NEW TRIGONOMETRY FOR SCHOOLS

with answers

BORCHARDT & PERROTT

A NEW TRIGONOMETRY
FOR SCHOOLS

Published by
BELL & HYMAN LIMITED
Denmark House
37–39 Queen Elizabeth Street
London SE1 2QB

First published in 1904 by
G. Bell & Sons Ltd
Forty-sixth impression 1979

ISBN 0 7135 0128 6

Printed in Great Britain by
ROBERT MACLEHOSE AND CO. LTD
Printers to the University of Glasgow

A NEW TRIGONOMETRY FOR SCHOOLS

BY

W. G. BORCHARDT, M.A., B.Sc.

Formerly Assistant Master at Cheltenham College
Sometime Scholar of St John's College, Cambridge

AND

The Rev. A. D. PERROTT, M.A.

Formerly Scholar of Gonville and Caius College, Cambridge

BELL & HYMAN
LONDON

PREFACE

THE recent changes in the methods of teaching Elementary Mathematics (so largely due to the genius of Prof. Perry) have considerably affected Plane Trigonometry. Students are expected to have a good *practical* knowledge of the subject, while great skill in the solution of artificial problems and identities has ceased to be regarded as the aim and object of the subject.

This book has been written with a view to these changes and to supply the need felt for a School Trigonometry based on the use of *Four Figure Logarithms*, in which Logarithms, the Solution of Triangles and the more practical parts of the subject are introduced as early as possible. For this reason the expansions of sin $(A+B)$, etc. and harder identities are deferred until after the Solution of Triangles, Heights and Distances, etc.

Seeing that incommensurable quantities are now omitted in Elementary Geometry and consequently no difficulty is found with the various theorems relating to arcs and sectors of circles, it has been thought advisable to place the Circular Measurement of Angles immediately after the measurement in degrees, etc.

Graphical Methods and *Squared Paper* are largely employed in the approximation to trigonometrical ratios of a given angle, in finding angles from given ratios, in the variations of trigonometrical expressions and logarithms.

Students are advised always to *check* their results in the Solution of Triangles, Heights and Distances, etc., by drawing figures to scale.

The more *theoretical* parts are treated with fullness for the benefit of those intending to proceed to higher branches of mathematics.

Part I includes Solution of Triangles, Heights and Distances, and Functions of Compound Angles, and is sufficient for the Oxford and Cambridge Junior Local, Mathematics I of the Woolwich and Sandhurst Examination, etc. It contains over 1200 examples.

Part II contains chapters on De Moivre's Theorem, the Exponential Theorem and the expansion of sin θ and cos θ in terms of θ, etc.

Considerable care has been given to the selection of examples, many of which are taken from recent Army and Navy Entrance and the various Cambridge Examinations.

An appendix on the *Slide Rule* will be found useful for students preparing for the Entrance Examinations to Woolwich and Sandhurst.

It is hoped that the sets of *Test Papers*, which have been very carefully graduated to fit in with the sequence of chapters in the book, will prove useful for revision. Harder questions will be found in the Miscellaneous Examples.

The examples have all been verified from the proof sheets and it is hoped that very few errors remain; in the use of four figure tables, answers vary slightly according to the precise method of working; *e.g.* log 4 is not exactly the same as 2 log 2; such variations occur chiefly in solving triangles when there are several formulae applicable; the authors have in many cases indicated which formulae should be used to obtain the answers in the book.

The authors wish to express their gratitude for many suggestions received from Mr T. Hyett of Cheltenham College.

CHELTENHAM COLLEGE, *September* 1904

PREFACE TO THE SIXTEENTH IMPRESSION

THIS edition contains an Appendix on Projection and a new set of questions on Graphical Solutions, Exercise XXXIV A.

