



# **COLLECTED PAPERS**

**Marston Morse**

VOLUME 5



**World Scientific**

*Published by*

**World Scientific Publishing Co. Pte. Ltd.**  
**P. O. Box 128, Farrer Road, Singapore 9128**

The publisher would like to thank the publishers of the reprints for their permissions granted to include the papers found in this volume.

Library of Congress Cataloging-in-Publication data is available.

#### **COLLECTED PAPERS OF MARSTON MORSE**

Copyright © 1987 by World Scientific Publishing Co Pte Ltd.

*All rights reserved. This book, or parts thereof, may not be reproduced in any form or by any means, electronic or mechanical, including photocopying, recording or any information storage and retrieval system now known or to be invented, without written permission from the Publisher.*

ISBN 9971-978-94-6

Printed in Singapore by Fong and Sons Printers Pte. Ltd.

# **COLLECTED PAPERS**

**Marston Morse**



*With family at Princeton, 1968. From left, standing: William, Louise II, Peter, Julia, Dryden; seated: Elizabeth, Louise, Marston, Meroe.*

# **COLLECTED PAPERS**

**Marston Morse**

VOLUME 5

# TABLE OF CONTENTS

## Volume 5

[114]	Fields of geodesics issuing from a point, <i>Proc. Nat. Acad. Sci. U. S. A.</i> <b>46</b> (1960) 105–111 . . . . .	2377
[115]	The existence of polar non-degenerate functions on differentiable manifolds, <i>Ann. of Math.</i> <b>71</b> (1960) 352–383 . . . . .	2384
[116]	A reduction of the Schoenflies extension problem, <i>Bull. Amer. Math. Soc.</i> <b>66</b> (1960) 113–115 . . . . .	2416
[117]	The existence of non-degenerate functions on a compact differentiable $m$ -manifold $M$ , <i>Ann. Mat. Pura Appl.</i> <b>49</b> (1960) 117–128 . . . . .	2419
[118]	Recurrent geodesics on a surface of negative curvature, <i>Trans. Amer. Math. Soc.</i> <b>22</b> (1921) 84–110. (Multilithed with Historical Note, 1960.) . . . . .	2432
[119]	An explicit solution of the Schoenflies extension problem (with W. Huebsch), <i>J. Math. Soc. Japan</i> <b>12</b> (1960) 271–289 . . . . .	2451
[120]	A singularity in the Schoenflies extension (with W. Huebsch), <i>Proc. Nat. Acad. Sci. U. S. A.</i> <b>46</b> (1960) 1100–1102 . . . . .	2470
[121]	On elevating manifold differentiability, <i>J. Indian Math. Soc.</i> <b>24</b> (1960) 379–400 . . . . .	2473
[122]	The dependence of the Schoenflies extension on an accessory parameter (with W. Huebsch), <i>J. Analyse Math.</i> <b>8</b> (1960–61) 209–271 . . . . .	2495
[123]	Abstracts: A characterization of an analytic $n$ -ball, and A Schoenflies extension of a real analytic diffeomorphism of $S$ into $E$ (with W. Huebsch) (Multilithed, April 17, 1961, 1 p.) . . . . .	2558
[124]	Boundary values of partial derivatives of Poisson integral, <i>Ann. Acad. Brasil. Ci.</i> <b>33</b> (1961) 131–139. . . . .	2559
[125]	Conical singular points of diffeomorphisms (with W. Huebsch), <i>Bull. Amer. Math. Soc.</i> <b>67</b> (1961) 490–493 . . . . .	2568
[126]	Schoenflies extensions of analytic families of diffeomorphisms (with W. Huebsch), <i>Math. Ann.</i> <b>144</b> (1961) 162–174. . . . .	2572

