same amplitude and frequency of the oscillation following a perturbation, indicates that from a mathematical point of view it should be useful to think of the cardiac oscillation as a stable limit cycle oscillation. A stable limit cycle oscillation represents a periodic solution of a differential equation that is attracting in the limit  $t \rightarrow \infty$ , for points in the neighbourhood of the cycle.

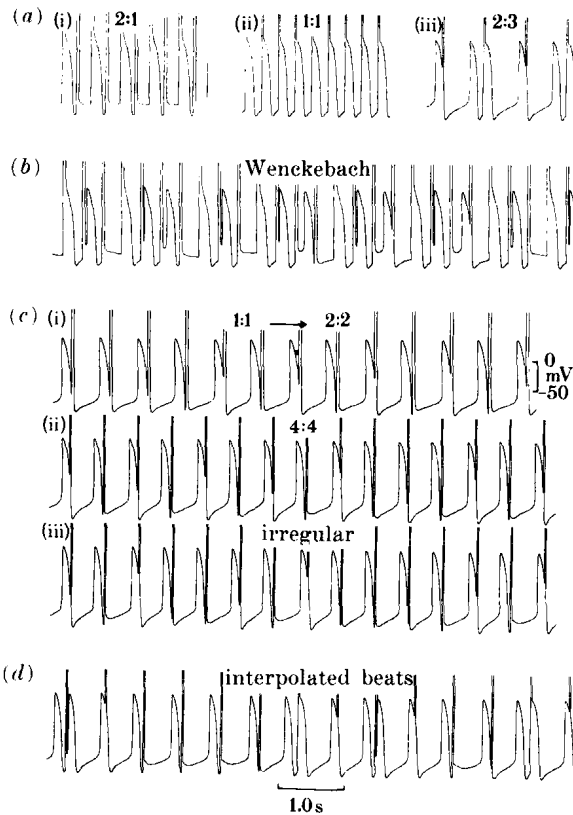


FIGURE 1. Representative transmembrane recordings showing the effects of intracellular periodic stimulation *in vitro* of spontaneously beating embryonic heart cells from chick. The stimulus artifact is observed as a narrow upward deflection. The broader complex is the action potential which corresponds to the contraction of the aggregate. (a) Stable phase-locked rhythms; (b) rhythms in which the time from the stimulus artifact to the action potential progressively increases until a beat is dropped; this is analogous to the Wenckebach phenomenon in electrocardiology (Pick & Langendorf 1979); (c) period-doubling bifurcations and irregular chaotic dynamics; (d) irregular rhythm in which there are more action potentials than stimuli. From Guevara *et al.* (1981).

Theoretical analysis of this system is possible by assuming that following a stimulus, the return to the cycle is extremely rapid (figure 2). Thus, if a periodic train of stimuli is delivered to the system with a time interval of  $T$  between the stimuli, then the effects of periodic stimulation can be computed from the finite difference equation

$$\phi_{i+1} = g(\phi_i) + \tau \pmod{1}, \quad (1)$$

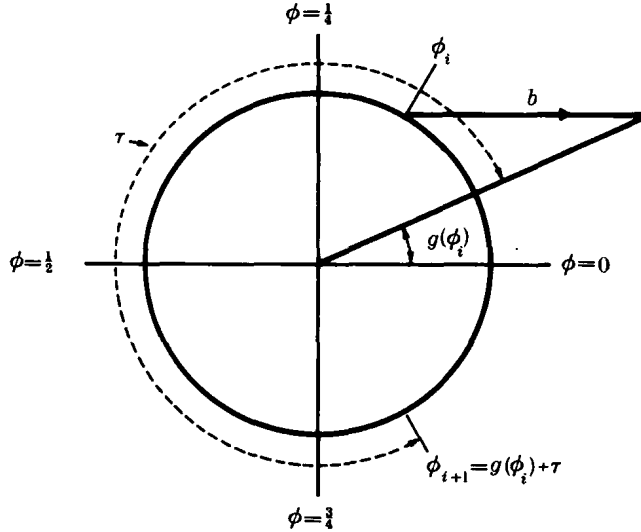


FIGURE 2. A schematic model for the perturbation of a limit cycle oscillation by a periodic stimulus. Provided that the relaxation to the limit cycle following a stimulus is rapid, (1) can be derived.

