Lecture Notes in Mathematics 1676

Pilar Cembranos José Mendoza

Banach Spaces of Vector-Valued Functions

Springer

Pilar Cembranos   José Mendoza

Banach Spaces of Vector-Valued Functions

Springer

Authors

Pilar Cembranos
José Mendoza
Departamento de Análisis Matemático
Facultad de Ciencias Matemáticas
Universidad Complutense de Madrid
E-28040 Madrid, Spain
e-mail: cembrp@sunam1.mat.ucm.es
mendoza@sunam1.mat.ucm.es

Cataloging-in-Publication Data applied for

Die Deutsche Bibliothek - CIP-Einheitsaufnahme

Cembranos, Pilar:
Banach spaces of vector valued functions / Pilar Cembranos ; José Mendoza. - Berlin ; Heidelberg ; New York ; Barcelona ; Budapest ; Hong Kong ; London ; Milan ; Paris ; Santa Clara ; Singapore ; Tokyo : Springer, 1997
(Lecture notes in mathematics ; 1676)
ISBN 3-540-63745-1

Mathematics Subject Classification (1991): 46E40, 46B20, 28B05

ISSN 0075-8434
ISBN 3-540-63745-1 Springer-Verlag Berlin Heidelberg New York

Typesetting: Camera-ready $\TeX$ output by the authors
SPIN: 10553403 46/3142-543210 - Printed on acid-free paper

Table of Contents

Introduction

"... there came wise men from the east (...) saying: (...) we have seen His star..."

Matthew 2

The study of vector-valued function Banach spaces is a very active field of research, and it is already very old, too. It begins soon after Banach's book, in the thirties and early forties, with the classical works of Birkhoff, Boas, Bochner, Clarkson, Day, Dunford, Gelfand, Pettis, Phillips, etc., and continues its developing through last decades with the contribution of mathematicians as Grothendieck, Dinculeanu, Diestel and Uhl, Kwapień, Maurey, Pisier, Bourgain, Talagrand, and many more.

We focus our attention here on $L_p(\mu, X)$ and $C(K, X)$ spaces (for notations, see below "Some Notations and Conventions"), because we believe they are the most important and representative examples of vector-valued function Banach spaces. They are the points of reference in any study on the subject. We are interested in a problem (or collection of problems) which has deserved a lot of attention in last years. To be precise we are interested in the following:

Problem 1: *When does $L_p(\mu, X)$ contain a copy of c_0, ℓ_1 or ℓ_∞? What about complemented copies? Same questions about $C(K, X)$.*

The aim of this monograph is to provide a detailed exposition of the answers to these questions, giving an unified and self-contained treatment. It continues in some way a small part of the big work intiated in Diestel & Uhl's "Vector Measures" [39], which in turn was a continuation of some chapters of Dunford & Schwartz's "Linear Operators" (Part I) [51].

Of course, the questions in Problem 1 are just the classical ones concerning the study of the subspaces of a given Banach space. So we are led to the Bessaga-Pełczyński's and Rosenthal's old theorems characterizing the Banach spaces which have copies of c_0 or ℓ_1 (see [35] or [93]). All these theorems give a very good insight in the internal structure of the Banach spaces involved. This is why we believe that Problem 1 is not only interesting by itself, but has also contributed very much to a better understanding of the structure of $L_p(\mu, X)$ and $C(K, X)$ spaces. In particular, it has made clear that their structure is much more than a simple addition of the structures of X and $L_p(\mu)$ or $C(K)$. The problem has also motivated the development of useful techniques, which are the fruit of the work of many authors during the last twenty-five years.

Let us add that we have now an almost complete solution to Problem 1.

