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Edited by A. Dold and B. Eckmann

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Introduction

An Einstein metric is a Riemannian metric on a smooth manifold of constant Ricci curvature, i.e. a Riemannian metric whose Ricci tensor is proportional to the metric tensor. As the curvature tensor is expressed in terms of second derivatives of the metric tensor, finding an Einstein metric is reduced to a system of partial differential equations. This system of equations corresponds to the Einstein field equation in general relativity although in this case the metric is Lorentzian. A geometric meaning of an Einstein metric may be that it is a metric which makes the manifold the best shape. When one studies the geometry or topology of a manifold, if the existence of an Einstein metric on the manifold is known, it is helpful to use the Einstein metric. A typical example is the following result of Yau. In complex geometry the Ricci form, i.e. the (1,1)-form associated to the Ricci curvature, represents the first Chern class by the Chern-Weil theory. Using this fact and the existence of a Kähler-Einstein metric, Yau [YS1] proved

$$(-1)^m \frac{2(m+1)}{m} c_1^{m-2} c_2 \geq (-1)^m c_1^m$$

for any compact complex manifold of dimension m and with negative first Chern class.

These notes are concerned with the case where the manifold is a compact complex manifold and the metric is a Kähler metric, i.e. a Riemannian metric compatible with the complex structure. In this case there is a natural necessary condition for the existence of a Kähler-Einstein metric, namely, if M is a compact Kähler-Einstein manifold, the first Chern class $c_1(M)$ must be negative, zero or positive, i.e. $c_1(M)$ is represented by a negative, zero or positive real (1,1)-form, according to the sign of the Ricci curvature. One then can ask the converse. The negative and zero cases are already known. Namely, there exists a Kähler-Einstein metric unique up to homothety in the case where $c_1(M) < 0$, and also unique in the case where $c_1(M) = 0$ if one fixes a Kähler class ([AT2] and [YS1]). Here the Kähler class means the de Rham class represented by the Kähler form.

The purpose of these notes is to discuss the existence and nonexistence problem of Kähler-Einstein metrics on compact complex manifolds of *positive* first Chern class. A first obstruction to this existence problem in the positive case was found by Matsushima [MY1], who proved that if M is a compact Kähler-Einstein manifold, the Lie algebra $\mathfrak{h}(M)$ consisting of all holomorphic vector fields on M is reductive. There are many examples of compact complex manifolds of positive first Chern class with nonreductive $\mathfrak{h}(M)$, e.g. the blow-up of the complex projective plane at a point. Besides Matsushima's obstruction the author [FA1] found another obstruction. This is defined as an integral invariant on compact complex manifolds of positive first Chern class, which can be viewed as a Lie algebra character $f : \mathfrak{h}(M) \to \mathbb{C}$ and whose vanishing is a necessary condition for the existence of a Kähler-Einstein metric. In these notes we gather recent results around these obstructions; their formulations, origins, generalizations, interpretations, sufficiency for the existence problem, the lifting problem of f to a group character, and prospects for the future. It should be mentioned that there is a third obstruction which will not be dealt with in these notes. In fact a theorem of Kobayashi-Lübke ([KS3],[LM]) says that if E is a holomorphic vector bundle over a compact Kähler manifold then E is semistable. Applying this theorem to the holomorphic tangent bundle we get the third necessary condition: if M admits a Kähler-Einstein metric the tangent bundle must be semistable. For this topic we refer the reader to [KS4].

In the positive case, Calabi conjectures that if M is a compact complex manifold with $c_1(M) > 0$ and $\mathfrak{h}(M) = 0$ there would exist a Kähler-Einstein metric (see [YS3]). At the same time he studies the following variational problem. Let M be a compact Kähler manifold and $\mathfrak{M}_\Omega$ be the set of all Kähler metrics whose Kähler forms represent a given Kähler class Ω: find a critical point of $\Phi : \mathfrak{M}_\Omega \to \mathbb{R}$ defined by