The authors take the opportunity of thanking Mr R. C. Chevalier, of Manchester Grammar School, for much valuable advice and criticism. In particular, the new proofs given on pages 117 *a*, 117 *b*, Art. 84, and the Alternative Proofs in the Appendix on Projection are due to him. Mr J. M. Child has independently evolved proofs similar to those on page 117 *b* and they are published in Barnard and Child's 'New Geometry', while Mr W. Clark of Uddingston Grammar School kindly sent the authors a proof similar to that in Art. 84.

November 1926.

CONTENTS

PART I.

PART II.

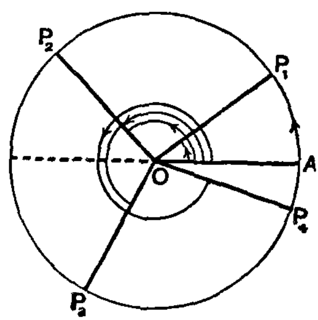
CHAPTER I.

MEASUREMENT OF ANGLES.

1. THE student is expected to be familiar with the definitions and explanations of an angle and a right angle as found in ordinary geometrical text-books.

2. The following slight extensions occur in Trigonometry.

Let a line OP revolve about the point O so that the point P traces out a circle, the direction of rotation being that shown in the figure, *i.e.* counter-clockwise. Let OA be the position from which OP starts to revolve.



When in the position OP_1 the angle described by OP may be either P_1OA or P_1OA + any multiple of 4 right angles, for OP might revolve any number of times in a complete circle before finally taking up the position OP_1.

In the positions OP_2, OP_3 and OP_4 the angles are as shown in the figure or these angles + any multiple of 4 right angles.

If OP revolved in the opposite direction to OP_4, then the angle P_4OA would be a negative angle and *e.g.* would be written $-30°$.

3. The geometrical unit is a right angle, but in Trigonometry this is subdivided.

1st method.

a right angle is divided into 90 equal parts called **Degrees**
a degree „ „ „ 60 „ „ „ **Minutes**
a minute „ „ „ 60 „ „ „ **Seconds**

Thus 1 right angle $= 90^\circ$ (degrees)
$1^\circ = 60'$ (minutes)
$1' = 60''$ (seconds).

This is called the English or Sexagesimal method and in practice is universally employed.

2nd method.

a right angle is divided into 100 equal parts called Grades
a grade „ „ „ „ „ „ „ Minutes
a minute „ „ „ „ „ „ „ Seconds

Thus 1 right angle $= 100^{g}$ (grades)
1 grade $= 100^{\backprime}$ (minutes)
1 minute $= 100^{\backprime\backprime}$ (seconds).

This is called the French or Centesimal method and is never employed in practice.

Rule. To convert Sexagesimal into Centesimal measure or vice versa, express the angle as a decimal of a right angle, then reduce to the new measure. (See examples 5 and 6.)

TYPICAL EXAMPLES.

Ex. 1. Reduce $21^\circ\ 13'\ 5''$ to seconds.

$$
\begin{array}{rl}
21^\circ\ 13' & 5'' \\
60 & \\
\hline
1273 & \text{minutes} \\
60 & \\
\hline
76385 & \text{seconds.}
\end{array}
$$

Ex. 2. Reduce 82097″ to degrees, etc.

$$\begin{array}{r|l} 60 & 82097'' \\ \hline 60 & 1368'\ 17'' \\ \hline & 22^\circ\ 48'\ 17''. \end{array}$$

Ex. 3. Reduce $21^{g}\ 13^{\backprime}\ 5^{\backprime\backprime}$ to centesimal seconds.

