COLLECTED PAPERS

Marston Morse

VOLUME 5

World Scientific

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Published by

World Scientific Publishing Co. Pte. Ltd.
P. O. Box 128, Farrer Road, Singapore 9128

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Library of Congress Cataloging-in-Publication data is available.

COLLECTED PAPERS OF MARSTON MORSE

ISBN 9971-978-94-6

Printed in Singapore by Fong and Sons Printers Pte. Ltd.

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Marston Morse

With family at Princeton, 1968. From left, standing: William, Louise II, Peter, Julia, Dryden; seated: Elizabeth, Louise, Marston, Meroe.

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FIELDS OF GEODESICS ISSUING FROM A POINT

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THE INSTITUTE FOR ADVANCED STUDY

Communicated November 6, 1959

§*1*. *Introduction.*—We shall be concerned with a 1-parameter family of open Riemannian μ-manifolds H_α of class C^∞, where the parameter α varies on an open interval I. The properties of fields of geodesics issuing from points p_α of the respective manifolds H_α are not fully stated or derived in mathematical literature known to us, at least for the ends we have in mind. This note is designed to fill these needs and will find an application in reference 1.

To simplify the exposition we begin with the more familiar case of a single Riemannian μ-manifold H, $\mu > 1$, leaving to §4 the main theorem of the paper. Let p be an arbitrary fixed point in H. Let $(x^1, \ldots, x^\mu)$ be rectangular coordinates in a Euclidean μ-space E_μ. Set

$$(x^1, \ldots, x^\mu) = \mathbf{x}, \|\mathbf{x}\| = (x^i x^i)^{1/2} \qquad (i = 1, \ldots, \mu)$$

employing the summation convention of tensor algebra. We introduce the μ-disk

$$\Delta_\rho^x = \{\mathbf{x} \mid \|\mathbf{x}\| < \rho\} \tag{1.0}$$

where ρ is any positive constant.

Let W be an open *coordinate domain* on H which contains p. We suppose that there exists a diffeomorphism

$$\varphi : \Delta_\rho^x \to W : \mathbf{x} \to \varphi(\mathbf{x}) \tag{1.1}$$

of Δ_ρ^x onto W by virtue of which Δ_ρ^x becomes (ref. 2, §0) the *range* of the local coordinates (x). The term "diffeomorphism" in this paper will always mean a diffeomorphism of class C^∞. Since H is supposed to be a Riemannian manifold we include the condition that at the point on W with local coordinates (x)

$$ds^2 = g_{ij}(\mathbf{x})dx^i dx^j \qquad (i, j = 1, \ldots, \mu), \tag{1.2}$$

where the coefficients g_{ij} are functions of class C^∞ for $\mathbf{x} \in \Delta_\rho^x$, and the invariant quadratic form is positive definite. As far as this paper is concerned H may be supposed identified with W. The compatibility of the above representation φ of W with other representations of open subdomains of W is presupposed in the sense usual for Riemannian manifolds of class C^∞.

We suppose that under the diffeomorphism φ, the point p is represented by the origin in E_μ. We shall also suppose that $g_{ij}(\mathbf{O}) = \delta_{ij}$. If this condition were not satisfied, it would be satisfied for a new set of local coordinates on a neighborhood of p in H, on making a suitable non-singular linear transformation of the initial coordinates (x).

Understanding that $(r^1, \ldots, r^\mu) = \mathbf{r}$ is a contravariant vector associated with the point $(\mathbf{x})$, set

$$f(\mathbf{x}, \mathbf{r}) = [g_{ij}(\mathbf{x})r^i r^j]^{1/2}. \tag{1.3}$$

A geodesic g on W is represented on Δ_ρ^x by a regular curve

$$t \to \mathbf{x}(t) = (x^1(t), \ldots, x^\mu(t)) \qquad (t \in J) \tag{1.4}$$

of class C^2 (at least), satisfying the Euler equations associated with the integral,

$$\int f(\mathbf{x}, \dot{\mathbf{x}})\, dt.$$

These Euler equations do not uniquely determine the parameterization of g. We shall admit only those representations of geodesics in which the parameter $t = cs$, where s is the arc length measured along the geodesic in the sense of increasing t, and c is a positive constant. We shall limit our study to geodesics issuing from $p \in H$ and shall measure s from p. It is easy to verify the fact that a regular curve, of class C^2, of the form (1.4), represents a geodesic and is of class C^∞, with $t = cs$, if and only if it is a non-null solution of the system of μ differential equations,

$$2\frac{d}{dt}(g_{hj}(\mathbf{x})\dot{x}^j) = \frac{\partial g_{ij}}{\partial x^h}(\mathbf{x})\dot{x}^i\dot{x}^j \qquad (h, i, j = 1, \ldots, \mu). \tag{1.5}$$

