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The Core Model

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PREFACE

This monograph is intended to give a self-contained presentation of the core model, adding details to the brief account that Ronald Jensen and I presented in [10] and [11]. By "self-contained" I mean that any result not proved should be easy to find in a textbook: I have used Jech's book ([25]) for references wherever possible. I have included just about everything I know about K, sometimes without proofs, and a few details about larger core models.

I am conscious, looking over the text, of many places where the explanations could be clearer, of important results that never quite surface as lemmas and of clumsiness in some of the proofs. I hope that proofs of everything that needs proving can be extracted from the text by a careful reader: I shall be most grateful for comments from anyone who finds the presentation incomprehensible (or erroneous). Two particular comments may help: firstly the exercises range from the very easy to the unsolved. Sometimes they are essential to later results in the text; I hope these ones are not too difficult! Secondly there is no good reason why appendix II is not inserted at appropriate points in the text, as, apart from one result in descriptive set theory, it is self-contained. Were I rewriting the text there would be more to be said about models of ZFC and less about R; but the reader will have to extract general proofs from those given in chapter 12.

Numbers in square brackets thus [1] refer to the bibliography. I have added to this a set of notes on the history of the results in the text, and used this as an excuse not to give attributions of results elsewhere. But I should add the traditiional disclaimer about not being a historian of the subject; it will be apparent that I have relied heavily on the notes in Jech's book. There is one historical point on which I can offer no help: people occasionally ask why mice are so-called. I am afraid that neither Jensen nor I can remember why, but plausible explanations would be welcomed.

Many thanks are due to those who have helped by their own explanations or by their criticism of mine. I should mention particularly Keith Devlin, Bill Mitchell and Philip Welch; and among the discoverers of errors in previous editions Peter Koepke and Lee Stanley. Robin Gandy has been a constant inspirer of improvements from the time when as my supervisor he cast a critical eye over the almost incomprehensible first draft of my thesis up to a recent seminar talk on rudimentary functions which compelled me completely to rewrite part one.

It is appropriate that I should dedicate this work to Ronald Jensen, since so much of it is his work. How much is not apparent from the references: one must also take into account his patient explanations in answer to my endless questionning. I hope the references do make it clear that, although I alone am responsible for the exposition, the major results are our joint work and many of the others are his alone.

I was supported while writing by Junior Research Fellowships first at New College and then from Merton College. To have been able to work in two such delightful communities has meant a great deal to me and I owe more than I can express to the friendship and generosity of the fellows of each.

Professor Ioan James first encouraged me to write this monograph, and to him, as well as to David Tranah and the Cambridge University Press, who gave much advice and help, and waited very patiently for the result, I express my gratitude.

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PRELIMINARIES

My intention in writing this book was that it should be accessible to anyone who had a reasonable background in axiomatic set theory up to Gödel's consistency results; and, if this failed, at least that readers should not be expected to hunt through piles of old journals in search of alleged folklore. The appearance of Jech's book ([25]), and more recently of that of Kunen, have made it easier to find these results and I have not hesitated to give references for results that lie away from the main development of the fine structural theory. Various odd facts have, no doubt, slipped in along the way unproved, but consultation of one of the textbooks mentioned should help. Chapter 0 of Devlin [6] also contains many of the things I feel I ought to have said here. A few notes on particular points are necessary.

LANGUAGE

The language L is the usual first order language for set theory. On official occasions its primitives are $-, \wedge, \exists, (,)$ with binary relation symbols $=$ and $\in$ and variables v_i. Formulae are also treated as objects (definition 1.16). A bounded quantifier is one of the form $(\exists v_i \in v_j)$, where $(\exists v_i \in v_j)\phi$ abbreviates $(\exists v_i)(v_i \in v_j \wedge \phi)$; a formula all of whose quantifiers are bounded is called restricted or Σ_0. In induction on Σ_0 structure, therefore, $(\exists v_i \in v_j)\phi$ is more complex than $v_i \in v_j \wedge \phi$. Generally if variables are displayed after a formula then all, but not necessarily only, free variables are displayed. But this is not a hard and fast rule, and the absence of a variable list is certainly not an indication that we are dealing with a sentence.

The language L_N is L together with N additional unary predicates, usually called $A_1 \ldots A_N$. If they are to be given other names, or we are to add binary relations or constants or anything else, or if we want to rename the $\in$ predicate then the complete list of non-logical symbols is displayed thus: $L_{R,f,c}$.

AXIOMS

A sequence of weak theories is introduced in part one. For reference the axioms of ZF are the universal closures of the following:

(1) $x=y \leftrightarrow \forall z(z\in x \leftrightarrow z\in y)$ (extensionality)

(2) $x\neq\phi \rightarrow (\exists y\in x)(y\cap x=\phi)$ (foundation)

(3) $\exists z(z=\{x,y\})$ (pairing)

(4) $\exists z(z=\cup x)$ (union)

$(5)_\phi$ $\exists y\forall z(z\in y \leftrightarrow z\in x\wedge\phi)$ (separation)

$(6)_\phi$ $\forall x\exists y\phi(x,y,p) \rightarrow \forall u\exists v(\forall x\in u)(\exists y\in v)\phi(x,y,p)$ (replacement)

(7) $\exists y(y=P(x))$ (power set)

(8) $\exists y(y=\omega)$ (infinity).