Since the system is a decimal system the answer can be written down at sight:

$$21^{g}\ 13^{\backprime}\ 5^{\backprime\backprime} = 21^{g}\ 13^{\backprime}\ 05^{\backprime\backprime} = 211305^{\backprime\backprime}.$$

Ex. 4. Reduce $320827^{\backprime\backprime}$ to Grades, etc.

$$320827^{\backprime\backprime} = 32^{g}\ 08^{\backprime}\ 27^{\backprime\backprime} = 32^{g}\ 8^{\backprime}\ 27^{\backprime\backprime}.$$

Ex. 5. Convert 64° 11′ 33″ to Centesimal measure.

$$\begin{array}{r|l} 60 & 33'' \\ \hline 60 & 11'\cdot55 \\ \hline 90 & 64^\circ\cdot1925 \\ \hline & \cdot71325 \text{ right angle.} \end{array}$$

Ans. $71^{g}\ 32^{\backprime}\ 50^{\backprime\backprime}$.

Ex. 6. Convert $64^{g}\ 11^{\backprime}\ 33^{\backprime\backprime}$ to Sexagesimal measure.

The angle = ·641133 of a right angle

$$\begin{array}{l} 90 \\ \hline 57^\circ\cdot70197 \\ 60\cdot \\ \hline 42'\cdot1182 \\ 60\cdot \\ \hline 7''\cdot092. \end{array}$$

Ans. 57° 42′ 7″·092.

EXAMPLES I*.

1. Convert to minutes

$5^{\circ}\ 12'$; $60^{\circ}\ 28'$; $132^{\circ}\ 52'$.

2. Convert to seconds

$12^{\circ}\ 13'\ 8''$; $48^{\circ}\ 37'\ 29''$; $105^{\circ}\ 24'\ 31''$.

3. Convert to centesimal minutes

$5^{g}\ 12^{\backprime}$; $60^{g}\ 9^{\backprime}$; $132^{g}\ 98^{\backprime}$.

4. Convert to centesimal seconds

$12^{g}\ 13^{\backprime}\ 8^{\backprime\backprime}$; $54^{g}\ 92^{\backprime}\ 94^{\backprime\backprime}$; $112^{g}\ 2^{\backprime}\ 4^{\backprime\backprime}$.

Convert to centesimal measure

5. $6^{\circ}\ 18'$; $18^{\circ}\ 27'$; $57^{\circ}\ 19'{\cdot}8$.

6. $30^{\circ}\ 46'\ 48''$; $63^{\circ}\ 39'\ 35''$; $35^{\circ}\ 10'\ 39''$.

Convert to sexagesimal measure

7. $5^{g}\ 19^{\backprime}$; $17^{g}\ 23^{\backprime}$; $56^{g}\ 22^{\backprime}$.

8. $31^{g}\ 47^{\backprime}\ 41^{\backprime\backprime}$; $64^{g}\ 3^{\backprime}\ 5^{\backprime\backprime}$; $79^{g}\ 91^{\backprime}\ 7^{\backprime\backprime}$.

* The examples on the Centesimal Measure may be omitted.

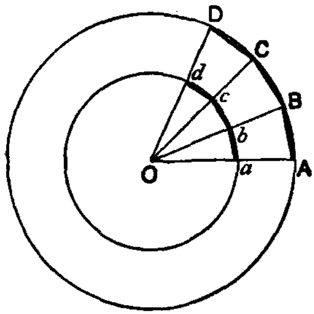
CIRCULAR MEASURE.

4. Theorem.

In all circles the ratio

$$\frac{\textit{circumference}}{\textit{diameter}} \textit{ is always the same.}$$

Take any two circles radii R and r and place them so that they have the same centre O. Divide the circles into n equal sectors by the lines OaA, ObB, OcC, etc. Join ab, bc, cd... and AB, BC, CD..., we then have 2 regular polygons of n sides inscribed in the circles.



$$\therefore \frac{\text{Perimeter of outer polygon}}{\text{Perimeter of inner polygon}} = \frac{n\,.\,\mathrm{AB}}{n\,.\,ab} = \frac{\mathrm{AB}}{ab} = \frac{\mathrm{R}}{r}$$

since ab and AB are parallel.