- [127] The Schoenflies extension in the analytic case (with W. Huebsch),  
*Ann. Mat. Pura Appl.* **54** (1961) 359–378 . . . . . 2585
- [128] Schoenflies problems, *Fund. Math.* **50** (1962) 319–332 . . . . . 2606
- [129] Schoenflies extensions without interior differential singularities  
(with W. Huebsch), *Ann. of Math.* **76** (1962) 18–54 . . . . . 2620
- [130] Topological, differential, and analytic formulations of Schoenflies  
problems, *Rend. Circ. Mat. Palermo* **21** (1962) 286–295. (Lecture  
delivered at Rome, April 1962.) . . . . . 2658
- [131] Analytic diffeomorphisms approximating  $C^m$ -diffeomorphisms  
(with W. Huebsch), *Rend. Circ. Mat. Palermo* **11** (1962)  
25–46. . . . . 2668
- [132] Diffeomorphisms of manifolds (with W. Huebsch), *Rend. Circ.*  
*Mat. Palermo* **11** (1962) 291–318 . . . . . 2691
- [133] Schoenflies extensions and differentiable isotopies, *J. Math.*  
*Pures Appl.* **42** (1963) 29–41 . . . . . 2719
- [134] An arbitrarily small analytic mapping into  $R_+$  of a proper, regular,  
analytic  $r$ -manifold in  $E_m$ , *Rend. Sci. Inst. Lombardo, Milano (A)*  
**97** (1963) 650–660. . . . . 2732
- [135] Harmonic extensions, *Monatsh. Math.* **67** (1963) 317–325 . . . . . 2743
- [136] The dependence of the Schoenflies extension on an accessory  
parameter (the topological case) (with W. Huebsch), *Proc. Nat.*  
*Acad. Sci. U. S. A.* **50** (1963) 1036–1037 . . . . . 2752
- [137] The bowl theorem and a model nondegenerate function  
(with W. Huebsch), *Proc. Nat. Acad. Sci. U. S. A.* **51**  
(1964) 49–51. . . . . 2754
- [138] Conditioned differentiable isotopies (with W. Huebsch),  
*Differential Analysis Colloq., Bombay* (1964) 1–25. . . . . 2757
- [139] The elimination of critical points of a non-degenerate function  
on a differential manifold, *J. Analyse Math.* **13** (1964)  
257–316. . . . . 2782
- [140] Bowls,  $f$ -fibre-bundles and the alteration of critical values,  
*Ann. Acad. Brasil. Ci.* **36** (1964) 245–259. . . . . 2842



- [141] Bows of a non-degenerate function on a compact differentiable manifold, *Differential and Combinatorial Topology* (A Sympos. in honor of M. Morse), Princeton University Press, Princeton (1965) 81–103. . . . . 2858
- [142] Quadratic forms  $\Theta$  and  $\Theta$ -fibre-bundles, *Ann. of Math.* **81** (1965) 303–340. . . . . 2881
- [143] A model non-degenerate function (with W. Huebsch), *Rev. Roumaine Math. Pures Appl.* **10** (1965) 691–722 . . . . . 2919
- [144] The reduction of a function near a nondegenerate critical point, *Proc. Nat. Acad. Sci. U. S. A.* **54** (1965) 1759–1764 . . . . . 2951

# TABLE OF CONTENTS

## Volume 5

[114]	Fields of geodesics issuing from a point . . . . .	2377
[115]	The existence of polar non-degenerate functions on differentiable manifolds. . . . .	2384
[116]	A reduction of the Schoenflies extension problem . . . . .	2416
[117]	The existence of non-degenerate functions on a compact differentiable $m$ -manifold $M$ . . . . .	2419
[118]	Recurrent geodesics on a surface of negative curvature . . . . .	2432
[119]	An explicit solution of the Schoenflies extension problem (with W. Huebsch) . . . . .	2451
[120]	A singularity in the Schoenflies extension (with W. Huebsch) . . . . .	2470
[121]	On elevating manifold differentiability . . . . .	2473
[122]	The dependence of the Schoenflies extension on an accessory parameter (with W. Huebsch) . . . . .	2495
[123]	Abstracts: A characterization of an analytic $n$ -ball, and A Schoenflies extension of a real analytic diffeomorphism of $S$ into $E$ (with W. Huebsch) . . . . .	2558
[124]	Boundary values of partial derivatives of Poisson integral . . . . .	2559
[125]	Conical singular points of diffeomorphisms (with W. Huebsch) . . . . .	2568
[126]	Schoenflies extensions of analytic families of diffeomorphisms (with W. Huebsch) . . . . .	2572
[127]	The Schoenflies extension in the analytic case (with W. Huebsch) . . . . .	2585
[128]	Schoenflies problems . . . . .	2606
[129]	Schoenflies extensions without interior differential singularities (with W. Huebsch) . . . . .	2620