$$\Phi(g) = \int_M \sigma_g^2 \, dv_g$$

where σ_g and dv_g are respectively the scalar curvature and the volume form with respect to $g \in \mathfrak{M}_\Omega$. A critical point g is called an extremal Kähler metric, and the first variation formula shows that g is an extremal Kähler metric if and only if the gradient vector field of σ_g is a holomorphic vector field. If $\Omega = c_1(M) > 0$ and $f = 0$, then an extremal Kähler metric is a Kähler-Einstein metric; in particular if $\Omega = c_1(M) > 0$ and $\mathfrak{h}(M) = 0$, we have the same conclusion. Calabi expected that this variational problem could be solved on any compact Kähler manifold with any Kähler class Ω. However a necessary condition for the existence of an extremal Kähler metric was found by himself ([CE2]). His condition determines the structure of $\mathfrak{h}(M)$, which generalizes Matsushima's theorem and its generalization due to Lichnerowicz [LA1] for compact Kähler manifolds of constant scalar curvature. Examples which do not satisfy Calabi's condition were exhibited by Levine [LM]. Further, although Calabi's condition says nothing about the case when $\mathfrak{h}(M) = 0$, Burns recently found a surprising example of a compact Kähler manifold with $\mathfrak{h}(M) = 0$ which does not carry any extremal Kähler metric. The example of Burns makes us somewhat pessimistic about the above conjecture of Calabi. However, so far there is no known example of a compact complex manifold with $c_1(M) > 0$ which does not carry any extremal Kähler metric.

The character f is originally defined on compact complex manifolds with $c_1(M) > 0$. It can however be generalized to a Kählerian invariant: more precisely, it can be defined on any compact Kähler manifold using a Kähler metric but depending only on the Kähler class. The most general formulation in this direction is due to Bando [BS1] who defines $f_k : \mathfrak{h}(M) \to \mathbb{C}$, $1 \le k \le m = \dim M$, such that f_k is a Lie algebra character, depends only on the Kähler class and vanishes if there is a Kähler metric with harmonic k-th Chern form. The case $k = 1$ is due to the author [FA2] and Calabi [CE2]. It can be proved that if $f_1 = 0$, any extremal Kähler metric is a metric of constant scalar curvature.

On the other hand Morita and the author [FA-MS1,2] generalized f to an invariant which is defined on any compact complex manifold M and depends only on the complex structure of M. More generally, let G be a complex Lie group and $I^k(G)$ be the set of all holomorphic symmetric G-invariant polynomials of degree k. Then we can define a linear map $F : I^{m+p}(GL(m,\mathbb{C})) \to I^p(H(M))$ where $H(M)$ is the group of all automorphisms of M. F depends only on the complex structure of M and $(m+1)f = F(c_1^{m+1})$; hence the members of the image of F are considered as generalizations of f. Thus f belongs to the intersection of the invariants of compact complex manifolds and of the Kählerian invariants.

We see that F appears naturally in the context of classical theories such as those of the Lefschetz numbers, the equivariant cohomology and the Chern-Simons invariants. The merits of this observation are, first of all, that we can obtain a localization formula of f to compute $f(X)$ fairly easily in terms of the zero set of the vector field X, and secondly that we can find how f lifts to a group character. In fact we can see that f lifts to a group character $\mathcal{F} : H(M) \to \mathbb{C}/\mathbb{Z}$. In [FA3] the author derived an explicit formula of the imaginary part of $\mathcal{F}$, but the explicit formula of the real part is still missing. The author thinks , without strong conviction, that the real part is more important. The reason is as follows. If $\mathfrak{h}(M) = 0$, $H(M)$ is a finite group; any character of a finite group into $\mathbb{R}$ is trivial, but a character into $\mathbb{R}/\mathbb{Z}$ may not be; thus the imaginary part of $\mathcal{F}$ may obstruct the existence of a

Kähler-Einstein metric even in the case when $\mathfrak{h}(M) = 0$. More recently Mabuchi and the author [FA-MT] proved that $\mathcal{F}$ can be obtained in the form $\mathcal{F} = (\det \phi)^{\gamma}$ where $\phi : H(M) \to V$ is a representation with V a a finite-dimensional vector space of the form $V = \bigoplus_{\nu} m_{\nu} H^{0}(M, K_{M}^{-\nu})$ and γ is a positive rational constant. However we have not given a decisive answer to the above question.