If n becomes indefinitely great and consequently AB, BC, CD... and ab, bc, cd... indefinitely small, the perimeters of the polygons become the circumferences of the circles;

$$\therefore \frac{\text{circumference of outer circle}}{\text{circumference of inner circle}} = \frac{\mathrm{R}}{r} = \frac{\text{diam. of outer circle}}{\text{diam. of inner circle}}.$$

$$\text{Hence } \frac{\text{circumference of any circle}}{\text{diameter of that circle}} = \text{a constant ratio.}$$

This constant ratio is an incommensurable number and is denoted by π.

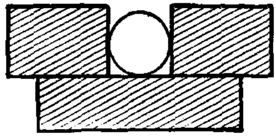
$\therefore$ **circumference** of a circle $= \pi\,.\,\mathrm{D} = \mathbf{2\pi r}$.

N.B. $\pi = 3{\cdot}1416$ approx.

$= \frac{22}{7}$ roughly.

5. The above result may be experimentally verified thus:

(i) Find the diameters of various coins by placing them between three rectangular blocks as in the figure.



(ii) Find the circumferences

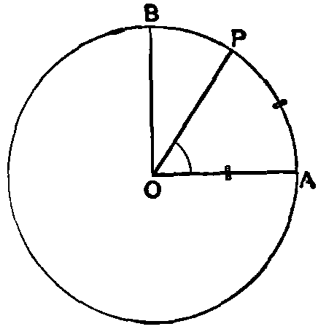
(*a*) with cotton

or (*b*) by making a small blot of ink on the rim of the coin, rolling it down a piece of cardboard and measuring the distance between two consecutive blots.

6. Theorem.

In any circle centre **O** *suppose an arc* **AP** *taken whose length = the radius.*

The angle **POA** *is constant for all circles.*



Draw **OB** perpendicular to **OA**.

Then

$$\frac{\hat{POA}}{\hat{BOA}} = \frac{\text{arc PA}}{\text{arc BA}}$$

$$= \frac{\text{radius OA}}{\text{one quarter of circumference}} = \frac{r}{\frac{2\pi r}{4}}$$

$$= \frac{2}{\pi};$$

$$\therefore \hat{POA} = \frac{2}{\pi} \text{ of a right angle} = \text{a constant}$$

$$= \frac{180^\circ}{\pi} = \frac{180^\circ}{3{\cdot}1416} = 57^\circ\ 17'\ 44'' \text{ approx}$$

This constant angle is called a **Radian**.

Thus $$1 \text{ Radian} = \frac{2}{\pi} \text{ of a right angle},$$

$$\therefore\ \pi \text{ radians} = 2 \text{ right angles}$$
$$= 180^\circ.$$

DEF. The angle subtended at the centre of a circle by an arc equal in length to the radius is called a *Radian*.

DEF. The *Circular Measure* of an angle is the number of radians it contains.

N.B. The ratio of one angle to another is the same whatever the units used.

$$\therefore\ \frac{\text{no. of degrees in an angle}}{90} = \frac{\text{no. of grades in same angle}}{100}$$
$$= \frac{\text{no. of radians}}{\frac{\pi}{2}}.$$

Ex. 1. Convert $\frac{\pi}{12}$ radians to sexagesimal measure.

$$\pi \text{ radians} = 180^\circ,$$
$$\therefore\ \frac{\pi}{12} \text{ radians} = \frac{180^\circ}{12} = 15^\circ.$$

Ex. 2. Convert 1·76 radians to sexagesimal measure.

$$\pi \text{ radians} = 180^\circ,$$
$$\therefore\ 1 \text{ radian} = \frac{180}{\pi},$$
$$\therefore\ 1{\cdot}76 \text{ radians} = \frac{180^\circ}{\pi} \times 1{\cdot}76$$
$$= \frac{7 \times 180}{22} \times 1{\cdot}76 \text{ approx.}$$
$$= 100^\circ{\cdot}8 \text{ approx.}$$

For greater accuracy π should be taken as 3·1416.