On any solution of (1.5)

$$\frac{ds}{dt} = f(\mathbf{x}, \dot{\mathbf{x}}) = c, \tag{1.6}$$

where c is a constant. Moreover $c > 0$ if the solution represents a geodesic, and $c = 0$ if the solution reduces to a point.

§*2. Solutions of (1.5) Vanishing at the Origin.*—Let S be the class of solutions $t \to \mathbf{x}(t)$ of (1.5) such that $\mathbf{x}(t)$ is in $\overset{x}{\Delta}_p$ and reduces to the null vector at $t = 0$. In S we include the trivial solution such that $\|\mathbf{x}(t)\| = 0$ for all t. Since the determinant $|g_{hj}(\mathbf{x})| \neq 0$ for $\mathbf{x} \in \Delta_p$ the following is true. Corresponding to an arbitrary point $(z^1, \ldots, z^\mu) = \mathbf{z}$ in a μ-plane $\mathcal{E}_\mu$ there exists a maximum positive constant $b(\mathbf{z})$, possibly infinite, such that there is a solution of (1.5) in the class S of the form, for fixed (z),

$$t \to \mathbf{A}(t, \mathbf{z}) \qquad (0 \leq t < b(\mathbf{z})) \tag{2.1}$$

with initial derived vector,

$$\mathbf{A}_t(0, \mathbf{z}) = \mathbf{z}. \tag{2.2}$$

If $|\mathbf{z}| \neq 0$ the solution (2.1) of (1.5) represents a geodesic g on H issuing from p with a tangent vector at p given by (z). In this context we regard (z) as a contravariant vector associated with the origin in the coordinate system (x), or equivalently with the point p in H represented by the origin.

The limits $b(\mathbf{z})$ and domain Ω. The limits $b(\mathbf{z})$ are positive, and the function $\mathbf{z} \to b(\mathbf{z})$, defined for every $\mathbf{z} \in \mathcal{E}_\mu$, is lower semicontinuous. This is readily verified. Let Ω be the union of the pairs $(t, \mathbf{z})$ such that

$$0 \leq t < b(\mathbf{z}). \tag{2.3}$$

The set Ω is an open subset of the product of $\mathcal{E}_\mu$ and the positive t-axis. Classical existence theorems on differential equations applied to (1.5) yield the following:

(i) *The function* $\mathbf{A}$ *is of class* C^∞ *over* Ω.

For $(t, \mathbf{z}) \in \Omega$ and $\mathbf{z}$ fixed, set $\mathbf{x}(t) = \mathbf{A}(t, \mathbf{z})$. With $k \geq 0$ set $\mathbf{x}(kt) = y(t)$. Then $t \to \mathbf{y}(t)$ is a solution of (1.5) for t in an appropriate maximal open subinterval J of the positive t-axis. The initial derived vector on this solution is $k\mathbf{z}$. Such a solution of (1.5) is given by the function $t \to \mathbf{A}(t, k\mathbf{z})$ for the given fixed k and $\mathbf{z}$, and for $t \in J$. Hence we have the fundamental relation

$$\mathbf{A}(t, k\mathbf{z}) = \mathbf{A}(kt, \mathbf{z}) \qquad (t, k\mathbf{z}) \in \Omega. \tag{2.4}$$

This relation leads to the following:

(ii) *For $k \geq 0$, the pair $(t, k\mathbf{z})$ is in Ω if and only if $(kt, \mathbf{z})$ is in Ω. If $(t, \mathbf{z})$ is in Ω then for $(0 \leq k \leq 1)$, $(kt, \mathbf{z})$ and $(t, k\mathbf{z})$ are in Ω.*