where  $\phi_i$  is the phase of the  $i$ th stimulus and  $\tau = T/T_0$ , where  $T_0$  is the control cycle length of the aggregate. The function  $g$ , called the phase transition curve, depends on the strength of the electrical current and can be measured from the phase resetting resulting from a single stimulus (Perkel *et al.* 1964; Pavlidis 1973; Guevara *et al.* 1981; Glass *et al.* 1983, 1984).

Equation (1) is a finite difference equation and the analysis of bifurcations of such equations is a topic of much current interest. In the present case, the finite difference equation takes a point on the circumference of a circle,  $\phi_i$ , and generates a new point also on the circumference of a circle,  $\phi_{i+1}$  (it is called a circle map). The analysis of circle maps was initiated by Poincaré and major advances in analysing the bifurcations of circle maps were made by Arnol'd (1965) for the case of invertible (for each  $\phi_i$  there is a unique  $\phi_{i+1}$  and vice versa) circle maps. In the practical situations that arise in the experimental system the circle maps are not always invertible and an extension of the theory of invertible circle maps was carried out (Guevara & Glass 1982; Glass *et al.* 1983, 1984; Keener & Glass 1984; Belair & Glass 1985). The analysis of bifurcations of noninvertible circle maps provides a fertile field for mathematical research (for a recent study and references to other work see MacKay & Tresser 1986).

From (1) it is possible to compute the effects of periodic stimulation at any

frequency once  $g$ , which is measured experimentally, is determined (figure 3). The following are the main conclusions derived from the experimental and theoretical studies. (i) There is a well-defined ordering of phase-locked rhythms corresponding to theoretical predictions based on the analysis of circle maps; (ii) for some stimulation parameters for which one theoretically computes that there should be

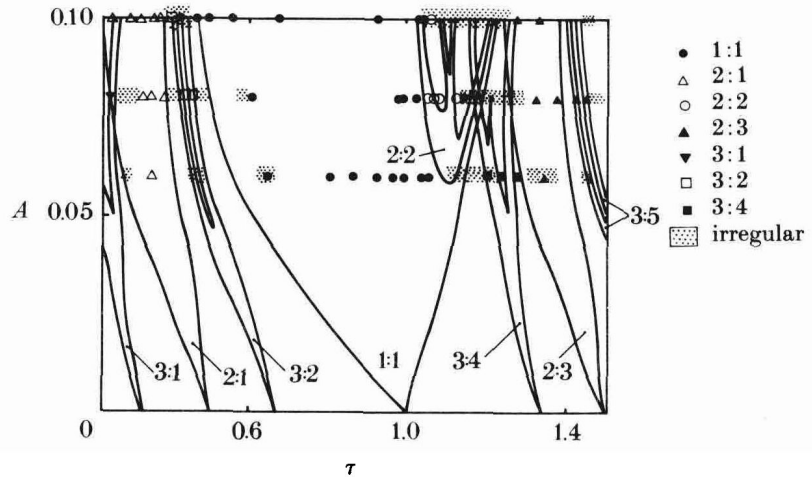


FIGURE 3. Experimentally observed dynamics for periodically stimulated aggregates of chick heart cells superimposed on theoretically computed phase locking zones. The computations use (1) and experimentally measured phase transition curves as described in Glass *et al.* (1984).  $\tau$  represents the period of the stimuli divided by the period of the oscillations in the aggregates.  $A$  is the amplitude of the stimulus in arbitrary units. The circle map in (1) is invertible for  $0 < A < 0.039$  and the Arnol'd tongue structure is observed. From Glass *et al.* (1984).

chaotic dynamics, aperiodic dynamics are experimentally observed; (iii) for situations in which the dynamics are believed to be chaotic, if  $\phi_{i+1}$  is plotted as a function of  $\phi_i$  from experimental data then the results are in good agreement with maps calculated based on single pulse phase resetting studies. Thus, our ability to compute theoretically the bifurcations for this system, and the strong agreement between theory and experiment, gives us confidence that the aperiodic dynamics in some regions of parameter space would still be present even if it were possible to eliminate all environmental noise (i.e. the dynamics is chaotic for some parameter values).