It is interesting to have in mind a standard way of thinking in vector-valued function spaces. Since X and $L_p(\mu)$ are (isometrically isomorphic to) complemented subspaces of $L_p(\mu, X)$, it is clear that if either X or $L_p(\mu)$ contains a copy of a Banach space Y, then so does $L_p(\mu, X)$. Of course, the same can be said about *complemented* copies. So, very often our main problem is to know whether the converse is true or not. The analogous situation also happens for $C(K, X)$. This is why the first problem we usually have to face is the following:

Problem 2: *Assume F is any of the spaces: c_0, ℓ_1 or ℓ_∞. Determine which of the following implications are true*

$$L_p(\mu, X) \supset F \overset{?}{\Longrightarrow} L_p(\mu) \supset F \text{ or } X \supset F$$

$$L_p(\mu, X) \underset{(c)}{\supset} F \overset{?}{\Longrightarrow} L_p(\mu) \underset{(c)}{\supset} F \text{ or } X \underset{(c)}{\supset} F$$

Consider the analogous questions for $C(K, X)$, that is,

$$C(K, X) \supset F \overset{?}{\Longrightarrow} C(K) \supset F \text{ or } X \supset F$$

$$C(K, X) \underset{(c)}{\supset} F \overset{?}{\Longrightarrow} C(K) \underset{(c)}{\supset} F \text{ or } X \underset{(c)}{\supset} F$$

When one of the implications in Problem 2 is true, we obtain a complete and natural solution to Problem 1 of the following kind:

$$L_p(\mu, X) \supset \ell_1 \Longleftrightarrow L_p(\mu) \supset \ell_1 \text{ or } X \supset \ell_1$$

We will see that in many cases, but not always, the answers are analogous to the preceding one.

Let us give a look to the content of the monograph.

In Chapter 1, Preliminaries, we give some fundamental results which we will need. We have done an effort in the careful selection of these results. In the first three sections we recall the main characterizations of Banach spaces containing copies (or *complemented* copies) of c_0, ℓ_1 or ℓ_∞. In the remaining four sections we include different basic facts about $L_p(\mu, X)$ and $C(K, X)$ spaces. In particular, in Section 1.5 we study in detail the representation of the dual of $L_p(\mu, X)$ spaces provided by lifting theory. Of course, the main references here are Dinuleanu's "Vector Measures" [40] and A. and C. Ionescu Tulcea's "Topics in the Theory of Liftings" [75]. However, in these books some results on the dual of $L_p(\mu, X)$ spaces are quite dispersed. This is why we have preferred to provided quite a complete and unified treatment of the subject.

The Second Chapter is devoted to Kwapień's and Pisier's Theorems characterizing when $L_p(\mu, X)$ contains copies of c_0 and ℓ_1, respectively. These theorems were proved in the mid seventies and they were the first solutions to our problems. They are deep and difficult, and for this reason they take an important part of our monograph. Concerning Kwapień's Theorem, we should remark the important contribution of Hoffmann-Jørgensen with his preliminary work. We will explain this better in the Notes and Remarks of the Chapter. To prove the theorem we do not follow Kwapień's approach but Bourgain's. This is very important for us because we will need Bourgain's results in subsequent Chapters.

Chapter Three is devoted to $C(K, X)$ spaces. Theorems of Saab and Saab, Cembranos-Freniche and Drewnowski are studied. They were proved in the eighties and are concerned with complemented copies of ℓ_1, complemented copies of c_0 and copies of ℓ_∞, respectively. The easiest of them is Cembranos-Freniche's, which could be viewed as an exercise on Josefson-Nissenzweig theorem. However, it was quite surprising and had a particular influence when it was proved, because it provided the first negative answer to Problem 2 above. Notice that while Kwapień's and Pisier's Theorems were very difficult, they gave a natural answer.

Chapter Four is devoted to the problems on $L_p(\mu, X)$ spaces ($1 \leq p < +\infty$) not solved by Kwapień's and Pisier's Theorems: complemented copies of c_0, complemented copies of ℓ_1 and copies of ℓ_∞. We give here contributions of Bombal, Emmanuelle and Mendoza, obtained between 1988 and 1992. A curious fact to be mentioned here is that we find a clear difference between the behavior of purely atomic and *non* purely atomic measure spaces, in contrast with the results of the preceding Chapters. This happens when we consider *complemented* copies of c_0 in $L_p(\mu, X)$, and it will happen again in the next Chapter when dealing with $L_\infty(\mu, X)$.