Let $\|\mathbf{z}\|$ be the Euclidean length of the vector $\mathbf{z}$ in the μ-plane $\mathcal{E}_\mu$. Since $g_{ij}(\mathbf{O}) = \delta_{ij}$ it follows that $f(\mathbf{O}, \mathbf{z}) = \|\mathbf{z}\|$ when $\|\mathbf{z}\| \neq 0$. When $\|\mathbf{z}\| = 0$, $f(\mathbf{O}, \mathbf{z})$ is not defined. The set of points $\mathbf{z} \in \mathcal{E}_\mu$ such that $\|\mathbf{z}\| = 1$, is compact. Since the function $\mathbf{z} \to b(\mathbf{z})$ is lower-semicotninuous one can affirm the existence of the

$$\min\ (b(\mathbf{z}) \mid \|\mathbf{z}\| = 1) = \eta, \qquad \text{(introducing } \eta) \tag{2.6}$$

The minimum η is positive. We shall prove (iii).

(iii) *The pairs* $(t, \mathbf{z})$ *of the form* $(1, \mathbf{z})$ *for which* $\|\mathbf{z}\| < \eta$ *are in* Ω.

Pairs $(1, \mathbf{z})$ for which $\|\mathbf{z}\| = 0$ are clearly in Ω. Suppose then that $0 < \|\mathbf{z}\| < \eta$. Note that

$$(1, \mathbf{z}) = \left(\frac{\|\mathbf{z}\|}{\|\mathbf{z}\|}, \mathbf{z}\right) \tag{2.7}$$

By (*ii*) the right-hand pair in (2.7) is in Ω if the pair

$$\left(\|\mathbf{z}\|, \frac{\mathbf{z}}{\|\mathbf{z}\|}\right) \tag{2.8}$$

is in Ω. But the norm of the vector in this pair is 1, and the numerical value of the first element is less than η. By definition of η the pair (2.8) is in Ω, and (iii) follows.

We shall presently use the following fact. If $\|\mathbf{a}\| \ddagger 0$ and if one sets $\mathbf{x}(t) = \mathbf{A}(t, \mathbf{a})$, then for $0 \leq t < b(\mathbf{a})$,

$$\frac{ds}{dt} = f(\mathbf{x}(t), \dot{\mathbf{x}}(t)) = f(\mathbf{x}(0), \dot{\mathbf{x}}(0)) = f(\mathbf{O}, \mathbf{a}) = \|\mathbf{a}\|. \tag{2.9}$$

§*3*. *A Geodesic Isometry*.—Let a be a positive constant and let Δ_a^z be the subset of points $\mathbf{z}$ in $\mathcal{E}_\mu$ such that $\|\mathbf{z}\| < a$. With η defined as in (2.6) we shall map Δ_η^z into the μ-plane E_μ of coordinates $\mathbf{x}$ by setting

$$\mathbf{A}(1, \mathbf{z}) = \mathbf{x} \qquad (\mathbf{z} \in \Delta_\eta^z) \tag{3.1}$$

or, in terms of the components A^i of $\mathbf{A}$, by setting

$$A^i(1, \mathbf{z}) = x^i \qquad (i = 1, \ldots, \mu) \tag{3.2}$$

The mapping (3.2) is well-defined for $\mathbf{z} \in \Delta_\eta^z$ in accord with (iii) of §2. We shall show that

$$\frac{\partial A^i}{\partial z^j}(1, \mathbf{z}) \,|(\|\mathbf{z}\| = 0) = \delta_j^i \qquad (i, j = 1, \ldots, \mu) \tag{3.3}$$

Proof of (3.3): If $0 \leq t \leq 1$ and $\|\mathbf{z}\| < \eta$ the pair $(1, t\mathbf{z})$ is in Ω according to (ii) and (iii), and

$$A^i(1, t\mathbf{z}) = A^i(t, \mathbf{z}) \qquad (i = 1, \ldots, \mu). \tag{3.4}$$

by virtue of (2.4). Differentiating the members of (3.4) with respect to z^j gives

$$\frac{\partial A^i}{\partial z^j}(1, t\mathbf{z}) = \frac{\partial A^i}{\partial z^j}(t, \mathbf{z}) \qquad (i, j = 1, \ldots, \mu). \tag{3.5}$$

Differentiating the members of (3.5) with respect to t and setting $t = 0$ gives

$$\frac{\partial A^i}{\partial z^j}(1, \mathbf{z}) \,|(\|\mathbf{z}\| = 0) = \frac{\partial^2 A^i}{\partial t \partial z^j}(t, \mathbf{z}) \,|(t = 0)$$

(by |(2.2)).

$$= \frac{\partial^2 A^i}{\partial z^j \partial t}(t, \mathbf{z}) \,|(t = 0) = \frac{\partial z^i}{\partial z^j} = \delta_j^i. \tag{3.6}$$

This establishes (3.3).

