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R. J. Williams

Introduction to Stochastic Integration

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PREFACE

The contents of this monograph approximate the lectures I gave in a graduate course at Stanford University in the first half of 1981. But the material has been thoroughly reorganized and rewritten. The purpose is to present a modern version of the theory of stochastic integration, comprising but going beyond the classical theory, yet stopping short of the latest discontinuous (and to some distracting) ramifications. Roundly speaking, integration with respect to a local martingale with continuous paths is the primary object of study here. We have decided to include some results requiring only right continuity of paths, in order to illustrate the general methodology. But it is possible for the reader to skip these extensions without feeling lost in a wilderness of generalities. Basic probability theory inclusive of martingales is reviewed in Chapter 1. A suitably prepared reader should begin with Chapter 2 and consult Chapter 1 only when needed. Occasionally theorems are stated without proof but the treatment is aimed at self-containment modulo the inevitable prerequisites. With considerable regret I have decided to omit a discussion of stochastic differential equations. Instead, some other applications of the stochastic calculus are given; in particular Brownian local time is treated in detail to fill an unapparent gap in the literature.

The applications to storage theory discussed in Section 8.4 are based on lectures given by J. Michael Harrison in my class. The material in Section 8.5 is Ruth Williams's work, which has now culminated in her dissertation [32].

At the start of my original lectures, I made use of Métivier's lecture notes [21] for their ready access. Later on I also made use of unpublished notes on continuous stochastic integrals by Michael J. Sharpe, and on local time by John B. Walsh. To these authors we wish to record our indebtedness. Some oversights in the references have been painstakingly corrected here. We hope any oversight committed in this book will receive similar treatment.

A methodical style, due mainly to Ruth Williams, is evident here. It is not always easy to strike a balance between utter precision and relative readability, and the final text represents a compromise of sorts. As a good author once told me, one cannot really hope to achieve consistency in writing a mathematical book—even a small book like this one.

K. L. Chung

December 1982

ABBREVIATIONS AND SYMBOLS

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$\mathbf{B}$	12
L	127, 133
$\mathbf{L}$	158
P	5
Z	147
$\mathbf{Z}$	158
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$J(t, x)$	133
L^p	3
M^k	19
$E(X)$	5
$PR(x)$	144
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$e(t)$	120
$h_n(x)$	114
$x^+ \equiv x \vee 0$	
$x^- \equiv (-x \vee 0)$	
$\mathbb{R}$	1
$\overline{\mathbb{R}}$	3
$\mathbb{R}_+$	1
$\mathbb{R}^d$	2
$\mathcal{A}$	28
$\mathcal{B}$	1
$\mathcal{C}$	144
$\mathcal{D}$	140
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$\mathcal{L}^2\ (\mu_M)$	35
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$\mathcal{L}^2_{\mathcal{P}}$		62
$\mathcal{L}^2_{\mathcal{B}\times\mathcal{F}}$		62
$\tilde{\mathcal{P}}$		60
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$\delta\pi_t$		66
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$\lambda_{(M)^2}$		34
λ_Z		33
μ_M		35
$\tilde{\mu}_M$		60
$\emptyset$	empty set	
Ω		5
$\Lambda(\mathcal{P}, M)$		46
$\Lambda(\tilde{\mathcal{P}}, M)$		62
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$[M]$		66
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$\int X\,dM$		36, 39
$\int_{[0,t]} X\,dM$		42, 48, 50
$\int_{(s,t]} X\,dM$		42, 50
$\int_0^t X\,dM$		42, 50
$\int_s^t X\,dM$		42, 50
c.		53
r.c.		53
l.c.		53
r.c.l.l.		53
l.c.r.l.		54
i.o.	infinitely often	
▮	end of proof	

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1

PRELIMINARIES

1.1 Notations and Conventions

For each interval I in $\mathbb{R} = (-\infty, \infty)$ let $\mathcal{B}(I)$ denote the σ-field of Borel subsets of I. For each $t \in \mathbb{R}_+ = [0, \infty)$, let $\mathcal{B}_t$ denote $\mathcal{B}([0, t])$ and let $\mathcal{B}$ denote $\mathcal{B}(\mathbb{R}_+) = \vee_{t \in \mathbb{R}_+} \mathcal{B}_t$ — the smallest σ-field containing $\mathcal{B}_t$ for all t in $\mathbb{R}_+$. Let $\overline{\mathbb{R}}_+ = [0, \infty]$ and $\overline{\mathcal{B}}$ denote the Borel σ-field of $\overline{\mathbb{R}}_+$ generated by $\mathcal{B}$ and the singleton $\{\infty\}$. Let λ denote the Lebesgue measure on $\mathbb{R}$.

Whenever t appears without qualification it denotes a generic element of $\mathbb{R}_+$. The collection $\{x_t, t \in \mathbb{R}_+\}$ is frequently denoted by $\{x_t\}$. The parameter t is sometimes referred to as time.

