K.L.Chung R.J.Williams

Introduction to Stochastic Integration

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This book was typeset at Stanford University using the TeX formatting system. Kathleen Flynn entered the manuscript into the computer and formatted all text, mathematical equations and figures. Ms. Flynn was also responsible for proofreading, editing and photocomposing the book and coordinating production arrangements with the publisher. Artists at Stanford Word Graphics, especially Willum Terluin, added final touches to some of the figures. Special thanks go to Donald E. Knuth for the use of his TeX system, and to David Fuchs for his generous advice and assistance with the many technical details. Finally, acknowledgements are due to all of the people involved in maintaining the computing machinery.

PREFACE

At the start of my original lectures, I made use of Métivier's lecture notes [21] for their ready access. Later on I also made use of unpublished notes on continuous stochastic integrals by Michael J. Sharpe, and on local time by John B. Walsh. To these authors we wish to record, our indebtedness. Some oversights in the references have been painsfalcingly corrected here. We hope any oversight committed in this book will

The contents of this monograph approximate the lectures I gave in a graduate course at Stanford University in the first half of 1981. But the material has been thoroughly reorganized and rewritten. The purpose is to present a modern version of the theory of stochastic integration, comprising but going beyond the classical theory, yet stopping short of the latest discontinuous (and to some distracting) ramifications. Roundly speaking, integration with respect to a local martingale with continuous paths is the primary object of study here. We have decided to include some results requiring only right continuity of paths, in order to illustrate the general methodology. But it is possible for the reader to skip these extensions without feeling lost in a wilderness of generalities. Basic probability theory inclusive of martingales is reviewed in Chapter 1. A suitably prepared reader should begin with Chapter 2 and consult Chapter 1 only when needed. Occasionally theorems are stated without proof but the treatment is aimed at self-containment modulo the inevitable prerequisites. With considerable regret I have decided to omit a discussion of stochastic differential equations. Instead, some other applications of the stochastic calculus are given; in particular Brownian local time is treated in detail to fill an unapparent gap in the literature.

The applications to storage theory discussed in Section 8.4 are based on lectures given by J. Michael Harrison in my class. The material in Section 8.5 is Ruth Williams's work, which has now culminated in her dissertation [32].

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A methodical style, due mainly to Ruth Williams, is evident here. It is not always easy to strike a balance between utter precision and relative readability, and the final text represents a compromise of sorts. As a good author once told me, one cannot really hope to achieve consistency in writing a mathematical book—even a small book like this one.

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December 1982

ABBREVIATIONS AND SYMBOLS

	page
	11
B B B	138
В	12
L	127, 133
L es Add	158
P 80 mg X dM	5
Z gg (and X dat	147
Z gel XdM	158
$H_n(x,y)$	113
J(t,x)	133
LP COLO	3
M^k	19
E(X)	5
PR(x)	
[1] [1] [1] [1] [1] [1] [1] [1] [1] [1]	144
Z ^a galo. Infinitely often	109

ABBREVIATIONS AND SYMBOLS

ar since the in Manday Nave and in the short	page
e(t)	120
$h_n(x)$	114
$x^+ \equiv x \lor 0$	
$x^- \equiv (-x \lor 0)$	
IR	Sharpe, 1
R sans It Walsh. To these authors we w	
R	
\mathbb{R}^d	
A	28
	1
B which and , due mainly to Ruth Williams,	144
C anside to strike a balance between utter pro	
Description of the state of the	36
one in the same and the many regime house to hi	5
to a small book I work - even a small book I	
M	60
N	29
0 881	28
P	32
Q	28
R	
S	32
WELL	62
\mathcal{F}_t	6, 23
\mathcal{F}_{t+}	6
\mathcal{L}^2	35
$\mathcal{L}^{2}\left(\mu_{M}\right)$	35
L2 2	62

	page
$\mathcal{L}^2_{\widetilde{\mathcal{P}}}$	62
$\mathcal{L}^{2}_{\tilde{\mathcal{P}}}$ $\mathcal{L}^{2}_{\mathcal{B}} \times \mathcal{F}$ $\tilde{\mathcal{P}}$	62
P	60
π_t	_ 66
$\delta\pi_t$	66
λ	1
$\lambda_{(M)^2}$	34
λ_Z	33
μ_M	35
$\tilde{\mu}_M$	60
Ø empty set	
Ω	5
$\Lambda(P,M)$	46
$\Lambda(\tilde{\mathcal{P}},M)$	62
$\Lambda^2(P,M)$	39
[M]	66
	(0,00), let \$1 deno 91'8
$\int X dM$	
	12, 48, 50
$\int_{(s,t]} X dM$	42, 50
$\int_0^t X dM$	42, 50
$\int_{s}^{t} X dM$	42, 50
	is frequently demon 53
r.c.	59
l.c.	53
r.c.l.l. the set of natural mu	robers, IVo denote 53
Lc.r.l.	54
i.o. infinitely often	
end of proof	