The axiom of choice is

(9) $\forall x(\forall z,z'(z,z'\in x \rightarrow z=z'\vee z\cap z'=\phi) \rightarrow \exists y\forall z\in x\exists! w w\in y\cap z)$.

(1)-(9) is called ZFC. (1)-(6),(8) is called ZF^-. The theory KP consists of axioms (1)-(4) together with $(5)_\phi$ and $(6)_\phi$ for restricted ϕ. Its models are called admissible. Admissible sets are too narrow a collection for the fine-structural theory used in part one - the theory R which we use is weaker than KP - but similar considerations motivate the removal of, for example, full replacement from the theory.

FUNCTIONS

The ordered pair $\langle x,y\rangle$ is $\{\{x,y\},\{x\}\}$. A relation is a collection of ordered pairs. A relation R is a function provided $Rxy\wedge Rxz \rightarrow y=z$; then instead of Rxy we may write $R(x)=y$. This rather trivial point needs attention: usually the literature of fine structure has followed Gödel in identifying the function f with $\{\langle f(x),x\rangle : x\in dom(f)\}$. I have used the different notation here because it seems to have become the more commonly used one. It is natural, given this usage, to define the ordered n+1-tuple so that $\langle x_1\ldots x_{n+1}\rangle=\langle\langle x_1\ldots x_n\rangle,x_{n+1}\rangle$; thus n+1-ary functions are functions. Parts of the theory - lemma 1.5 for example - are sensitive to the choice of convention - although they would not be if we were to redefine $\langle x,y\rangle=\{\{x,y\};\{y\}\}$.

I have tried to avoid reliance on the convention by adding unnecessary $\langle\ \rangle$ in places. Sometimes an inverse defined by $x=\langle (x)_0\ldots(x)_{n-1}\rangle$ is used, but care is needed as this is ambiguous. $\vec{x}$ represents a list and not a sequence: so $\vec{x}\in X$ means $x_1\ldots x_n\in X$ and not $\langle x_1\ldots x_n\rangle\in X$. Exceptions occur if there is no danger of confusion.

id is the function $id(x)=x$.

ORDINALS AND CARDINALS

$\bar{\bar{X}}$ denotes the cardinality of X: this is always identified with the least ordinal in one-one correspondence with X (in all our weak systems other than R this can be shown to exist). The order-type of

$\langle X,<\rangle$ is written $\mathrm{otp}(\langle X,<\rangle)$; $<$ may be omitted when it is $\in$. $\aleph_\alpha$ and ω_α are used interchangeably.

If κ is regular and uncountable and $C\subseteq\kappa$ then C is closed in κ if $X\subseteq C \Rightarrow \sup X\in C\cup\{\kappa\}$. It is unbounded if $\sup C=\kappa$.

$S\subseteq\kappa$ is stationary provided $S\cap C\neq\phi$ whenever C is closed and unbounded in κ. If C,C' are closed unbounded in κ then so is $C\cap C'$. Indeed, if $\gamma<\kappa$ and $\delta<\gamma \Rightarrow C_\delta$ closed unbounded in κ then $\bigcap_{\delta<\gamma} C_\delta$ is closed unbounded in κ. Fodor's theorem states that if S is stationary in κ and $f:S\to\kappa$ is regressive (i.e. for $x\in S$ $f(x)<x$) then there is $\beta\in\kappa$ and stationary $S'\subseteq S$ such that $f''S'=\{\beta\}$.

STRUCTURES

A structure for L is a pair $\langle M,E\rangle$. We distinguish the structure $\langle M,E\rangle$ from the set M by underlining: $\underline{M}$ denotes $\langle M,E\rangle$. There are some exceptions (other than carelessness) to this rule, though. One is discussed in chapter 4; another will be introduced in a moment. But we have tried to adhere strictly to the rule that underlining should not be used for any other purpose: so $\underline{M}$ and M will never be used simply as different variables.

V denotes the universe, even if the theory is weaker than ZF; this looks bizarre until you get used to it. When $V\models ZFC$, we call a structure $\underline{M}$ an inner model of a theory T provided $\underline{M}\models T$ and M is a transitive class containing On. It is usually safe to ignore the underlining convention with inner models, and in particular with V itself (as we did at the start of the previous sentence).

Another abuse of notation is the writing of $\langle M,A\rangle$ to denote $\langle M,A\cap M\rangle$. This never causes confusion. If $\underline{M}=\langle M,E,A_1\ldots A_N\rangle$ then $\underline{M}|X$ denotes $\langle M\cap X, E\cap X^2, A_1\cap X,\ldots,A_N\cap X\rangle$. In listing structures E may be omitted if it is clear what it should be. If t is a term then $t^{\underline{M}}(\vec{y})$ denotes that x such that $\underline{M}\models x=t(\vec{y})$. If convenient a subscript rather than a superscript may be used.

