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Edited by
TARUN KANT

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IN
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PREFACE

The computer-based finite element method has now emerged as a unified procedure for numerical solution of physical problems in high technology engineering. It has gained acceptance as a powerful analysis and design aid in diverse areas due to its versatility and adaptability. In order to consolidate the recent advances in this area of activity and to provide an international forum for interaction between the active researchers of east and west, the idea of FEICOM-85 was born sometime in the beginning of 1983.

This book contains the texts of contributed papers and most of the invited lectures. All the authors prepared their manuscripts in camera-ready form. However, in some cases a good amount of editing and retyping had to be undertaken for clarity sake. The editor however does not accept responsibility for comments and opinions expressed in these papers.

Tarun Kant

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GENERAL

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ADAPTIVE FINITE ELEMENT METHODS FOR COMPLEX PROBLEMS IN SOLID AND FLUID MECHANICS

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ABSTRACT

This paper addresses the general topic of adaptive methods for automatically enhancing the quality of numerical solutions to linear and nonlinear boundary-value problems in solid and fluid mechanics, and reviews some of the recent work of the author and his collaborators on this subject.

KEYWORDS

Adaptive Finite Element Methods, A Posteriori Error Estimates.

INTRODUCTION

The basic objective of an adaptive finite element method is to improve the quality of an initial finite element approximation by automatically changing the model: refining the mesh, moving mesh nodal points, enriching the local order of approximation, etc. Thus, all adaptive methods must attempt to resolve two basic issues: 1) how is the quality of the approximate solution to be measured? and 2) how does one adapt the model to improve the quality of the approximation?

The first question is generally resolved by attempting to measure the local approximation error in some appropriate norm. The error, of course, is the difference between the exact solution u and a finite element approximation u_h of u on a given mesh. Since u is not known, the problem of assessing the quality of an approximation reduces to one of a-posteriori error estimation: the determination of estimates of the error using computed finite element solutions. A number of important papers on various schemes for a-posteriori error estimation has been contributed by Babuska and his collaborators (1978a, 1978b, 1984).

Once an estimate of the distribution of the error is available, the difficult question of how to best modify the model to improve accuracy arises. There are three general approaches:

h-Methods. Here the mesh is refined; the mesh size h is reduced and the number of elements in the mesh is increased in regions of large error.

p-Methods. Here the mesh is fixed but the degree p of the polynomial shape functions is increased over elements in which a high error is indicated.

Moving Mesh Methods. In these methods, the number of nodes and the type of finite element remains constant during the adaptive process and the nodal points are moved to regions of high error.

Of course, one can also employ combinations of these strategies. But the correct strategy for use of combined methods is apparently a delicate issue and one in which much additional study needs to be done.

We shall describe here two methods for error estimation and show how these can be implemented in each of the three adaptive schemes listed above.

A-POSTERIORI ERROR ESTIMATES

We describe two classes of a-posteriori error estimation, one based on the computation of element residuals and the other based on interpolation error estimates. The former class of methods was introduced by Demkowicz, Oden, and Strouboulis (1984) and includes several of the results of Babuska and colleagues (1978a, 1978b, 1983), and the latter was first advocated by Diaz, Kikuchi, and Taylor (1983) and investigated by Demkowicz and Oden (1985a, 1985b, 1985c).

Residual Methods

Consider the abstract boundary-value problem, Find u in V such that

$$\langle Au, v\rangle = \langle f, v\rangle \quad \text{for all } v \text{ in } V \tag{1}$$

where

- A = a (possibly nonlinear) operator from a reflexive Banach space of admissible functions V into its dual V^*
- v = an arbitrary test function in V
- f = given data in V^*
- $\langle \cdot, \cdot \rangle$ = duality pairing on $V^* \times V$

This problem is equivalent to the abstract problem: $Au = f$ in V^*.

A Galerkin approximation of (1) consists of seeking a function u_h in a finite dimensional subspace V_h of V such that

$$\langle Au_h, v_h\rangle = \langle f, v_h\rangle \quad \text{for all } v_h \text{ in } V_h \tag{2}$$

The *residual* r_h is the degree with which the approximation u_h fails to satisfy the original conditions on the solution:

$$r_h = Au_h - f \neq 0, \qquad r_h \in V^* \tag{3}$$

Since the residual belongs to the dual space V^* and not necessarily V, its magnitude must be measured with respect to the norm $\|\cdot\|_*$ on V^*:

$$\| r_h \|_* = \sup_{v \in V} \frac{\langle r_h, v \rangle}{\|v\|} = \sup_{\|v\| \leq 1} \langle r_h, v \rangle \tag{4}$$

where $\| \cdot \|$ is the norm in V

In some of the error estimators that we have developed, we use the following procedure to approximate the supremum in (4):

- The original finite element approximation u_h is computed in a space $V_h = V_h^1$ of spanned by low-order (say, linear) piecewise polynomial shape functions, resulting in the residual r_h^1.