Chapter Five is devoted to $L_\infty(\mu, X)$. We wish to characterize when this space contains complemented copies of c_0 or ℓ_1. This is shown by the results of Leung and Räbiger, Díaz and Kalton. They were obtained from 1990 until very recently (see the "Notes and Remarks" of the Chapter). In the ℓ_1 case, we find a connection between local theory and the problems we are studying. We believe that this is the last important finding in our subject.

Next, we have included a table summarizing the results of the monograph, that is, the solutions to Problem 1. We see that we have a complete solution to our problem with the only exception of the $L_\infty(\mu, X)$ case, and even in this case, for finite (or σ-finite) measures the problem is completely solved.

Finally, we have devoted a short Chapter to comment some open and related problems of the theory. It is clear that in our original Problem 1, we can change the spaces $L_p(\mu, X)$ and $C(K, X)$ for some other more or less similar spaces, like Köthe-Bochner spaces or tensor products, to mention two simple examples. On the other hand, we can also change the candidate subspaces c_0, ℓ_1 or ℓ_∞ for some other else, like ℓ_p $(1 < p < +\infty)$ or $L_1([0,1])$. There are also connections between the problems we have been considering and some other problems, related to classical properties of Banach spaces. In this final chapter, we comment some of these aspects.

This work is intended for those mathematicians interested in Banach Space Theory, Functional Analysis, and in general Abstract Analysis. The subject is crossroad of many branches in Banach space theory, and so one has occasion of applying many fundamental and classical results. For this reason, we think that the techniques developed here are interesting for anyone working on Banach spaces and absolutely fundamental for those interested in the study of vector-valued function spaces and related fields.

It is written at a graduate student level. In fact we have followed the first draft of this work to teach our 30 hours Seminar during the Course 96-97, and we have taken a great advantage of that experience. We think that it might be a first step to begin a research in the field. We assume the reader knows the basics on Banach space theory, and we mean for this to be familiar, for instance, with most parts of J. Diestel's "Sequences and Series in Banach Spaces " [35].

Of course, many of the proofs given here are different to the original ones. And we believe that some of them are much easier. In fact, we have done an effort trying to give the best points of view and the best proofs to understand the theory. We would say that in some cases we give surprisingly easy proofs and one could think that we are actually proving almost trivial results. However, if one reads the original papers one does not have the same impression. We would say that this is so because of two main reasons.

On one hand, as we have already mentioned, we have chosen carefully the preliminary results we need, and even the precise versions which are the most suitable for our purposes. Although these results are "well known", most of them were quite dispersed and very often they could not be found easily in the literature.

On the other hand, it is normal that after several years we can see a theory in a much clearer way, that is, we understand better the results. When they were discovered, neither their relationship with other results nor the right tools to prove them were so clear. In our case for instance, Saab & Saab's Theorem 3.1.4 on complemented copies of ℓ_1 might seem now to be very easy. However, if we reflect a little bit about the main ingredients in its proof: lifting theory and Bourgain's Theorem 2.1.1, one realizes immediately that in the early eighties this theorem of Bourgain was not very well known, and it was not at all usual to apply lifting theory to get a theorem on $C(K, X)$ spaces.

Bourgain's name has just appeared. We would like to emphasize that our monograph contains a detailed exposition of some fundamental results of Bourgain which were not very accessible until now, at least for a beginner in the subject. His ideas are all over this monograph and are very important in our study.

We must add that this is not just a compillation of known results. We have tried to put some important ideas into an accessible expository form and we have also included: (a) some results which appear here for the first time, and (b) some results which we believe have been quite unnoticed and are difficult to find in the literature. We give now a list of the most important of these results. The first three are in case (b), and the other two in case (a).