[130]	Topological, differential, and analytic formulations of Schoenflies problems . . . . .	2658
[131]	Analytic diffeomorphisms approximating $C^m$ -diffeomorphisms (with W. Huebsch) . . . . .	2668
[132]	Diffeomorphisms of manifolds (with W. Huebsch) . . . . .	2691
[133]	Schoenflies extensions and differentiable isotopies . . . . .	2719
[134]	An arbitrarily small analytic mapping into $R_+$ of a proper, regular, analytic $r$ -manifold in $E_m$ . . . . .	2732
[135]	Harmonic extensions . . . . .	2743
[136]	The dependence of the Schoenflies extension on an accessory parameter (the topological case) (with W. Huebsch) . . . . .	2752
[137]	The bowl theorem and a model nondegenerate function (with W. Huebsch) . . . . .	2754
[138]	Conditioned differentiable isotopies (with W. Huebsch) . . . . .	2757
[139]	The elimination of critical points of a non-degenerate function on a differential manifold . . . . .	2782
[140]	Bowls, $f$ -fibre-bundles and the alteration of critical values . . . . .	2842
[141]	Bowls of a non-degenerate function on a compact differentiable manifold . . . . .	2858
[142]	Quadratic forms $\Theta$ and $\Theta$ -fibre-bundles . . . . .	2881
[143]	A model non-degenerate function (with W. Huebsch) . . . . .	2919
[144]	The reduction of a function near a nondegenerate critical point . . . . .	2951

## FIELDS OF GEODESICS ISSUING FROM A POINT

BY MARSTON MORSE

THE INSTITUTE FOR ADVANCED STUDY

*Communicated November 6, 1959*

§1. *Introduction.*—We shall be concerned with a 1-parameter family of open Riemannian  $\mu$ -manifolds  $H_\alpha$  of class  $C^\infty$ , where the parameter  $\alpha$  varies on an open interval  $I$ . The properties of fields of geodesics issuing from points  $p_\alpha$  of the respective manifolds  $H_\alpha$  are not fully stated or derived in mathematical literature known to us, at least for the ends we have in mind. This note is designed to fill these needs and will find an application in reference 1.

To simplify the exposition we begin with the more familiar case of a single Riemannian  $\mu$ -manifold  $H$ ,  $\mu > 1$ , leaving to §4 the main theorem of the paper. Let  $p$  be an arbitrary fixed point in  $H$ . Let  $(x^1, \dots, x^\mu)$  be rectangular coordinates in a Euclidean  $\mu$ -space  $E_\mu$ . Set

$$(x^1, \dots, x^\mu) = \mathbf{x}, \|\mathbf{x}\| = (x^i x^i)^{1/2} \quad (i = 1, \dots, \mu)$$

employing the summation convention of tensor algebra. We introduce the  $\mu$ -disk

$$\Delta_\rho^x = \{\mathbf{x} \mid \|\mathbf{x}\| < \rho\} \quad (1.0)$$

where  $\rho$  is any positive constant.

Let  $W$  be an open coordinate domain on  $H$  which contains  $p$ . We suppose that there exists a diffeomorphism

$$\varphi: \Delta_\rho^x \rightarrow W: \mathbf{x} \rightarrow \varphi(\mathbf{x}) \quad (1.1)$$

of  $\Delta_\rho^x$  onto  $W$  by virtue of which  $\Delta_\rho^x$  becomes (ref. 2, §0) the range of the local coordinates  $(x)$ . The term "diffeomorphism" in this paper will always mean a diffeomorphism of class  $C^\infty$ . Since  $H$  is supposed to be a Riemannian manifold we include the condition that at the point on  $W$  with local coordinates  $(x)$

$$ds^2 = g_{ij}(\mathbf{x}) dx^i dx^j \quad (i, j = 1, \dots, \mu), \quad (1.2)$$

where the coefficients  $g_{ij}$  are functions of class  $C^\infty$  for  $\mathbf{x} \in \Delta_\rho^x$ , and the invariant quadratic form is positive definite. As far as this paper is concerned  $H$  may be supposed identified with  $W$ . The compatibility of the above representation  $\varphi$  of  $W$  with other representations of open subdomains of  $W$  is presupposed in the sense usual for Riemannian manifolds of class  $C^\infty$ .