Meanwhile Mabuchi [MT1] and Bando and Mabuchi [BS-MT1] also found independently a formula for the imaginary part of the lift. Their formula uses certain homotopy between Kähler metrics. Fixing a reference point it defines a functional, called the K-energy map, on the space of Kähler forms in a fixed Kähler class. Further, in [BS-MT2], they proved that if M is a compact complex manifold of positive first Chern class the space $\mathcal{E}$ of all Kähler-Einstein forms on M is connected, and moreover that the K-energy map takes the minimum on $\mathcal{E}$. As a corollary to this result, if $H(M)$ is discrete then a Kähler -Einstein metric, if any, is unique. Another corollary is that if M admits a Kähler-Einstein metric then the K-energy map is bounded from below. Using the heat equation method Bando [BS2] further proved that, under the assumption that the K-energy map is bounded from below, for any $\varepsilon > 0$ there exists a Kähler metric g such that $(1-\varepsilon)m < \sigma_g < (1+\varepsilon)m$. These results are inspired by the work of Donaldson [DS] on the existence of a Hermitian-Einstein metric on a stable vector bundle.

Concerning the sufficiency of the vanishing of f for the existence of a Kähler-Einstein metric, Sakane [SY] and Koiso-Sakane [KN-SY1] proved that, for some class of $\mathbb{P}^1$-bundles, the vanishing of f is a necessary and sufficient condition for the existence of a Kähler-Einstein metric. Sakane's existence result produced first examples of nonhomogeneous Kähler-Einstein manifolds with positive Ricci curvature (see also [KN-SY2] for further examples).

These notes treat only the Kähler case. For general Einstein manifolds there is a recent monograph by Besse [BA]. The problem of finding a Kähler-Einstein metric in the zero case is related to the Calabi conjecture proved by Yau [YS1], which states that any real closed (1,1)-form representing $c_1(M)$ is a Ricci form with respect to some Kähler metric. In fact the Calabi conjecture says that if $c_1(M) = 0$ there is a Kähler metric with vanishing Ricci tensor, which is a Kähler-Einstein metric in the zero case. Moreover both the Calabi conjecture and the problem of finding a Kähler-Einstein metric can be reduced to a same single nonlinear elliptic partial differential equation, the so-called Monge-Ampère equation ([AT1]). We refer the reader to the Astérisque volume [BJ2] for the Calabi conjecture and Aubin's book [AT4] for the detailed analysis of the Monge-Ampère equation.

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Chapter I Preliminaries

§1.1 Kählerian geometry

In this section we review elementary facts about Kählerian geometry. There are two ways to introduce Kählerian geometry; one is from Riemannian geometry and another is from Hermitian geometry for holomorphic vector bundles. In this section we employ the first, and the second will be explained in the next section.

Let M be a smooth manifold of dimension n, TM the tangent bundle and T^*M the cotangent bundle. Let $E \to M$ be a real or complex vector bundle, $C^\infty(E)$ the set of all smooth sections of E and $C^\infty(M)$ (resp. $C^\infty_{\mathbb{C}}(M)$) the set of all real (resp. complex) valued smooth functions on M.

Definition 1.1.1: A *connection* on E is a bilinear map $\nabla : C^\infty(TM) \times C^\infty(E) \to C^\infty(E)$, denoted by $(X,s) \to \nabla_X s$, such that

(i) $\nabla_{fX} s = f\, \nabla_X s$

(ii) $\nabla_X(fs) = (Xf)s + f\, \nabla_X s$

where $X \in C^\infty(TM)$, $s \in C^\infty(E)$ and $f \in C^\infty(M)$ or $C^\infty_{\mathbb{C}}(M)$.

If one chooses a local frame field $e_1,\cdots,e_r$, $r = \operatorname{rank} E$, the matrix valued 1-form (θ^i_j) defined by $\nabla e_j = \sum_{i=1}^{n} \theta^i_j\, e_i$ is called the *connection form* of ∇ with respect to $e_1,\cdots,e_r$. The connection form is defined only locally and depends on the choice of the local frame.