Let X denote the coordinate range Δ_ρ^x in E_μ. To simplify statement (*i*) which follows we regard X as a Riemannian manifold X^* with a differential form given by (1.2).

(i) *If* ϵ *is a sufficiently small positive constant the mapping*

$$\mathbf{z} \to \mathbf{A}(1, \mathbf{z}) \qquad (\mathbf{z} \in \Delta_\epsilon^z) \tag{3.7}$$

is a diffeomorphism of Δ_ϵ^z *into* X* *in which a ray* λ *issuing from the origin in* Δ_ϵ^z *is mapped isometrically onto a geodesic arc in* X* *issuing from the origin.*

Let $\epsilon < \eta$ be chosen as a positive number so small that the mapping (3.7) is a diffeomorphism. This is possible because of (3.3).

Let $\mathbf{a}$ now be a point in $\mathcal{E}_\mu$ such that $||\mathbf{a}|| = 1$. The ray λ may be given as a set of points

$$\{\mathbf{z} \,|\, \mathbf{z} = t\mathbf{a}, 0 \leq t < \epsilon\}.$$

The image of $\mathbf{z} = t\mathbf{a}$ under the diffeomorphism (3.7) is the point

$$\mathbf{x} = \mathbf{A}(1, t\mathbf{a}) = A(t, \mathbf{a}) \qquad (0 \leqq t < \epsilon)$$

This is a point on the geodesic arc γ in X^* with initial derived vector $\mathbf{a}$, and distance s, measured along γ from the origin, such that $s = t||\mathbf{a}|| = ||z||$. Cf. 2.9. This establishes (i)

Since the diffeomorphism φ in (1.1) may be regarded as an isometry of X^* onto H, the preceding statement (*i*) gives us the following. Cf. ref. 3, pp. 97–100.

LEMMA 3.1. *Corresponding to the point* $p \in H$ *and a sufficiently small positive constant* ϵ, *there exists a diffeomorphism*

$$\psi: \Delta_\epsilon^z \to H; \mathbf{z} \to \psi(\mathbf{z})$$

such that each ray λ *in* Δ_ϵ^z *issuing from the origin with* $0 = ||\mathbf{z}|| < \epsilon$ *on* λ *is mapped isometrically onto a geodesic arc issuing from* $p \in H$.

This lemma is preliminary to the principal theorem of §4. We *term the mapping* ψ *in the lemma a radial-geodesic isometry of* Δ_ϵ^z *onto a geodesic* μ*-disk in* H *with pole* p.

§*4*. *A 1-Parameter Family of* μ*-Manifolds* H_α.—We shall consider a 1-parameter family of μ-manifolds regularly embedded in a Riemannian n-manifold M of class C^∞ with $n = \mu + 1$. We suppose that the parameter α varies on an open interval I which includes $\alpha = 0$. Our family of μ-manifolds H_α is defined by a diffeomorphism

$$\Phi: \Delta_\rho^x \times I \to M; \mathbf{x} \times \alpha \to p(\mathbf{x}, \alpha), \tag{4.0}$$

where $p(\mathbf{x}, \alpha)$ is a point in M, and where, for fixed $\alpha \in I$, Φ reduces to a diffeomorphism,

$$\Phi_\alpha: \Delta_\rho^x \to M; \mathbf{x} \to p(\mathbf{x}, \alpha) \tag{4.1}$$

which serves to define H_α as the image of Δ_ρ^x under Φ_α. Since our principal lemma is local in character we can suppose without loss of generality that Φ is a map *onto* M. Thus M is given as a single coordinate domain. The sets $(x^1, \ldots, x^\mu, \alpha)$ in $\Delta_\rho^x \times$ I are thus local coordinates on M.