1. Proposition 1.6.3 on measurability. The result is quite an easy application of fundamental results about the Souslin operation, but we have not seen such an application in this context in the literature.
2. Lemma 2.1.3. It is a result due to Rosenthal which we have called Kadec-Pełczyński-Rosenthal's Lemma because it is a refinement of known ideas of the classical Kadec-Pełczyński paper [77]. We think that at the moment it is the best "subsequence splitting lemma" in $L_1(\mu)$.
3. Theorem 2.1.4. It gives a large class of subspaces of $L_p(\mu, X)$ in which the L_p-norm and the L_1-norm are equivalent. We obtain the result as a quite easy consequence of the just mentioned Kadec-Pełczyński-Rosenthal's Lemma 2.1.3.
4. Díaz-Kalton's Theorem 5.2.3 characterizing when $L_\infty(\mu, X)$ contains a complemented copy of ℓ_1. When we write these lines, neither Díaz's contribution nor Kalton's have been published yet. However, they kindly have allowed us to use their ideas and preprints to give a complete account of their results.
5. Theorem characterizing when $L_\infty(\mu, X)$ contains a complemented copy of $L_1([0, 1])$. It may be found in the Notes and Remarks of Chapter 5. It is

actually an immediate consequence of Díaz-Kalton's Theorem 5.2.3 and Hagler-Stegall's theorem [67] characterizing those duals which contain a complemented copy of $C([0,1])^*$.

ACKNOWLEDGEMENTS

First of all we wish to express our gratitude to F. Bombal, our teacher, because he guided our first steps in the research and introduced us in this subject. We also have to thank B. Rodríguez-Salinas, because he also had a great influence in our formation. J. Diestel's books, conferences, seminars and friendly conversations have been very important for us, too.

We may say that the origin of this monograph goes back to November of 1990. Then, K. Sundaresan invited the second named author to deliver a talk at Cleveland State University. The subject of that talk coincided exactly with the one of this monograph and K. Sundaresan encouraged him to continue to work on this field. A few years later, the same author was invited by K. Jarosz to deliver a talk at the Second Conference on Function Spaces held at Edwarsville in the Spring of 1994. This was a new motivation to continue and bring to date the work which had been exposed at Cleveland. As a result he wrote the survey [98], which has become a very condensed preliminary version of this monograph. For these reasons, we are indebted with K. Sundaresan and K. Jarosz for their encouragement and contribution to the birth of this work.

We are also indebted with J.M.F. Castillo who, may be unaware, encouraged us to write this monograph.

We have already mentioned above that N. J. Kalton and S. Díaz have allowed us to include here some important results of them which will appear here for the first time. We have also had several interesting conversations with them on different aspects of this work. We are indeed very grateful to them.

We are also grateful to many colleagues which have gently provided us some preprints and with whom we have had interesting conversations. Among others, we have to mention: L. Drewnowski, Z. Hu and B.-L. Lin.

J. Bourgain and H. Rosenthal kindly told us about the history of the key lemma 2.1.3, since they knew we were wrong about its origin. H. Rosenthal gave us very interesting advices, too. They have our gratitude.

We have to thank I. Villanueva, student of our seminar of 96-97, for several valuable suggestions and comments.

It is a pleasure to recall also our colleagues and friends of our Department who have helped and encouraged us very often.

J. Esquinas answered very patiently many questions about TEX we posed him. Also Springer-Verlag staff, and particularly, F. Holzwarth, gave us all the help we asked them.

Finally, during the preparation of the manuscript we have asked D. H. Fremlin about different aspects of our work. He has always been extremely kind and his help can not be overestimated. We are very grateful to him.

Of course, we are the only responsible of any possible mistakes in this work.

This work has been partially supported by D.G.I.C.Y.T. grant PB94-0243

Madrid, September 1997

Some Notations and Conventions

In general, we will use standard notation as in [35], [39] or [93].

We will work with real or complex Banach spaces.