We suppose that under the diffeomorphism  $\varphi$ , the point  $p$  is represented by the origin in  $E_\mu$ . We shall also suppose that  $g_{ij}(\mathbf{O}) = \delta_{ij}$ . If this condition were not satisfied, it would be satisfied for a new set of local coordinates on a neighborhood of  $p$  in  $H$ , on making a suitable non-singular linear transformation of the initial coordinates  $(x)$ .

Understanding that  $(r^1, \dots, r^\mu) = \mathbf{r}$  is a contravariant vector associated with the point  $(\mathbf{x})$ , set

$$f(\mathbf{x}, \mathbf{r}) = [g_{ij}(\mathbf{x}) r^i r^j]^{1/2}. \quad (1.3)$$

A geodesic  $g$  on  $W$  is represented on  $\Delta_\rho^x$  by a regular curve

$$t \rightarrow \mathbf{x}(t) = (x^1(t), \dots, x^\mu(t)) \quad (t \in J) \quad (1.4)$$

of class  $C^2$  (at least), satisfying the Euler equations associated with the integral,

$$\int f(\mathbf{x}, \dot{\mathbf{x}}) dt.$$

These Euler equations do not uniquely determine the parameterization of  $g$ . We shall admit only those representations of geodesics in which the parameter  $t = cs$ , where  $s$  is the arc length measured along the geodesic in the sense of increasing  $t$ , and  $c$  is a positive constant. We shall limit our study to geodesics issuing from  $p \in H$  and shall measure  $s$  from  $p$ . It is easy to verify the fact that a regular curve, of class  $C^2$ , of the form (1.4), represents a geodesic and is of class  $C^\infty$ , with  $t = cs$ , if and only if it is a non-null solution of the system of  $\mu$  differential equations,

$$2 \frac{d}{dt} (g_{hj}(\mathbf{x}) \dot{x}^j) = \frac{\partial g_{ij}}{\partial x^h} (\mathbf{x}) \dot{x}^i \dot{x}^j \quad (h, i, j = 1, \dots, \mu). \quad (1.5)$$

On any solution of (1.5)

$$\frac{ds}{dt} = f(\mathbf{x}, \dot{\mathbf{x}}) = c, \quad (1.6)$$

where  $c$  is a constant. Moreover  $c > 0$  if the solution represents a geodesic, and  $c = 0$  if the solution reduces to a point.

§2. *Solutions of (1.5) Vanishing at the Origin.*—Let  $S$  be the class of solutions  $t \rightarrow \mathbf{x}(t)$  of (1.5) such that  $\mathbf{x}(t)$  is in  $\Delta_p^x$  and reduces to the null vector at  $t = 0$ . In  $S$  we include the trivial solution such that  $\|\mathbf{x}(t)\| = 0$  for all  $t$ . Since the determinant  $|g_{hj}(\mathbf{x})| \neq 0$  for  $\mathbf{x} \in \Delta_p$  the following is true. Corresponding to an arbitrary point  $(z^1, \dots, z^\mu) = \mathbf{z}$  in a  $\mu$ -plane  $\mathcal{E}_\mu$  there exists a maximum positive constant  $b(\mathbf{z})$ , possibly infinite, such that there is a solution of (1.5) in the class  $S$  of the form, for fixed  $(z)$ ,

$$t \rightarrow \mathbf{A}(t, \mathbf{z}) \quad (0 \leq t < b(\mathbf{z})) \quad (2.1)$$

with initial derived vector,

$$\mathbf{A}_t(0, \mathbf{z}) = \mathbf{z}. \quad (2.2)$$

If  $\|\mathbf{z}\| \neq 0$  the solution (2.1) of (1.5) represents a geodesic  $g$  on  $H$  issuing from  $p$  with a tangent vector at  $p$  given by  $(z)$ . In this context we regard  $(z)$  as a contravariant vector associated with the origin in the coordinate system  $(x)$ , or equivalently with the point  $p$  in  $H$  represented by the origin.