TABLE OF CONTENTS

	ix
ABBREVIATIONS AND SYMBOLS	xi
1. PRELIMINARIES	1
	1
1.2 Measurability and L^p Spaces	2
1.3 Functions of Bounded Variation and Stieltjes Integrals	4
1.4 Probability Space, Random Variables, Filtration	5
1.5 Convergence, Conditioning	6
1.6 Stochastic Processes	8
	9
1.8 Two Canonical Processes	
1.9 Martingales	3
경기를 보고 있다. 그런데 그는 그들은 이번 보고 있다. 그는 그들은 사람들은 사람들은 그들은 사람들이 되었다. 그렇게 되었다는 그를 가지 않는 것이다. 그를 가지 않는 것이다.	9
2. DEFINITION OF THE STOCHASTIC INTEGRAL . 2	5
2.1 Introduction notion male word to notice the man 19	5

2.2 Predictable Sets and Processes					28
2.3 Stochastic Intervals					29
2.4 Measure on the Predictable Sets					33
2.5 Definition of the Stochastic Integral					36
2.6 Extension to Local Integrators and Integrands	. 1				46
3. EXTENSION OF THE PREDICTABLE INTE	GR	A	N	DS	53
3.1 Introduction					53
3.2 Relationship between P, O, and Adapted Processes .					53
3.3 Extension of the Integrands					59
3.4 An Historical Note					63
4. QUADRATIC VARIATION PROCESS	AI	i.	·	ists	65
4.1 Introduction					65
4.2 Definition and Characterization of Quadratic Variati	on				66
4.3 Properties of Quadratic Variation for an L2-marting	ale				70
4.4 Direct Definition of μ_M	anje	Į,		4	74
4.5 Decomposition of $(M)^2$	de			V	78
4.6 A Limit Theorem					82
Space, Random Varables Pilitations of the second					
5. THE ITO FORMULA	100		10	0	85
5.1 Introduction					85
5.2 One-dimensional Itô Formula					86
5.3 Mutual Variation Process					91
5.4 Multi-dimensional Itô Formula					101
The second secon					
6. APPLICATIONS OF THE ITO FORMULA.		131		IST.	105
6.1 Characterization of Brownian Motion					105
6.2 Exponential Processes					109

	TABLE OF CONTENTS	vii
	Generated by M	13
6.4 Feynman-Kac Functiona	and the Schrödinger Equation 12	20
7. LOCAL TIME AND	ANAKA'S FORMULA 12	27
7.1 Introduction		27
7.2 Local Time		28
7.3 Tanaka's Formula		37
7.4 Proof of Lemma 7.2		19
8. REFLECTED BROW	NIAN MOTIONS 14	13
8.1 Introduction		3
8.2 Brownian Motion Reflect	ed at Zero	4
8.3 Analytical Theory of Z		8
8.4 Approximations in Stora	ge Theory	0
	ons in a Wedge 16	4
8.6 Alternative Derivation of	Equation (8.7) 16	
	FORMULA	
	D CHANGE OF TIME 17	
9.1 Introduction	77 Artists at Stanford W	3
9.2 Generalized Itô Formula	a beobs multion multiw yilais	1

9.3 Change of Time .

PRELIMINARIES

1.1 Notations and Conventions Washington and bas

For each interval I in $\mathbb{R} = (-\infty, \infty)$ let $\mathcal{B}(I)$ denote the σ -field of Borel subsets of I. For each $t \in \mathbb{R}_+ = [0, \infty)$, let \mathcal{B}_t denote $\mathcal{B}([0, t])$ and let \mathcal{B} denote $\mathcal{B}(\mathbb{R}_+) = \bigvee_{t \in \mathbb{R}_+} \mathcal{B}_t$ — the smallest σ -field containing \mathcal{B}_t for all t in \mathbb{R}_+ . Let $\overline{\mathbb{R}}_+ = [0, \infty]$ and $\overline{\mathcal{B}}$ denote the Borel σ -field of $\overline{\mathbb{R}}_+$ generated by \mathcal{B} and the singleton $\{\infty\}$. Let λ denote the Lebesgue measure on \mathbb{R} .

Whenever t appears without qualification it denotes a generic element of \mathbb{R}_+ . The collection $\{x_t, t \in \mathbb{R}_+\}$ is frequently denoted by $\{x_t\}$. The parameter t is sometimes referred to as time.