Let X, Y be Banach spaces. As usual, we say that X has a (complemented) copy of Y if X has a (complemented) subspace which is isomorphic to Y. We will denote

$$X \supset Y \quad \text{and} \quad X \underset{(c)}{\supset} Y$$

Let X be a Banach space, let (Ω, Σ, μ) be a positive measure space , and let $1 \leq p \leq \infty$. We denote by $L_p(\mu, X)$ the Banach space of all X-valued p-Bochner μ-integrable (μ-essentially bounded, when $p = \infty$) functions with its usual norm. That is, the vectors of $L_p(\mu, X)$ are (equivalence classes of) μ-measurable functions f such that

$$\| f \|_p = \left(\int_\Omega \| f(\omega) \| \, d\mu(\omega) \right)^{\frac{1}{p}} < +\infty$$

in the case $1 \leq p < +\infty$, and

$$\| f \|_\infty = \text{ess sup}\{\| f(\omega) \| : \omega \in \Omega\} < +\infty$$

in the case $p = +\infty$. In case X is the scalar field we simply denote $L_p(\mu)$.

If K is a compact Hausdorff space, we denote by $C(K, X)$ the Banach space of all continuous X-valued functions defined on K, endowed with the supremum norm. In case X is the scalar field we simply denote $C(K)$.

In order to avoid trivial situations, *we will always suppose that* X, $C(K)$ *and* $L_p(\mu)$ *are infinite dimensional.* Notice that in the case of $C(K)$ this simply means that K is infinite, and in the case of $L_p(\mu)$, that the corresponding measure space (Ω, Σ, μ) is not trivial, i.e. it has infinitely many disjoint measurable sets of finite positive measure.

For a vector measure $m : \Sigma \to X$, $\|m\|$ and $\tilde{m}$ denote the variation and semivariation of m respectively. The symbol $cabv(\Sigma, X)$ stands for the Banach space of all X-valued countably additive measures with bounded variation defined on Σ, endowed with the variation norm. If X is the scalar field this space is simply denoted by $cabv(\Sigma)$.

Recall that if (Ω, Σ, μ) is a measure space and we consider its completion $(\Omega, \bar{\Sigma}, \bar{\mu})$, then the corresponding $L_p(\mu, X)$ and $L_p(\bar{\mu}, X)$ spaces coincide. For this reason *we will always assume our measure spaces are complete.*

1. Preliminaries

Since we wish to study when the spaces $L_p(\mu, X)$ or $C(K, X)$ contain copies or complemented copies of c_0, ℓ_1 or ℓ_∞, we will devote the first part of the chapter to recall the main characterizations of when a general Banach space enjoys any of these properties. In the second part of the chapter we will give some general facts about the spaces $L_p(\mu, X)$ and $C(K, X)$.

1.1 Banach spaces containing c_0

Bessaga and Pełczyński's classical results [4] provide the main characterizations of Banach spaces containing c_0 we will need in our work (see [35, Chapter V]).

The basic sequences equivalent to the canonical basis of c_0 will be simply called c_0*-sequences*. Later on we will also use the expressions "ℓ_1*-sequence*" and in general "ℓ_p*-sequence*" in the same obvious sense. We will say that a sequence in X is *complemented* if its closed linear span is a complemented subspace of X. As usual, when we say that (x_n) is *seminormalized* we mean that $0 < inf \, \|x_n\| \leq sup \, \|x_n\| < +\infty$.

Let us remember that a series $\sum x_n$ in X is called *weakly unconditionally Cauchy* (or *weakly unconditionally convergent*), or in short w.u.C., if $\sum x^*(x_n)$ is absolutely convergent for each $x^* \in X^*$ (see [35, Theorem 6, Chapter V] for the main properties of w.u.C. series). The typical example of a non trivial w.u.C. series is the canonical c_0-basis, moreover we have (see [35, pages 42-45]):

Theorem 1.1.1 (Bessaga-Pełczyński). *Let (x_n) be a seminormalized sequence and suppose that the series $\sum x_n$ is w.u.C., then (x_n) has a c_0-subsequence. Moreover, if (x_n) is a basic sequence, then it is a c_0-sequence.*