- The full space V is approximated by a higher-order finite element space V_h^p, $p > 1$, spanned by piecewise polynomials of degree p.

An approximation of the residual r_h^1 is constructed according to

$$\| r_h^1 \|_* \leq C \| v_0 - v_h^p \| + \sup_{\| v_h^p \| \leq 1} \langle r_h^1, v_h^p \rangle \tag{5}$$

where C is a constant, v_0 is an element of V realizing the sup in (4) and v_h^p is an arbitrary element of V_h^p. If h is the mesh size (i.e. for a partition T_h of elements K,

$$h = \max_{K \in T_h} h_K, \quad h_K = \text{diameter } (K)$$

we generally have

$$\| v_p - v_h^p \| = O(h^p)$$

so that it makes sense asymptotically (as $h \to 0$) to approximate $\sup \langle r_h^1, v \rangle$ by $\sup \langle r_h^1, v_h^p \rangle$.

As an example of how this procedure can be implemented, consider the model problem,

$$\text{Find } u \text{ in } V = \{ v \in H^1(\Omega) ; v = 0 \text{ on } \Gamma_1 \} \text{ such that}$$

$$\int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega fv \, dx + \int_{\Gamma_2} gv \, ds \quad \text{for all } v \text{ in } V \tag{6}$$

This is the variational form of the model Poisson problem,

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega \subset \mathbb{R}^2 \\ u &= 0 \text{ on } \Gamma_1 \subset \partial\Omega \\ \frac{\partial u}{\partial n} &= g \text{ on } \Gamma_2 \subset \partial\Omega \end{aligned} \tag{7}$$

with $\Delta = \Delta^2$, $H^1(\Omega)$ the usual Sobolev space of functions with derivatives in $L^2(\Omega)$, and $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$.

We define

$$V_h = \{v_h \in V : v_h|_K \in Q_1(K) \ , \ \overline{\Omega} = \cup K\}$$

$$V_h^p(K) = \{v_h^p \in P_p(K)\} \tag{8}$$

$$V_{h0}^p(K) = \{v_h^p \in V_h^p(K) \ ; \ p > 1 \ , \ v_h^1 \text{ interpolant } v_h^p = 0\}$$

where $Q_1(K)$ is the usual set of bilinear functions defined on a quadrilateral element K and $P_p(K)$ is the space of polynomials of degree p defined on K.

The residual r_h^1 satisfies

$$\langle r_h^1, v_h^p\rangle = \sum_K \{\int_K (-\Delta u_h^1 - f)v_h^p \, dx + \int_{\partial K \setminus \partial\Omega} \tfrac{1}{2}\left(\frac{\partial u_h^1}{\partial n} - \frac{\partial u_{*h}^1}{\partial n}\right) v_h^p \, ds + \int_{\partial K \cap \Gamma_2} \left(\frac{\partial u_h^1}{\partial n} - g\right) v_h^p \, ds\} = \sum_K \langle R_K^{1h}, v_h^p\rangle \tag{9}$$

where u_h^1 is the (coarse-grid-initial) finite element approximation of u determined using the space V_h and R_K^{1h} is the functional on V_h^p defined as indicated, with $\partial u_{*h}^1/\partial n$ an approximation of $\partial u/\partial n$ obtained from an adjacent element to K.

It is not enough to simply calculate R_K^{1h} as an indicator of the error in element K. In general, we wish to have an indicator ϕ_K of the error which will bound the local error above and below and which will converge to zero at the same rate as the actual error; e.g.

$$C\|\phi_K\|_{1,K} \leq \|\text{error}\|_{1,K} \leq \|\phi_K\|_{1,K}$$

$$\|\phi_K\|_{1,K}^2 = \int_K \nabla\phi_K \cdot \nabla\phi_K \, dx \tag{10}$$

such an error indicator is obtained as a solution of the auxiliary problem,

$$\int_K \nabla\phi_K \cdot \nabla v_h^p \, dx = \langle R_K^{1h}, v_h^p\rangle \text{ for all } v_h^p \text{ in } V_{h0}^p(K) \tag{11}$$

We generally compute the solution of (11) using the concept of hierarchic elements in which the stiffness matrices are only modified by the addition of a row and column with the addition of each degree of freedom (see, e.g., Carey and Oden, 1981 for details). Using (11), (9), and (5), we have (to within terms of $O(h^p)$)

$$\| r_h^1 \|_* \leq \{ C \sum_K \int_K |\nabla \phi_K|^2 \, dx \}^{\frac{1}{2}} \tag{12}$$

where C is a (hopefully) known constant. Though this estimate is global, we use $\| \phi_K \|_{1,K}$ as an estimate of the local error over each element K. In general, reducing $\| \phi_K \|_{1,K}$ implies a reduction in $\| r_h^1 \|_*$ which (particularly for linear self-adjoint problems on Hilbert spaces) implies a reduction in $\| u-u_h^1 \|$.