Let $I\!N$ denote the set of natural numbers, $I\!N_0$ denote $I\!N \cup \{0\}$, and $I\!N_\infty$ denote $I\!N \cup \{\infty\}$. Whenever n, k, or m, appears without

qualification, it denotes a generic element of $I\!N$. A sequence $\{x_n, n \in I\!N\}$ is frequently denoted by $\{x_n\}$. We write $x_n \to x$ when $\{x_n\}$ converges to x. A sequence of real numbers $\{x_n\}$ is said to be increasing (decreasing) if $x_n \leq x_{n+1}$ ($x_n \geq x_{n+1}$) for all n. The notation $x_n \uparrow x$ ($x_n \downarrow x$) means $\{x_n\}$ is increasing (decreasing) with limit x.

For each $d \in \mathbb{N}$, the components of $x \in \mathbb{R}^d$ are denoted by x_i , $1 \le i \le d$, and the Euclidean norm of x by $|x| = \left(\sum_{i=1}^d (x_i)^2\right)^{\frac{1}{2}}$.

The symbol 1_A denotes the indicator function of a set A, i.e., $1_A(x) = 1$ if $x \in A$ and = 0 if $x \notin A$. The symbol \emptyset denotes the empty set.

For each n, $C^n(\mathbb{R})$ or simply C^n denotes the set of all real-valued continuous functions defined on \mathbb{R} for which the first n derivatives exist and are continuous. We use $C(\mathbb{R})$ to denote the set of real-valued continuous functions on \mathbb{R} and $C^\infty(\mathbb{R})$ or C^∞ to denote $\bigcap_{n\in\mathbb{N}}C^n$, the set of infinitely differentiable real-valued functions on \mathbb{R} .

We use the words "positive", "negative", "increasing", and "decreasing", in the loose sense. For example, "x is positive" means " $x \geq 0$ "; the qualifier "strictly" is added when "x > 0" is meant. The infimum of an empty set of real numbers is defined to be ∞ . A sum over an empty index set is defined to be zero.

1.2 Measurability and IP Spaces

Suppose (S, Σ) is a measurable space, consisting of a non-empty set S and a σ -field Σ of subsets of S. A function $X: S \to \mathbb{R}^d$ is called Σ -measurable if $X^{-1}(A) \in \Sigma$ for all Borel sets A in \mathbb{R}^d , where X^{-1} denotes the inverse image. A similar definition holds for a function X:

 $S \to \overline{\mathbb{R}} = [-\infty, \infty]$. We use " $X \in \Sigma$ " to mean "X is Σ -measurable" and " $X \in b\Sigma$ " to mean "X is bounded and Σ -measurable".

If Γ is a sub-family of Σ , a function $X:S\to I\!\!R^d$ is called Γ -simple if $X=\sum_{k=1}^n c_k 1_{\Lambda_k}$ for some constants c_k in $I\!\!R^d$, sets $\Lambda_k\in\Gamma$, and $n\in I\!\!N$. Such a function is Σ -measurable. Conversely, any Σ -measurable function is a pointwise limit of a sequence of Σ -simple functions. For example, a Σ -measurable function $X:S\to I\!\!R$ is the pointwise limit of the sequence $\{X^n\}$ of Σ -simple functions defined by

$$X^{n} = \sum_{k=0}^{n2^{n}} \frac{k}{2^{n}} 1_{\{k2^{-n} \le X < (k+1)2^{-n}\}} + \sum_{k=-1}^{-n2^{n}} \frac{(k+1)}{2^{n}} 1_{\{k2^{-n} \le X < (k+1)2^{-n}\}}$$
(appending)

and $|X^n| \uparrow |X|$. In the above we have suppressed the argument of X, as we often do in the text.

Suppose ν is a (positive) measure on (S, Σ) . A set in Σ of ν -measure zero is called a ν -null set. For $p \in [1, \infty)$, $L^p(S, \Sigma, \nu)$ denotes the vector space of Σ -measurable functions $X : S \to \mathbb{R}$ for which

$$\parallel X \parallel_p \equiv \left(\int_S |X(s)|^p \nu(ds) \right)^{\frac{1}{p}}$$

is finite. We use " ν -a.e." to denote " ν -almost everywhere". If functions which are equal ν -a.e. are identified, then $L^p(S, \Sigma, \nu)$ is a Banach space with norm $\|\cdot\|_p$. In the case p=2, it is also a Hilbert space with inner product (\cdot,\cdot) given by $(X,Y)=\int_S X(s)Y(s)\nu(ds)$ for X and Y in $L^2(S,\Sigma,\nu)$. Whenever we view these spaces in this way, it will be implicit that we are identifying functions which are equal ν -a.e.