Let $\mathbb{N}$ denote the set of natural numbers, $\mathbb{N}_0$ denote $\mathbb{N} \cup \{0\}$, and $\mathbb{N}_\infty$ denote $\{\mathbb{N}\} \cup \{\infty\}$. Whenever n, k, or m, appears without qualification, it denotes a generic element of $\mathbb{N}$. A sequence $\{x_n, n \in \mathbb{N}\}$ is frequently denoted by $\{x_n\}$. We write $x_n \to x$ when $\{x_n\}$ converges to x. A sequence of real numbers $\{x_n\}$ is said to be increasing (decreasing) if $x_n \leq x_{n+1}$ ($x_n \geq x_{n+1}$) for all n. The notation $x_n \uparrow x$ ($x_n \downarrow x$) means $\{x_n\}$ is increasing (decreasing) with limit x.

For each $d \in \mathbb{N}$, the components of $x \in \mathbb{R}^d$ are denoted by x_i, $1 \leq i \leq d$, and the Euclidean norm of x by $|x| = \left(\sum_{i=1}^{d} (x_i)^2\right)^{\frac{1}{2}}$.

The symbol 1_A denotes the indicator function of a set A, i.e., $1_A(x) = 1$ if $x \in A$ and $= 0$ if $x \notin A$. The symbol $\emptyset$ denotes the empty set.

For each n, $C^n(\mathbb{R})$ or simply C^n denotes the set of all real-valued continuous functions defined on $\mathbb{R}$ for which the first n derivatives exist and are continuous. We use $C(\mathbb{R})$ to denote the set of real-valued continuous functions on $\mathbb{R}$ and $C^\infty(\mathbb{R})$ or C^∞ to denote $\bigcap_{n \in \mathbb{N}} C^n$, the set of infinitely differentiable real-valued functions on $\mathbb{R}$.

We use the words "positive", "negative", "increasing", and "decreasing", in the loose sense. For example, "x is positive" means "$x \geq 0$"; the qualifier "strictly" is added when "$x > 0$" is meant. The infimum of an empty set of real numbers is defined to be ∞. A sum over an empty index set is defined to be zero.

1.2 Measurability and L^p Spaces

Suppose (S, Σ) is a measurable space, consisting of a non-empty set S and a σ-field Σ of subsets of S. A function $X : S \to \mathbb{R}^d$ is called Σ-measurable if $X^{-1}(A) \in \Sigma$ for all Borel sets A in $\mathbb{R}^d$, where X^{-1} denotes the inverse image. A similar definition holds for a function X :

$S \to \overline{\mathbb{R}} = [-\infty, \infty]$. We use "$X \in \Sigma$" to mean "$X$ is Σ-measurable" and "$X \in b\Sigma$" to mean "X is bounded and Σ-measurable".

If Γ is a sub-family of Σ, a function $X : S \to \mathbb{R}^d$ is called *Γ-simple* if $X = \sum_{k=1}^n c_k 1_{\Lambda_k}$ for some constants c_k in $\mathbb{R}^d$, sets $\Lambda_k \in \Gamma$, and $n \in \mathbb{N}$. Such a function is Σ-measurable. Conversely, any Σ-measurable function is a pointwise limit of a sequence of Σ-simple functions. For example, a Σ-measurable function $X : S \to \mathbb{R}$ is the pointwise limit of the sequence $\{X^n\}$ of Σ-simple functions defined by

$$X^n = \sum_{k=0}^{n2^n} \frac{k}{2^n} 1_{\{k2^{-n} \le X < (k+1)2^{-n}\}} + \sum_{k=-1}^{-n2^n} \frac{(k+1)}{2^n} 1_{\{k2^{-n} \le X < (k+1)2^{-n}\}}$$

and $|X^n| \uparrow |X|$. In the above we have suppressed the argument of X, as we often do in the text.

Suppose ν is a (positive) measure on (S, Σ). A set in Σ of ν-measure zero is called a *ν-null set*. For $p \in [1, \infty)$, $L^p(S, \Sigma, \nu)$ denotes the vector space of Σ-measurable functions $X : S \to \mathbb{R}$ for which

$$\| X \|_p \equiv \left(\int_S |X(s)|^p \nu(ds) \right)^{\frac{1}{p}}$$

is finite. We use "ν-a.e." to denote "ν-almost everywhere". If functions which are equal ν-a.e. are identified, then $L^p(S, \Sigma, \nu)$ is a Banach space with norm $\| \cdot \|_p$. In the case $p = 2$, it is also a Hilbert space with inner product $(\cdot\ , \ \cdot)$ given by $(X, Y) = \int_S X(s)Y(s)\nu(ds)$ for X and Y in $L^2(S, \Sigma, \nu)$. Whenever we view these spaces in this way, it will be implicit that we are identifying functions which are equal ν-a.e.