We also can give a characterization of the Banach spaces containing *complemented* copies of c_0. We need first the following simple result

Proposition 1.1.1 (Chapter VII, Exercise 4 of [35]). *The bounded linear operators from X into c_0 correspond precisely to the weak*-null sequences in X^*, where each weak*-null sequence (x_n^*) in X^* has associated the operator $x \mapsto (x_n^*(x))$.*

With the preceding proposition in mind the following result is trivial (as usual, $\delta_{nm} = 0$ if $n \neq m$ and $\delta_{nn} = 1$).

Proposition 1.1.2. *Let (x_n) be a c_0-sequence in X, then (x_n) is complemented if and only if there is a weak*-null sequence (x_n^*) in X^* such that*

$$x_n^*(x_m) = \delta_{nm}.$$

But the main criterion we will use to find complemented c_0-sequences will be the following (versions of it may be found in [52] or [122]):

Theorem 1.1.2. *Let $\sum x_n$ be a w.u.C. series in X, then (x_n) has a complemented c_0-subsequence if and only if there exists a weak*-null sequence (x_n^*) in X^* such that*

$$x_n^*(x_n) \not\to 0.$$

Proof. Let us show the non trivial implication. Let $\sum x_n$ be a w.u.C. series in X, and let (x_n^*) be a weak*-null sequence in X^* such that

$$x_n^*(x_n) \not\to 0.$$

Then, without loss of generality, we may assume that there exists $\delta > 0$ such that

$$| \, x_n^*(x_n) \, | > \delta$$

for all $n \in \mathbb{N}$. Let T be the bounded linear operator associated to (x_n^*) as in Proposition 1.1.1. Notice that $\sum x_n$ and $\sum T(x_n)$ satisfy the assumptions of Theorem 1.1.1, and then, taking subsequences if necessary, we may assume that (x_n) and $(T(x_n))$ are both c_0-sequences. Of course this says that $T|_F$ is an isomorphism, where F denotes the closed linear span of the x_n's. Observe now that Sobczyk's theorem implies that the closed linear span of the $T(x_n)$'s is complemented in c_0. Let P be a projection from c_0 onto this subspace. It is clear that $(T|_F)^{-1} \circ P \circ T$ is a projection from X onto F.

1.2 Banach spaces containing ℓ_1

Of course the main characterization of Banach spaces containing ℓ_1 is the classical Rosenthal's theorem (see Chapter XI of [35]). To give a criterion to find *complemented* ℓ_1-sequences we will use again w.u.C. series. The following result is quite easy.

Proposition 1.2.1 (Exercise 3 of Chapter VII of [35]). *The bounded linear operators from a Banach space X into ℓ_1 correspond precisely to the w.u.C. series in X^*, where each w.u.C. series $\sum x_n^*$ in X^* has associated the operator $x \mapsto (x_n^*(x))$.*

From the preceding result we get immediately the following:

Proposition 1.2.2. *Let* (x_n) *be an* ℓ_1*-sequence in* X*, then* (x_n) *is complemented if and only if there exists a w.u.C. series* $\sum x_n^*$ *in* X^* *such that*

$$x_n^*(x_m) = \delta_{nm}.$$

But to give a really useful criterion to find *complemented* ℓ_1-sequences we need another fundamental result of Rosenthal [110] which has many different applications: the disjointification lemma. Its proof may be found in [35, page 82].

Lemma 1.2.1 (Rosenthal's Disjointification lemma). *Let* (μ_n) *be a bounded sequence in* $cabv(\Sigma)$*. Then given* $\epsilon > 0$ *and a sequence* (A_n) *of pairwise disjoint members of* Σ *there exists and increasing sequence* (k_n) *of positive integers for which*

$$\|\mu_{k_n}\| \, (\bigcup_{j \neq n} A_{k_j}) < \epsilon$$

for all n*.*

We can now give the announced criterion. It is very similar to the one we have given for c_0-sequences (Theorem 1.1.2) in the preceding section.