1.3 Functions of Bounded Variation and Stieltjes Integrals

For a real-valued function g on \mathbb{R}_+ , the variation of g on [0,t] is given by

$$|g|_t \equiv \sup \left(\sum_{k=0}^{n-1} |g(t_{k+1}) - g(t_k)| \right)$$

where the supremum is over all partitions $0 = t_0 < t_1 < \ldots < t_n = t$ of [0,t]. The variation $|g|_t$ is increasing in t. If $|g|_t < \infty$, g is said to be of bounded variation on [0,t]. If this is true for all t in \mathbb{R}_+ , g is said to be locally of bounded variation on \mathbb{R}_+ ; and if $\sup_{t \in \mathbb{R}_+} |g|_t < \infty$, then g is of bounded variation on \mathbb{R}_+ . A (continuous) function is locally of bounded variation on \mathbb{R}_+ if and only if it is the difference of two (continuous) increasing functions (see Royden [25, p.100]).

A function g which is locally of bounded variation on \mathbb{R}_+ induces a signed measure μ on the σ -field \mathcal{B} , where

$$\mu((a,b]) = g(b) - g(a)$$
 for $a < b$ in \mathbb{R}_+ and $\mu(\{0\}) = 0$.

The measure μ is uniquely determined by the above since intervals of the form (a,b] together with $\{0\}$ generate \mathcal{B} . It is a positive measure if g is increasing and has no atoms if g is continuous. The variation $|\mu|$ of μ is the measure associated with the variation |g|. If $f \in L^1([0,t],\mathcal{B}_t,|\mu|)$, then the Lebesgue-Stieltjes integral of f with respect to g over [0,t] is defined by

$$\int_{[0,t]} f(s) \, dg(s) \equiv \int_{[0,t]} f \, d\mu$$

$$= \lim_{n \to \infty} \left(\sum_{k=0}^{\infty} \frac{k}{2^n} \mu \left(\left\{ s \in [0,t] : \frac{k}{2^n} \le f(s) < \frac{k+1}{2^n} \right\} \right) + \sum_{k=-1}^{-\infty} \frac{(k+1)}{2^n} \mu \left(\left\{ s \in [0,t] : \frac{k}{2^n} \le f(s) < \frac{k+1}{2^n} \right\} \right) \right)$$

and $|\int_{[0,t]} f(s) dg(s)| \leq \int_{[0,t]} |f| d|\mu|$. If the last integral is finite for all $t \in$ [0,T] and g is continuous, then $\int_{[0,t]} f(s) dg(s)$ is a continuous function of $t \in [0,T]$ and we denote it by $\int_0^t f(s) dg(s)$. If f is a continuous function on [0, t], then the Riemann-Stieltjes integral of f with respect to g on [0, t] is well-defined and equals the Lebesgue-Stieltjes integral, i.e.,

(1.1)
$$\int_{[0,t]} f(s) dg(s) = \lim_{n \to \infty} \sum_{k=1}^{N_n} f(s_k^n) (g(t_k^n) - g(t_{k-1}^n)),$$

for any sequence of partitions $0 = t_0^n < t_1^n < \ldots < t_{N_n}^n = t$ of [0,t]where $s_k^n \in [t_{k-1}^n, t_k^n]$ and $\max_{k=1}^{N_n} |t_k - t_{k-1}| \to 0$ as $n \to \infty$. If g is continuous, then $\int_0^t f(s) dg(s)$ is also given by (1.1) when f is right continuous on [0,t) with finite left limits on (0,t], or left continuous on (0,t] with finite right limits on [0,t).

1.4 Probability Space, Random Variables, Filtration

Throughout this book, (Ω, \mathcal{F}, P) denotes a given complete probability space. This means that (Ω, \mathcal{F}) is a measurable space and P is a probability measure on (Ω, \mathcal{F}) such that each subset of a P-null set in F is in F. The abbreviation "a.s." for "almost surely" means "P-a.e.". The symbol ω denotes a generic element of Ω . For a function $Y:\Omega\to$ \mathbb{R}^d (or $\overline{\mathbb{R}}$) and a set A in \mathbb{R}^d (or $\overline{\mathbb{R}}$), $Y^{-1}(A) = \{\omega : Y(\omega) \in A\}$ is also written as $\{Y \in A\}$. The symbol ω is also suppressed in similar

We write L^p for $L^p(\Omega, \mathcal{F}, P)$. For $X \in L^1$, $E(X) \equiv \int_{\Omega} X dP$ denotes the expectation of X. As an extension of notation, for $\Lambda \in \mathcal{F}$, $E(X;\Lambda)$ denotes $\int_{\Lambda} X dP$, and when Λ is of the form $\{Y \in A\}$ this is written as $E(X; Y \in A)$. notherogong animaliant the state was considered to the state of the