1.3 Functions of Bounded Variation and Stieltjes Integrals

For a real-valued function g on $\mathbb{R}_+$, the *variation* of g on $[0,t]$ is given by

$$|g|_t \equiv \sup\left(\sum_{k=0}^{n-1} |g(t_{k+1}) - g(t_k)|\right)$$

where the supremum is over all partitions $0 = t_0 < t_1 < \ldots < t_n = t$ of $[0,t]$. The variation $|g|_t$ is increasing in t. If $|g|_t < \infty$, g is said to be of *bounded variation* on $[0,t]$. If this is true for all t in $\mathbb{R}_+$, g is said to be *locally of bounded variation* on $\mathbb{R}_+$; and if $\sup_{t \in \mathbb{R}_+} |g|_t < \infty$, then g is of *bounded variation on* $\mathbb{R}_+$. A (continuous) function is locally of bounded variation on $\mathbb{R}_+$ if and only if it is the difference of two (continuous) increasing functions (see Royden [25, p.100]).

A function g which is locally of bounded variation on $\mathbb{R}_+$ induces a signed measure μ on the σ-field $\mathcal{B}$, where

$$\mu((a,b]) = g(b) - g(a) \text{ for } a < b \text{ in } \mathbb{R}_+ \text{ and } \mu(\{0\}) = 0.$$

The measure μ is uniquely determined by the above since intervals of the form $(a,b]$ together with $\{0\}$ generate $\mathcal{B}$. It is a positive measure if g is increasing and has no atoms if g is continuous. The variation $|\mu|$ of μ is the measure associated with the variation $|g|$. If $f \in L^1([0,t], \mathcal{B}_t, |\mu|)$, then the Lebesgue-Stieltjes integral of f with respect to g over $[0,t]$ is defined by

$$\int_{[0,t]} f(s)\, dg(s) \equiv \int_{[0,t]} f\, d\mu$$

$$= \lim_{n\to\infty}\left(\sum_{k=0}^{\infty} \frac{k}{2^n}\mu\left(\left\{s \in [0,t] : \frac{k}{2^n} \le f(s) < \frac{k+1}{2^n}\right\}\right)\right.$$
$$\left. + \sum_{k=-1}^{-\infty} \frac{(k+1)}{2^n}\mu\left(\left\{s \in [0,t] : \frac{k}{2^n} \le f(s) < \frac{k+1}{2^n}\right\}\right)\right)$$

and $|\int_{[0,t]} f(s)\,dg(s)| \le \int_{[0,t]} |f|\,d|\mu|$. If the last integral is finite for all $t \in [0,T]$ and g is continuous, then $\int_{[0,t]} f(s)\,dg(s)$ is a continuous function of $t \in [0,T]$ and we denote it by $\int_0^t f(s)\,dg(s)$. If f is a continuous function on $[0,t]$, then the Riemann-Stieltjes integral of f with respect to g on $[0,t]$ is well-defined and equals the Lebesgue-Stieltjes integral, i.e.,

$$\int_{[0,t]} f(s)\,dg(s) = \lim_{n\to\infty} \sum_{k=1}^{N_n} f(s_k^n)\big(g(t_k^n) - g(t_{k-1}^n)\big), \tag{1.1}$$

for any sequence of partitions $0 = t_0^n < t_1^n < \cdots < t_{N_n}^n = t$ of $[0,t]$ where $s_k^n \in [t_{k-1}^n, t_k^n]$ and $\max_{k=1}^{N_n} |t_k - t_{k-1}| \to 0$ as $n \to \infty$. If g is continuous, then $\int_0^t f(s)\,dg(s)$ is also given by (1.1) when f is right continuous on $[0,t)$ with finite left limits on $(0,t]$, or left continuous on $(0,t]$ with finite right limits on $[0,t)$.

1.4 Probability Space, Random Variables, Filtration

Throughout this book, $(\Omega, \mathcal{F}, P)$ denotes a given complete probability space. This means that $(\Omega, \mathcal{F})$ is a measurable space and P is a probability measure on $(\Omega, \mathcal{F})$ such that each subset of a P-null set in $\mathcal{F}$ is in $\mathcal{F}$. The abbreviation "a.s." for "almost surely" means "P-a.e.". The symbol ω denotes a generic element of Ω. For a function $Y : \Omega \to \mathbb{R}^d$ (or $\overline{\mathbb{R}}$) and a set A in $\mathbb{R}^d$ (or $\overline{\mathbb{R}}$), $Y^{-1}(A) = \{\omega : Y(\omega) \in A\}$ is also written as $\{Y \in A\}$. The symbol ω is also suppressed in similar expressions.

We write L^p for $L^p(\Omega, \mathcal{F}, P)$. For $X \in L^1$, $E(X) \equiv \int_\Omega X\,dP$ denotes the expectation of X. As an extension of notation, for $\Lambda \in \mathcal{F}$, $E(X\,;\Lambda)$ denotes $\int_\Lambda X\,dP$, and when Λ is of the form $\{Y \in A\}$ this is written as $E(X\,; Y \in A)$.