Theorem 1.2.1 (Rosenthal [110, 111]). *Let* (x_n) *be a bounded sequence in* X*. Then* (x_n) *has a complemented* ℓ_1*-subsequence if and only if there exists a weakly unconditionally Cauchy series* $\sum x_n^*$ *in* X^* *such that*

$$x_n^*(x_n) \not\to 0.$$

Proof. The condition is of course necessary. For the converse take (x_n) and (x_n^*) as in the statement. We may suppose, without loss of generality, that both sequences are in the corresponding unit balls. Taking subsequences and multiplying the x_n^*'s by adequate scalars if necessary, we may also assume that there exists $\delta > 0$ such that

$$x_n^*(x_n) \geq \delta$$

for all n. Let $\mathcal{P}(\mathbb{N})$ be the σ-algebra of all subsets of $\mathbb{N}$. For each $m \in \mathbb{N}$ consider the countably additive scalar measure μ_m defined on $\mathcal{P}(\mathbb{N})$ by

$$\begin{aligned} \mu_m : \mathcal{P}(\mathbb{N}) &\rightarrow \mathbb{K} \\ A &\rightarrow \sum_{n \in A} x_n^*(x_m) \end{aligned}$$

(this is just the usual way in which an element of ℓ_1 may be viewed as a member of $cabv(\mathcal{P}(\mathbb{N}))$). Since $\sum x_n^*$ is a w.u.C. series in X^* and (x_n) is bounded, using Proposition 1.2.1 it is clear that (μ_m) is bounded in $cabv(\mathcal{P}(\mathbb{N}))$. Hence we deduce from the preceding lemma that there are subsequences of (x_n) and (x_n^*), which we continue to denote again in the same way, such that

$$\|\mu_m\| \left(\bigcup_{n \neq m} \{n\} \right) = \sum_{n \neq m} |x_n^*(x_m)| < \delta/2$$

for all m. Let $R : X \to \ell_1$ be the bounded linear operator associated to $\sum x_n^*$ as in Proposition 1.2.1. Put $z_m = R(x_m) = (x_n^*(x_m))_n$. It is straightforward that the bounded linear operator

$$\begin{array}{rccc} S: & \ell_1 & \longrightarrow & \ell_1 \\ & (t_m) & \longrightarrow & \sum t_m z_m \end{array}$$

satisfies

$$\| I - S \| < 1 - \frac{\delta}{2},$$

where I is the identity operator in ℓ_1. Hence S is an isomorphism and we can deduce that $(z_m) = (R(x_m))$ is an ℓ_1-sequence. Therefore (x_m) is an ℓ_1-sequence, too. Finally, if we define $T : \ell_1 \to X$ by $T((t_m)) = \sum t_m x_m$, it is immediate that $T \circ S^{-1} \circ R$ is a projection from X onto the closed linear span of the x_n's.

1.3 Banach spaces containing ℓ_∞

The main criteria to determine if a given Banach space contains ℓ_∞ are also due to Rosenthal [110].

We denote by (e_n) the unit vector sequence of ℓ_∞.

Theorem 1.3.1. *The following are equivalent:*

(a) $X \supset \ell_\infty$.
(b) There exists a bounded linear operator $T : \ell_\infty \to X$ *such that*

$$\| T(e_n) \| \not\to 0.$$

(c) There exists a bounded linear operator $T : \ell_\infty \to X$ *which is not weakly compact.*

Observe that (c) implies (a) is the only non easy implication in this theorem (and it is just VI.1.3. of [39]). That (a) implies (b) is trivial, and to understand why (b) implies (c) it is enough to recall that every weakly compact operator is unconditionally converging (that is, it transforms w.u.C. series into unconditionally convergent ones [35, Chapter V, Exercise 8]). Since $\sum e_n$ is a w.u.C. series in ℓ_∞, it is clear that the operator T of condition (b) is not unconditionally converging, and so it can not be weakly compact either.