Interpolation Error Estimates

It is well known (see, e.g. Oden and Carey, 1983) that for linear elliptic problems the approximation error $\| e_h \|_V^2 = \| r-u_h^1 \|_V$ can be bounded above by the so-called interpolation error,

$$\| e_h \|_V \leq C \| u-v_h^1 \|_V \quad \text{for all} \quad v_h^1 \quad \text{in} \quad V_h^1 \tag{13}$$

For the model problem (6), for example,

$$\int_\Omega \nabla(u-u_h^1) \cdot \nabla(u-u_h^1) \, dx \Big)^{\frac{1}{2}}$$

$$|u-u_h^1|_{1,} \leq C \inf_{v_h \in V_h} |u-v_h^1|_{1,\Omega} \tag{14}$$

If u is smooth enough, a *local interpolation error estimate* can be derived of the type (for Q_1-elements)

$$|u-v_h^1|_{1,K} \leq C \, h_K |u|_{2,K} \tag{15}$$

where

$$|u|_{2,K}^2 = \int_K W \, dx$$

$$W \, dx \equiv \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} \right) dx \, dy \tag{16}$$

The basic problem we face when attempting to make use of any of these estimates is that we must calculate the higher order derivatives of the unknown solution using only available information, i.e., through use of the currently available finite element solution u_h^1. There are numerous a priori techniques for estimating the second derivatives u_{xx}, u_{xy} or u_{yy}, but many are somewhat intuitive and not all are based on rigorous estimates. Exceptions are the techniques based on so-called "extraction formulas" introduced by Babuska and Miller (1984a, 1984b). Following their idea one can prove that, if u is regular enough, then the second derivatives at an arbitrary point $(x_0,$

$y_0)$ satisfy

$$\frac{\partial^2 u}{\partial x^2}(x_0, y_0) - \frac{\partial^2 u}{\partial y^2}(x_0, y_0) = \int_\Omega \Delta\bar{\phi}\, u\, dxdy$$

$$\int_\Omega (\phi + \bar{\phi})\, f\, dxdy - \int_{\partial\Omega} u \frac{\partial}{\partial n}(\phi + \bar{\phi}) ds + \int_{\partial\Omega} (\phi + \bar{\phi}) \frac{\partial u}{\partial n} ds \qquad (17)$$

Here $\phi = \frac{1}{\pi}\frac{\cos 2}{r^2}$ where (r, θ) are the polar coordinates centered at the point (x_0, y_0) under consideration and $\bar{\phi}$ is an arbitrary, regular function. By the proper choice of $\bar{\phi}$, one can eliminate the boundary terms in (17). Of course, u on the right-hand side of (17) remains still unknown, but when replaced by its element approximations u_h^1 results in a formula for approximation of second-derivatives at (x_0, y_0) of the same order of accuracy as the L^2-error in the approximation of u by u_h^1. For example, for the first order approximation we can "extract" the difference of second order derivatives with $O(h^2)$ order of convergence! Formula (17), when combined with equation (7), allows us to calculate each of the derivatives separately. Also, by choosing $\phi = \frac{1}{\pi}\frac{\sin 2\theta}{r^2}$ in the same formula, we can "extract" the mixed derivative $\frac{\partial^2 u}{\partial x \partial y}(x_0, y_0)$.

One method we have used successfully in applying the estimate (15) is to construct the function ϕ using a bivariate blending function of Gordon and Hall (1971, 1973) type.

Note that we still have a global estimate although we "apply it" to K

$$|u - u_h^1|_{1,\Omega}^2 \leq C \sum_K h_K^2 |u|_{2,K}^2 \qquad (18)$$

MESH REFINEMENT STRATEGIES BASED ON THE A POSTERIORI ERROR ESTIMATES

While many issues remain open in the area of reliable a-posteriori error estimation, still further complications exist in designing efficient adaptive algorithms based on these estimates. The basic problem can assume the form of an optimal control problem in which one has to attain a discrete approximation which is optimal in some sense determined by the error measures and the strategy used to reduce error. The entire problem is further complicated by the fact that our a-posteriori estimates are global in nature (particularly the residual-type estimates discussed earlier) even though they are used locally as a basis for local enrichments of the solution.

In this section we describe three methods developed by Oden and colleagues, (1985), Demkowicz and Oden (1985), Demkowicz, Oden, and Strouboulis (1984), and Demkowicz, Oden, and Devloo (1985).

An h-Method

Consider a quadratic mesh and the associated error estimate (18). If we