*The limits  $b(\mathbf{z})$  and domain  $\Omega$ .* The limits  $b(\mathbf{z})$  are positive, and the function  $\mathbf{z} \rightarrow b(\mathbf{z})$ , defined for every  $\mathbf{z} \in \mathcal{E}_\mu$ , is lower semicontinuous. This is readily verified. Let  $\Omega$  be the union of the pairs  $(t, \mathbf{z})$  such that

$$0 \leq t < b(\mathbf{z}). \quad (2.3)$$

The set  $\Omega$  is an open subset of the product of  $\mathcal{E}_\mu$  and the positive  $t$ -axis. Classical existence theorems on differential equations applied to (1.5) yield the following:

(i) *The function  $\mathbf{A}$  is of class  $C^\infty$  over  $\Omega$ .*

For  $(t, \mathbf{z}) \in \Omega$  and  $\mathbf{z}$  fixed, set  $\mathbf{x}(t) = \mathbf{A}(t, \mathbf{z})$ . With  $k \geq 0$  set  $\mathbf{x}(kt) = \mathbf{y}(t)$ . Then  $t \rightarrow \mathbf{y}(t)$  is a solution of (1.5) for  $t$  in an appropriate maximal open subinterval  $J$  of the positive  $t$ -axis. The initial derived vector on this solution is  $k\mathbf{z}$ . Such a solution of (1.5) is given by the function  $t \rightarrow \mathbf{A}(t, k\mathbf{z})$  for the given fixed  $k$  and  $\mathbf{z}$ , and for  $t \in J$ . Hence we have the fundamental relation

$$\mathbf{A}(t, k\mathbf{z}) = \mathbf{A}(kt, \mathbf{z}) \quad (t, k\mathbf{z}) \in \Omega. \quad (2.4)$$

This relation leads to the following:

(ii) *For  $k \geq 0$ , the pair  $(t, k\mathbf{z})$  is in  $\Omega$  if and only if  $(kt, \mathbf{z})$  is in  $\Omega$ . If  $(t, \mathbf{z})$  is in  $\Omega$  then for  $(0 \leq k \leq 1)$ ,  $(kt, \mathbf{z})$  and  $(t, k\mathbf{z})$  are in  $\Omega$ .*

Let  $\|\mathbf{z}\|$  be the Euclidean length of the vector  $\mathbf{z}$  in the  $\mu$ -plane  $\mathcal{E}_\mu$ . Since  $g_{ij}(\mathbf{O}) = \delta_{ij}$  it follows that  $f(\mathbf{O}, \mathbf{z}) = \|\mathbf{z}\|$  when  $\|\mathbf{z}\| \neq 0$ . When  $\|\mathbf{z}\| = 0$ ,  $f(\mathbf{O}, \mathbf{z})$  is not defined. The set of points  $\mathbf{z} \in \mathcal{E}_\mu$  such that  $\|\mathbf{z}\| = 1$ , is compact. Since the function  $\mathbf{z} \rightarrow b(\mathbf{z})$  is lower-semicontinuous one can affirm the existence of the

$$\min (b(\mathbf{z}) \mid \|\mathbf{z}\| = 1) = \eta, \quad (\text{introducing } \eta) \quad (2.6)$$

The minimum  $\eta$  is positive. We shall prove (iii).

(iii) The pairs  $(t, \mathbf{z})$  of the form  $(1, \mathbf{z})$  for which  $\|\mathbf{z}\| < \eta$  are in  $\Omega$ .

Pairs  $(1, \mathbf{z})$  for which  $\|\mathbf{z}\| = 0$  are clearly in  $\Omega$ . Suppose then that  $0 < \|\mathbf{z}\| < \eta$ . Note that

$$(1, \mathbf{z}) = \left( \frac{\|\mathbf{z}\|}{\|\mathbf{z}\|}, \mathbf{z} \right) \quad (2.7)$$

By (ii) the right-hand pair in (2.7) is in  $\Omega$  if the pair

$$\left( \|\mathbf{z}\|, \frac{\mathbf{z}}{\|\mathbf{z}\|} \right) \quad (2.8)$$

is in  $\Omega$ . But the norm of the vector in this pair is 1, and the numerical value of the first element is less than  $\eta$ . By definition of  $\eta$  the pair (2.8) is in  $\Omega$ , and (iii) follows.