### 3. THEORETICAL MODELS FOR PARASYSTOLE

In parasystole there is competition between the normal sinus pacemaker and a pacemaker which is present at some ectopic (i.e. non-sinus) focus. Although the ectopic focus can be present in either the atria or ventricles, for the current discussion we assume that the ectopic focus is present in the ventricles. The recognition of the possibility of ventricular parasystole dates back at least as far as Fleming (1912) who based his work on the analysis of pulse pressure data. In



the ideal situation the two rhythms have their own set frequencies and there is no phase resetting of the ectopic focus by the sinus rhythm. This 'pure' parasystole has recently been analysed (Glass *et al.* 1986*a*) and we follow the treatment there. It is also possible that the sinus rhythm can act to modulate the ectopic rhythm (Jalife & Moe 1976; Moe *et al.* 1977; Swenne *et al.* 1981; Ikeda *et al.* 1983). For this case of 'modulated' parasystole we follow the basic ideas sketched out in these earlier papers, but try to place the analysis in the context of current studies in nonlinear dynamics and give some new computations. The above formulations assume that parameters remain constant. In realistic situations, the parameters may in fact fluctuate. Accordingly, we consider some effects of parameter fluctuation in the above models.

### (a) Pure parasystole

We assume the mechanism for parasystole considered by Fleming (1912) and Kaufman & Rothberger (1917); figure 4. There is a normal sinus rhythm with period  $t_s$  and an ectopic rhythm with a period  $t_e$ , where  $t_e > t_s$ . After each sinus beat there is a refractory period  $\theta$ . If the ectopic focus generates an impulse during the refractory period it is blocked, but otherwise it will lead to an ectopic beat which can be recognized on the electrocardiogram because of its abnormal morphology. After each ectopic beat, the next sinus beat is assumed to be blocked, resulting in a 'compensatory pause'.



FIGURE 4. Schematic model for pure parasystole. Sinus rhythm (s) and ectopic rhythm (e) are shown. Refractory time is represented as a shaded region. Any ectopic beat that falls outside the refractory time is conducted (filled arrows) and leads to a blocking of the subsequent sinus beat (dashed lines). Ectopic beats falling during the refractory time are blocked (open arrows). In the illustration  $\theta/t_s = 0.4$ ,  $t_e/t_s = 1.65$ , and there are either 1, 2 or 4 sinus beats between ectopic beats. From Glass *et al.* (1986*a*).

Remarkably, the hypothesized mechanism for pure parasystole is equivalent to a well-studied problem in number theory (Slater 1967) and a very detailed analysis of the dynamics for fixed  $t_e$ ,  $t_s$  and  $\theta$  can be given (Glass *et al.* 1986*a*). In particular, we have found the following rules for parasystole.

**Rule 1.** For any ratio of  $t_e/t_s$  there are at most three different values for the number of sinus beats between ectopic beats.

**Rule 2.** One and only one of these values is odd.

**Rule 3.** For any value of  $t_e/t_s$  at which there are three different values for the number of sinus beats between ectopic beats, the sum of the two smaller values is one less than the larger value.

**Rule 4.** Consider the sequence giving the number of sinus beats between ectopic beats. One and only one of these values can succeed itself.

To illustrate these rules we have numerically computed the sequences giving the number of sinus beats between ectopic beats for fixed parameter values. For any fixed set of parameters call  $p(a)$ ,  $p(b)$  and  $p(c)$  the probability that there are  $a$ ,  $b$