Ex. 3. Convert 64° 11′ 33″ to circular measure.

$$\begin{array}{r|l} 60 & 33'' \\ 60 & 11'{\cdot}55 \\ \hline & 64^\circ{\cdot}1925. \end{array}$$

Now $180° = \pi$ radians,

$$\therefore\ 64°\cdot1925 = \frac{\pi}{180} \times 64\cdot1925 \text{ radians}$$

$$= \frac{3\cdot1416}{180} \times 64\cdot1925 \text{ radians approx.}$$

$$= 1\cdot1204 \text{ radians approx.}$$

$$\left[\text{or more roughly} = \frac{22}{7 \times 180} \times 64\cdot1925 \text{ radians}\right.$$

$$\left.= 1\cdot1208 \text{ radians}\right].$$

EXAMPLES II.

Express in degrees, using $\pi = \frac{22}{7}$:

1. $\frac{\pi}{2}$ radians,
2. $\frac{\pi}{3}$ „
3. $\frac{\pi}{4}$ „
4. $\frac{\pi}{5}$ „
5. $\frac{\pi}{6}$ „
6. $\frac{\pi}{7}$ „
7. $\frac{\pi}{8}$ „
8. $\frac{\pi}{9}$ „
9. $\frac{2\pi}{3}$ radians,
10. $\frac{3\pi}{4}$ „
11. $\frac{5\pi}{24}$ „
12. 8·8 „
13. 1·65 „
14. 0·22 „
15. 1·1 „
16. 0·066 „

Express in circular measure as fractions of π:

17.	15°.	18.	18°.	19.	30°.	20.	36°.
21.	45°.	22.	54°.	23.	60°.	24.	75°.
25.	90°.	26.	120°.	27.	135°.	28.	180°.

Convert to circular measure, using $\pi = 3{\cdot}1416$ and working to 4 places:

29. $5^\circ\, 12'$; $17^\circ\, 33'$; $82^\circ\, 39'$.

30. $4^\circ\, 2'$; $19^\circ\, 24'$; $78^\circ\, 29'$.

31. Express in degrees and in radians the angle in a regular figure of 3, 4, 5, 6, or 8 sides.

32. Two angles are such that their difference is 20° and their sum $1\frac{1}{2}$ radians, find the angles in degrees, and radians ($\pi = \frac{22}{7}$).

33. In a triangle one angle is 30° and another $\frac{\pi}{4}$, find the third angle in degrees, and radians ($\pi = \frac{22}{7}$).

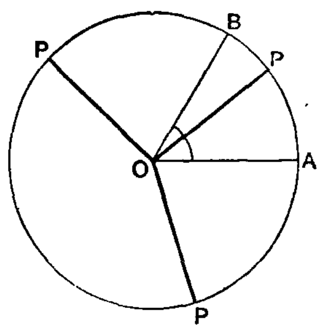
34. An angle contains x sexagesimal minutes or y radians, find the ratio $x : y$, and hence state the multiplier necessary to convert radians into sexagesimal minutes; use $\pi = \frac{22}{7}$.

7. Theorem.

The circular measure of an angle

$$= \frac{\text{The arc which the angle subtends when at the centre of any circle}}{\text{The radius of that circle}}.$$

Let AOP be any angle and AOB a radian.



Then circular measure of $A\hat{O}P$

$$= \frac{A\hat{O}P}{A\hat{O}B} \text{ (def. Art. 6)}$$

$$= \frac{\text{arc AP}}{\text{arc AB}}$$

$$= \frac{\text{arc AP}}{\text{radius}}.$$

Hence if an arc of any circle radius r subtends an angle θ radians at the centre

$$\textbf{arc} = r\theta.$$

This agrees with the result obtained in Art. 4, for putting $\theta = 2\pi$

$$\text{circumference} = r \,.\, 2\pi = 2\pi r.$$