We shall presently use the following fact. If  $\|\mathbf{a}\| \neq 0$  and if one sets  $\mathbf{x}(t) = \mathbf{A}(t, \mathbf{a})$ , then for  $0 \leq t \leq b(\mathbf{a})$ ,

$$\frac{ds}{dt} = f(\mathbf{x}(t), \dot{\mathbf{x}}(t)) = f(\mathbf{x}(0), \dot{\mathbf{x}}(0)) = f(\mathbf{O}, \mathbf{a}) = \|\mathbf{a}\|. \quad (2.9)$$

§3. *A Geodesic Isometry.*—Let  $a$  be a positive constant and let  $\Delta_a^*$  be the subset of points  $\mathbf{z}$  in  $\mathcal{E}_\mu$  such that  $\|\mathbf{z}\| < a$ . With  $\eta$  defined as in (2.6) we shall map  $\Delta_\eta^*$  into the  $\mu$ -plane  $E_\mu$  of coordinates  $\mathbf{x}$  by setting

$$\mathbf{A}(1, \mathbf{z}) = \mathbf{x} \quad (\mathbf{z} \in \Delta_\eta^*) \quad (3.1)$$

or, in terms of the components  $A^i$  of  $\mathbf{A}$ , by setting

$$A^i(1, \mathbf{z}) = x^i \quad (i = 1, \dots, \mu) \quad (3.2)$$

The mapping (3.2) is well-defined for  $\mathbf{z} \in \Delta_\eta^*$  in accord with (iii) of §2. We shall show that

$$\frac{\partial A^i}{\partial z^j}(1, \mathbf{z}) \big|_{(\|\mathbf{z}\| = 0)} = \delta_j^i \quad (i, j = 1, \dots, \mu) \quad (3.3)$$

*Proof of (3.3):* If  $0 \leq t \leq 1$  and  $\|\mathbf{z}\| < \eta$  the pair  $(1, t\mathbf{z})$  is in  $\Omega$  according to (ii) and (iii), and

$$A^i(1, t\mathbf{z}) = A^i(t, \mathbf{z}) \quad (i = 1, \dots, \mu). \quad (3.4)$$

by virtue of (2.4). Differentiating the members of (3.4) with respect to  $z^j$  gives

$$\frac{\partial A^i}{\partial z^j}(1, t\mathbf{z}) = \frac{\partial A^i}{\partial z^j}(t, \mathbf{z}) \quad (i, j = 1, \dots, \mu). \quad (3.5)$$

Differentiating the members of (3.5) with respect to  $t$  and setting  $t = 0$  gives

$$\begin{aligned} \frac{\partial A^i}{\partial z^j}(1, \mathbf{z}) \big|_{(\|\mathbf{z}\| = 0)} &= \frac{\partial^2 A^i}{\partial t \partial z^j}(t, \mathbf{z}) \big|_{(t = 0)} \\ (\text{by (2.2)}). &= \frac{\partial^2 A^i}{\partial z^j \partial t}(t, \mathbf{z}) \big|_{(t = 0)} = \frac{\partial z^i}{\partial z^j} = \delta_j^i. \end{aligned} \quad (3.6)$$

This establishes (3.3).

Let  $X$  denote the coordinate range  $\Delta_p^z$  in  $E_\mu$ . To simplify statement (i) which follows we regard  $X$  as a Riemannian manifold  $X^*$  with a differential form given by (1.2).

(i) If  $\epsilon$  is a sufficiently small positive constant the mapping

$$z \rightarrow A(1, z) \quad (z \in \Delta_\epsilon^z) \quad (3.7)$$

is a diffeomorphism of  $\Delta_\epsilon^z$  into  $X^*$  in which a ray  $\lambda$  issuing from the origin in  $\Delta_\epsilon^z$  is mapped isometrically onto a geodesic arc in  $X^*$  issuing from the origin.

Let  $\epsilon < \eta$  be chosen as a positive number so small that the mapping (3.7) is a diffeomorphism. This is possible because of (3.3).