$\nabla_X s$ is called the *covariant derivative* of s in the direction of

X. A *Riemannian metric* g on M is an element of $C^\infty(T^*M \otimes T^*M)$ such that at every point $p \in M$, $g(p)$ is a positive definite symmetric bilinear form on TM. Given a Riemannian metric there is a unique connection ∇ on TM with the defining properties

$$(1.1.2) \qquad X\, g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

$$(1.1.3) \qquad \nabla_X Y - \nabla_Y X = [X,Y].$$

This connection is called the *Levi-Civita connection.* ∇ in turn defines a connection on T^*M by duality, denoted by ∇ again:

$$\nabla_X \alpha\,(Y) = X(\alpha(Y)) - \alpha(\nabla_X Y)$$

where $\alpha \in C^\infty(T^*M)$ and $X, Y \in C^\infty(TM)$. Further ∇ defines, by derivation, a connection on vector bundles $(\otimes^p TM)\otimes(\otimes^q T^*M)$ for all p and q. For instance the condition (1.1.2) can be written as $\nabla g = 0$. In fact

$$\nabla_X g\,(Y,Z) = X\, g(Y,Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0.$$

The *curvature tensor* $R \in C^\infty(\otimes^4 T^*M)$ is defined by

$$(1.1.4) \qquad R(X,Y,Z,W) = g(\nabla_X\nabla_Y W - \nabla_Y\nabla_X W - \nabla_{[X,Y]}W,\ Z).$$

R satisfies the following relations:

$$(1.1.5) \qquad R(X,Y,Z,W) = R(Z,W,X,Y) = -\,R(Y,X,Z,W)$$

$$(1.1.6) \qquad R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0$$

$$(1.1.7) \qquad \nabla_X R\,(Y,Z,W,V) + \nabla_Y R\,(Z,X,W,V) + \nabla_Z R\,(X,Y,W,V) = 0.$$

(1.1.6) and (1.1.7) are respectively called the first and the second Bianchi identities.

To explain the notations of tensor calculus, we choose a local coordinate system $(x^1,\cdots,x^n)$. A tensor field $s \in C^\infty((\otimes^p TM)\otimes(\otimes^q T^*M))$ is locally written by

$$s = s^{i_1\cdots i_p}{}_{j_1\cdots j_q} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_q}.$$

Here the sum is taken if an index appears as an upper and lower index at the same time; this convention is called the Einstein convention. ∇s defines an element of $C^\infty(T^*M \otimes(\otimes^p TM)\otimes(\otimes^q T^*M))$ and is locally written by

$$(1.1.8) \qquad \nabla s = \nabla_i s^{i_1\cdots i_p}{}_{j_1\cdots j_q}\, dx^i \otimes \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_q}.$$

The upper and lower indices are lowered and raised using the isomorphism between the tangent space and the cotangent space induced by the Riemannian metric:

$$s^{i_1\cdots i_p}{}_{j_1\cdots j_q} = g^{ki_1} g_{\ell j_1} s_k{}^{i_2\cdots i_p \ell}{}_{j_2\cdots j_q}$$

where $g = g_{ij}\, dx^i \otimes dx^j$ and (g^{ij}) is the inverse matrix of (g_{ij}). Notice that this operation is compatible with the notation of (1.1.8) because of $\nabla g = 0$. With these notations (1.1.4) is written by

$$(1.1.9) \qquad \nabla_i \nabla_j s^k - \nabla_j \nabla_i s^k = R_{ij\ell m}\, s^m g^{\ell k} = R_{ij}{}^k{}_\ell\, s^\ell.$$

(1.1.9) is called the Ricci identity. Using (1.1.5) and (1.1.9) one can easily deduce

$$(1.1.10) \qquad \nabla_i \nabla_j s_k - \nabla_j \nabla_i s_k = - R_{ij}{}^k{}_\ell\, s_k.$$

An *almost complex structure* on a smooth manifold M is a field of endomorphisms J on TM such that $J^2 = J\circ J = - I$ where I denotes the identity endomorphism. If such a J exists, (M,J) is called an *almost complex manifold*. An almost complex manifold is necessarily even dimensional. Since J has two eigenvalues $\sqrt{-1}$ and

$-\sqrt{-1}$, J induces a splitting of the complexified tangent bundle $T_{\mathbb{C}}M$:

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$$

where $T^{1,0}M$ and $T^{0,1}M$ respectively consist of eigenvectors belonging to $\sqrt{-1}$ and $-\sqrt{-1}$. J is called *integrable* if $[\xi,\eta] \in C^{\infty}(T^{1,0}M)$ for all ξ, $\eta \in C^{\infty}(T^{1,0}M)$. If $\xi = \frac{1}{2}(X - \sqrt{-1}JX)$ and $\eta = \frac{1}{2}(Y - \sqrt{-1}JY)$ where $X, Y \in C^{\infty}(TM)$, then $[\xi,\eta] = 0$ if and only if

$$N(X,Y) = [X,Y] - [JX,JY] + J[JX,Y] + J[X,JY] = 0.$$

N is called the *Nijenhaus tensor*. Newlander-Nirenberg theorem [NA-NL] says that if $N = 0$ then M becomes a complex manifold in such a way that

$$J\frac{\partial}{\partial z^{i}} = \sqrt{-1}\frac{\partial}{\partial z^{i}} \tag{1.1.11}$$

where $(z^{1},\cdots,z^{m})$ is a local holomorphic coordinate system. When $N = 0$ J is called a *complex structure*. Conversely, given a complex manifold, we can define J by (1.1.11) and then N for this J naturally vanishes.

Let (M,J) be a complex manifold. A *Hermitian metric* on M is a J-invariant Riemannian metric g:

$$g(JX,JY) = g(X,Y) \qquad X, Y \in C^{\infty}(TM).$$

The *fundamental form* of g is a real 2-form ω defined by

$$\omega(X,Y) = \frac{1}{2\pi}g(JX,Y).$$

Since g is J-invariant, ω is indeed skew-symmetric and defines a 2-form. We call g a *Kähler metric* if ω is a closed form, and then ω is called a *Kähler form* of g. The significance of Kähler metrics may be understood by Proposition 1.1.14 below which says that the Levi-Civita connection of a Kähler metric is compatible with the complex structure. A complex manifold with a Kähler metric is called a *Kähler manifold*. Since a Kähler form ω is closed it defines a cohomology class Ω_g in the de Rham cohomology group $H^2_{DR}(M)$. Ω_g is called the *Kähler class*.

We extend g, ∇ and R in the $\mathbb{C}$-linear way. Let $(z^1, \cdots, z^m)$ be a local holomorphic coordinate system. We define g_{AB} by

$$g_{AB} = g\left(\frac{\partial}{\partial z^A}, \frac{\partial}{\partial z^B}\right)$$

where $A, B \in \{1, \cdots, m, \bar{1}, \cdots, \bar{m}\}$ and $z^A = z^i$ if $A = i$, $z^A = \overline{z^i}$ if $A = \bar{i}$. Then since g is J-invariant we have $g_{ij} = g_{\bar{i}\bar{j}} = 0$. In fact

$$g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) = g\left(J\frac{\partial}{\partial z^i}, J\frac{\partial}{\partial z^j}\right) = g\left(\sqrt{-1}\frac{\partial}{\partial z^i}, \sqrt{-1}\frac{\partial}{\partial z^j}\right) = -g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right).$$

Thus the Kähler form ω is locally written by

$$(1.1.12) \qquad \omega = \frac{\sqrt{-1}}{2\pi} g_{i\bar{j}}\, dz^i \wedge \overline{dz^j}.$$

We define Γ_{AB}^{C}, called the *Christoffel symbols*, by

$$\nabla_{\frac{\partial}{\partial z^A}} \frac{\partial}{\partial z^B} = \Gamma_{AB}^{C} \frac{\partial}{\partial z^C}$$

It is clear from the $\mathbb{C}$-linearity and (1.1.3) that

$$(1.1.13) \qquad \Gamma_{AB}^{C} = \Gamma_{BA}^{C} \quad \text{and} \quad \Gamma_{\bar{A}\bar{B}}^{\bar{C}} = \overline{\Gamma_{AB}^{C}}.$$

Proposition 1.1.14: Let (M, J) be a complex manifold and g a Hermitian metric. Then the following three conditions are equivalent.