Let a Riemannian metric be assigned the manifolds H_α by restriction of the first fundamental form on M. Thus at the point $p(\mathbf{x}, \alpha) \in H_\alpha$

$$ds^2 = g_{ij}(\mathbf{x}, \alpha)dx^i dx^j \qquad (i, j = 1, \ldots, \mu). \tag{4.2}$$

where the coefficients g_{ij} are functions of class C^∞ on $\Delta_\rho^x \times I$, and for each $\alpha \in I$ the form (4.2) is positive definite.

A curve κ traversing the family $\{H_\alpha | \alpha \in I\} = \Lambda$. Let κ be a simple, regular arc of class C^∞ on M, meeting each manifold H_α in a single point p_α and not tangent to H_α at p_α. With κ so defined the principal theorem of this paper follows.

THEOREM 4.1. *Let κ be a regular curve of class C^∞ traversing Λ. Corresponding to a sufficiently small positive constant ϵ and a sufficiently small subinterval J of I containing $\alpha = 0$, there exists a diffeomorphism Ψ of $\Delta_\epsilon^z \times J$ into M which for fixed $\alpha \in J$ reduces to a radial-geodesic isometry Ψ_α of Δ_ϵ^z onto a geodesic μ-disk in H_α with pole $H_\alpha \cap \kappa$.*

Since the sets $(x^1, \ldots, x^\mu, \alpha)$ in $\Delta_\epsilon^z \times I$ are admissible coordinates on M and the arc κ is not tangent to any of the manifolds H_α, on which α is constant, κ has a representation $x^i = \varphi^i(\alpha): i = 1, \ldots, \mu$, of class C^∞ for $\alpha \in I$. Without loss of generality we can suppose that each of the functions φ^i vanishes. Were that not the case a change of local coordinates of the form

$$y^i = x^i - \varphi^i(\alpha) \qquad (i = 1, \ldots, \mu)$$

would bring it about. Such a change would call for a proper new choice of Δ_ρ^y and interval I. We shall suppose however that κ is represented by the interval I of the α-axis in the space $E_\mu \times I$.

The proof of the theorem is similar to the proof of Lemma 3.1. It requires supplementary consideration of how the mappings introduced in §§1–3 depend upon α.

Without loss of generality in the proof of the theorem we can suppose that

$$g_{ij}(\mathbf{O}, \alpha) = \delta_{ij} \qquad (i, j = 1, \ldots, \mu) \tag{4.3}$$

provided α is restricted to some sufficiently small neighborhood of $\alpha = 0$. In fact if (4.3) did not hold for some such neighborhood of $\alpha = 0$, then for some sufficiently small constant $\delta > 0$ it would be possible, for each $\alpha \in (-\delta, \delta)$, to make a nonsingular, linear transformation of coordinates

$$L_\alpha: y^i = b_{ij}(\alpha)x^j \tag{4.4}$$

such that the coefficients b_{ij} are functions of class C^∞ for $\alpha \in (-\delta, \delta)$ and the condition (4.3) holds for the transformed coefficients. The transformations (4.4) are readily defined, making use of a modified Lagrangian method of reduction of a quadratic form. The reader is referred to a proof of Lemma 10.1 in Ref. 4 for an application of such a method.

We assume then that (4.3) holds for $\alpha \in I$.

The differential equations which here replace (1.5), have the same form, except that $g_{ij}(\mathbf{x}, \alpha)$ replaces $g_{ij}(\mathbf{x})$. Limits $b(\mathbf{z}, \alpha)$ are here defined for each $\alpha \in I$ and $\mathbf{z} \in \mathcal{E}_\mu$, as was $b(\mathbf{z})$ in §2. The resulting function $(\mathbf{z}, \alpha) \to b(\mathbf{z}, \alpha)$ is lower semicontinuous on the product $\mathcal{E}_\mu \times I$.

Let Ω^* be the union of the triples $(t, \mathbf{z}, \alpha)$ for $0 \leq t < b(\mathbf{z}, \alpha)$ and $\alpha \in I$. The function $\mathbf{A}$ with values $\mathbf{A}(t, \mathbf{z}, \alpha)$ and domain of definition Ω^*, is defined for fixed $\alpha \in I$, as was the function $\mathbf{A}$ in §2. The new function is of class C^∞ over Ω^*, and the identity