Let  $\mathbf{a}$  now be a point in  $\mathcal{E}_\mu$  such that  $\|\mathbf{a}\| = 1$ . The ray  $\lambda$  may be given as a set of points

$$\{z \mid z = t\mathbf{a}, 0 \leq t < \epsilon\}.$$

The image of  $z = t\mathbf{a}$  under the diffeomorphism (3.7) is the point

$$\mathbf{x} = A(1, t\mathbf{a}) = A(t, \mathbf{a}) \quad (0 \leq t < \epsilon)$$

This is a point on the geodesic arc  $\gamma$  in  $X^*$  with initial derived vector  $\mathbf{a}$ , and distance  $s$ , measured along  $\gamma$  from the origin, such that  $s = t\|\mathbf{a}\| = \|z\|$ . Cf. 2.9. This establishes (i).

Since the diffeomorphism  $\varphi$  in (1.1) may be regarded as an isometry of  $X^*$  onto  $H$ , the preceding statement (i) gives us the following. Cf. ref. 3, pp. 97-100.

LEMMA 3.1. Corresponding to the point  $p \in H$  and a sufficiently small positive constant  $\epsilon$ , there exists a diffeomorphism

$$\psi: \Delta_\epsilon^z \rightarrow H; z \rightarrow \psi(z)$$

such that each ray  $\lambda$  in  $\Delta_\epsilon^z$  issuing from the origin with  $0 = \|z\| < \epsilon$  on  $\lambda$  is mapped isometrically onto a geodesic arc issuing from  $p \in H$ .

This lemma is preliminary to the principal theorem of §4. We term the mapping  $\psi$  in the lemma a radial-geodesic isometry of  $\Delta_\epsilon^z$  onto a geodesic  $\mu$ -disk in  $H$  with pole  $p$ .

§4. A 1-Parameter Family of  $\mu$ -Manifolds  $H_\alpha$ .—We shall consider a 1-parameter family of  $\mu$ -manifolds regularly embedded in a Riemannian  $n$ -manifold  $M$  of class  $C^\infty$  with  $n = \mu + 1$ . We suppose that the parameter  $\alpha$  varies on an open interval  $I$  which includes  $\alpha = 0$ . Our family of  $\mu$ -manifolds  $H_\alpha$  is defined by a diffeomorphism

$$\Phi: \Delta_p^x \times I \rightarrow M; \mathbf{x} \times \alpha \rightarrow p(\mathbf{x}, \alpha), \quad (4.0)$$

where  $p(\mathbf{x}, \alpha)$  is a point in  $M$ , and where, for fixed  $\alpha \in I$ ,  $\Phi$  reduces to a diffeomorphism,

$$\Phi_\alpha: \Delta_p^x \rightarrow M; \mathbf{x} \rightarrow p(\mathbf{x}, \alpha) \quad (4.1)$$

which serves to define  $H_\alpha$  as the image of  $\Delta_p^x$  under  $\Phi_\alpha$ . Since our principal lemma is local in character we can suppose without loss of generality that  $\Phi$  is a map onto  $M$ . Thus  $M$  is given as a single coordinate domain. The sets  $(x^1, \dots, x^\mu, \alpha)$  in  $\Delta_p^x \times I$  are thus local coordinates on  $M$ .

Let a Riemannian metric be assigned the manifolds  $H_\alpha$  by restriction of the first fundamental form on  $M$ . Thus at the point  $p(\mathbf{x}, \alpha) \in H_\alpha$



$$ds^2 = g_{ij}(\mathbf{x}, \alpha) dx^i dx^j \quad (i, j = 1, \dots, \mu). \quad (4.2)$$

where the coefficients  $g_{ij}$  are functions of class  $C^\infty$  on  $\Delta_p^x \times I$ , and for each  $\alpha \in I$  the form (4.2) is positive definite.

A curve  $\kappa$  traversing the family  $\{H_\alpha | \alpha \in I\} = \Lambda$ . Let  $\kappa$  be a simple, regular arc of class  $C^\infty$  on  $M$ , meeting each manifold  $H_\alpha$  in a single point  $p_\alpha$  and not tangent to  $H_\alpha$  at  $p_\alpha$ . With  $\kappa$  so defined the principal theorem of this paper follows.