(i) $d\omega = 0$, i.e. g is Kähler

(ii) $\nabla J = 0$, i.e. $\nabla_X(JY) = J\nabla_X Y$

(iii) $\Gamma_{AB}^{C} = 0$ except for Γ_{ij}^{k} and $\Gamma_{\bar{i}\bar{j}}^{\bar{k}}$ $(= \overline{\Gamma_{ij}^{k}})$.

Proof: (ii) => (i) follows from

$$2\pi\, d\omega\,(X, Y, Z) = g((\nabla_X J)Y, Z) + g((\nabla_Y J)Z, X) + g((\nabla_Z J)X, Y).$$

(i) => (ii) follows from $N = 0$ and

$$\frac{1}{\pi} g((\nabla_X J)Y, Z) = d\omega\,(X,Y,Z) - d\omega\,(X,JY,JZ) - g(JX, N(Y,Z)).$$

(iii) => (ii) is trivial. To prove (ii) => (iii) it is sufficient to show $\Gamma_{ij}^{\bar{k}} = 0$ and $\Gamma_{i\bar{j}}^{k} = 0$ by (1.1.13). $\Gamma_{ij}^{\bar{k}} = 0$ follows from

$$\nabla_{\frac{\partial}{\partial z^i}} J\frac{\partial}{\partial z^j} = \sqrt{-1}\,\nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^j} = \sqrt{-1}\,\Gamma_{ij}^{k} \frac{\partial}{\partial z^k} + \sqrt{-1}\,\Gamma_{ij}^{\bar{k}} \frac{\partial}{\partial \bar{z}^k},$$

$$J\,\nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^j} = J(\Gamma_{ij}^{k} \frac{\partial}{\partial z^k} + \Gamma_{ij}^{\bar{k}} \frac{\partial}{\partial \bar{z}^k}) = \sqrt{-1}\,\Gamma_{ij}^{k} \frac{\partial}{\partial z^k} - \sqrt{-1}\,\Gamma_{ij}^{\bar{k}} \frac{\partial}{\partial \bar{z}^k}.$$

$\Gamma_{i\bar{j}}^{k} = 0$ follows from

$$\nabla_{\frac{\partial}{\partial z^i}} J\frac{\partial}{\partial \bar{z}^j} = -\sqrt{-1}\,\nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial \bar{z}^j} = -\sqrt{-1}\,\Gamma_{i\bar{j}}^{k} \frac{\partial}{\partial z^k} - \sqrt{-1}\,\Gamma_{i\bar{j}}^{\bar{k}} \frac{\partial}{\partial \bar{z}^k},$$

$$J\,\nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial \bar{z}^j} = J(\Gamma_{i\bar{j}}^{k} \frac{\partial}{\partial z^k} + \Gamma_{i\bar{j}}^{\bar{k}} \frac{\partial}{\partial \bar{z}^k}) = \sqrt{-1}\,\Gamma_{i\bar{j}}^{k} \frac{\partial}{\partial z^k} - \sqrt{-1}\,\Gamma_{i\bar{j}}^{\bar{k}} \frac{\partial}{\partial \bar{z}^k}.$$

q.e.d

Using (iii) of Proposition 1.1.14 we can deduce several formulae. First of all one has

(1.1.15) $$\Gamma_{ij}^{k} = g^{k\bar{l}} \frac{\partial g_{j\bar{l}}}{\partial z^i}$$

where $(g^{i\bar{j}})$ is the inverse matrix of $(g_{i\bar{j}})$: $g^{i\bar{j}} g_{k\bar{j}} = \delta^i{}_k$. Thus with respect to the local frame $\frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^m}$ of TM, the connection form θ of ∇ is written by $\theta = g^{-1}\partial g$.

Next one can compute R using (1.1.4) and (1.1.15):

(1.1.16) $$R_{ijk\bar{l}} = -R_{ij\bar{l}k} = \overline{R_{\bar{i}\bar{j}\bar{k}l}} = -\overline{R_{\bar{i}\bar{j}l\bar{k}}} = 0$$

(1.1.17) $$R_{i\bar{j}k\bar{l}} = \frac{\partial^2 g_{k\bar{l}}}{\partial z^i \partial \bar{z}^j} - g^{p\bar{q}} \frac{\partial g_{k\bar{q}}}{\partial z^i} \frac{\partial g_{p\bar{l}}}{\partial \bar{z}^j}.$$