**THEOREM 4.1.** *Let  $\kappa$  be a regular curve of class  $C^\infty$  traversing  $\Lambda$ . Corresponding to a sufficiently small positive constant  $\epsilon$  and a sufficiently small subinterval  $J$  of  $I$  containing  $\alpha = 0$ , there exists a diffeomorphism  $\Psi$  of  $\Delta_\epsilon^z \times J$  into  $M$  which for fixed  $\alpha \in J$  reduces to a radial-geodesic isometry  $\Psi_\alpha$  of  $\Delta_\epsilon^z$  onto a geodesic  $\mu$ -disk in  $H_\alpha$  with pole  $H_\alpha \cap \kappa$ .*

Since the sets  $(x^1, \dots, x^\mu, \alpha)$  in  $\Delta_\epsilon^z \times I$  are admissible coordinates on  $M$  and the arc  $\kappa$  is not tangent to any of the manifolds  $H_\alpha$ , on which  $\alpha$  is constant,  $\kappa$  has a representation  $x^i = \varphi^i(\alpha)$ ,  $i = 1, \dots, \mu$ , of class  $C^\infty$  for  $\alpha \in I$ . Without loss of generality we can suppose that each of the functions  $\varphi^i$  vanishes. Were that not the case a change of local coordinates of the form

$$y^i = x^i - \varphi^i(\alpha) \quad (i = 1, \dots, \mu)$$

would bring it about. Such a change would call for a proper new choice of  $\Delta_p^y$  and interval  $I$ . We shall suppose however that  $\kappa$  is represented by the interval  $I$  of the  $\alpha$ -axis in the space  $E_\mu \times I$ .

The proof of the theorem is similar to the proof of Lemma 3.1. It requires supplementary consideration of how the mappings introduced in §§1-3 depend upon  $\alpha$ .

Without loss of generality in the proof of the theorem we can suppose that

$$g_{ij}(\mathbf{0}, \alpha) = \delta_{ij} \quad (i, j = 1, \dots, \mu) \quad (4.3)$$

provided  $\alpha$  is restricted to some sufficiently small neighborhood of  $\alpha = 0$ . In fact if (4.3) did not hold for some such neighborhood of  $\alpha = 0$ , then for some sufficiently small constant  $\delta > 0$  it would be possible, for each  $\alpha \in (-\delta, \delta)$ , to make a non-singular, linear transformation of coordinates

$$L_\alpha: y^i = b_{ij}(\alpha) x^j \quad (4.4)$$

such that the coefficients  $b_{ij}$  are functions of class  $C^\infty$  for  $\alpha \in (-\delta, \delta)$  and the condition (4.3) holds for the transformed coefficients. The transformations (4.4) are readily defined, making use of a modified Lagrangian method of reduction of a quadratic form. The reader is referred to a proof of Lemma 10.1 in Ref. 4 for an application of such a method.

We assume then that (4.3) holds for  $\alpha \in I$ .

The differential equations which here replace (1.5), have the same form, except that  $g_{ij}(\mathbf{x}, \alpha)$  replaces  $g_{ij}(\mathbf{x})$ . Limits  $b(\mathbf{z}, \alpha)$  are here defined for each  $\alpha \in I$  and  $\mathbf{z} \in \mathcal{E}_\mu$ , as was  $b(\mathbf{z})$  in §2. The resulting function  $(\mathbf{z}, \alpha) \rightarrow b(\mathbf{z}, \alpha)$  is lower semi-continuous on the product  $\mathcal{E}_\mu \times I$ .

Let  $\Omega^*$  be the union of the triples  $(t, \mathbf{z}, \alpha)$  for  $0 \leq t < b(\mathbf{z}, \alpha)$  and  $\alpha \in I$ . The function  $\mathbf{A}$  with values  $\mathbf{A}(t, \mathbf{z}, \alpha)$  and domain of definition  $\Omega^*$ , is defined for fixed  $\alpha \in I$ , as was the function  $\mathbf{A}$  in §2. The new function is of class  $C^\infty$  over  $\Omega^*$ , and the identity