The Mostowski collapsing lemma (a theorem of ZF) says that if $\langle X,E\rangle\models$axiom (1) and E is well-founded then there is a unique transitive set M and a unique isomorphism π such that $\pi:\langle X,E\rangle\cong\langle M,\in\rangle$. If X is a proper class it must also be specified that for all $x\in X$ $\{y:yEx\}$ is a set.

THE LEVY HIERARCHY

Σ_0 has already been defined. Suppose Σ_n is defined: then Π_n is the collection of negations of formulae in Σ_n. Σ_{n+1} is the set of formulae of the form $\exists y\phi$ where $\phi\in\Pi_n$. If Γ is a set of formulae then $\Gamma^T=\{\phi: T\vdash\phi\leftrightarrow\psi$ for some ψ in $\Gamma\}$. T is always omitted; it is taken to be whatever set theory we are working in.

$\pi:\underline{N}\to_{\Sigma_m}\underline{M}$ means that for all Σ_m ϕ and $x_1\ldots x_k\in N$

$\underline{N}\models\phi(x_1\ldots x_k) \leftrightarrow \underline{M}\models\phi(\pi(x_1)\ldots\pi(x_k))$.

$X\prec_{\Sigma_m}\underline{M}$ (where $X\subseteq M$) means $id|X:\underline{M}|X\to_{\Sigma_m}\underline{M}$. A relation R is $\Sigma_m(\underline{M})$ with parameter p provided there is a Σ_m formula ϕ such that

$R(y) \leftrightarrow \underline{M}\models\phi(y,p)$.

$\Sigma_m(\underline{M})$ denotes the set of relations which are $\Sigma_m(\underline{M})$ in some parameter p. Really we should use a bold-face Σ for this, but instead we specify explicitly if there is a restriction on parameters.

Note that if $\underline{M},\underline{N}$ are models of ZF^-+AC and $\pi:\underline{M}\to_{\Sigma_1}\underline{N}$ maps On_M cofinally into On_N then $\pi:\underline{M}\to_{\Sigma_m}\underline{N}$ for all m. Suppose this proved for $n\leq m$ and let ϕ be $\exists y\psi$ with ψ Π_m. Say $\underline{N}\models\psi(y,\pi(x_1)\ldots\pi(x_k))$. So if $y\in V^M_{\pi(\beta)}$ $\underline{N}\models(\exists y\in V_{\pi(\beta)})\psi(y,\pi(x_1)\ldots\pi(x_k))$. Since $\pi:\underline{M}\to_{\Sigma_1}\mathbf{N}$ $\pi(V^M_\beta)=V^N_{\pi(\beta)}$; and $(\exists y\in z)\psi(y,\vec{x})$ is Π_m (see [25] lemma 14.2(ii): it is here that we need the full force of ZF^-) so $\underline{M}\models(\exists y\in V_\beta)\psi(y,x_1\ldots x_k)$, i.e. $\underline{M}\models\phi(x_1\ldots x_k)$. $\pi:\underline{M}\to_{\Sigma_m}\underline{N}$ for all m is abbreviated $\pi:\underline{M}\to_e\underline{N}$ (e for elementary). $id|X:\underline{M}|X\to_e\underline{M}$ is written $X\prec\underline{M}$. The usual terminology, $\underline{N}\prec\underline{M}$ may also be used.

$\Delta_n=\Sigma_n\cap\Pi_n$. Δ_1 formulae are <u>absolute</u>; that is, if ϕ is Δ_1 and $X\subseteq M$, $\vec{x}\in X$ then $\underline{M}\models\phi(\vec{x}) \leftrightarrow \underline{M}|X\models\phi(\vec{x})$. Δ_1^T are absolute between models of T; for example, if $\underline{M}$ is an inner model of ZF and $R\in M$ is a partial order then R is well-founded if and only if $M\models$R is well-founded. More results of this kind are in appendix II.

Note also that if $\underline{M},\underline{N}\models ZF$ and $j:\underline{M}\to_e\underline{N}$ and $j\neq id|M$ then there is an ordinal κ such that $j(\kappa)\neq\kappa$. For otherwise for all x $j(rank(x))=$ $=rank(x)$ so $rank(j(x))=rank(x)$ and an easy induction shows that for all $\alpha\in On_M$ $j|V^M_\alpha=id|V^M_\alpha$. The least such κ is called the critical point of j.

<u>DESCRIPTIVE SET THEORY</u>

The reals are identified with $P(\omega)$. A formula is arithmetic if all its quantifiers are restricted to range over ω: ϕ is Σ^1_1 if there is an arithmetic formula ψ such that $\phi(x) \leftrightarrow (\exists a\in P(\omega))\psi(a,x)$. Generally, it is Σ^1_{n+1} if this holds with "Π^1_n" in place of "arithmetic"; and it is Π^1_{n+1} if $-\phi$ is Σ^1_{n+1}. $\Delta^1_n=\Sigma^1_n\cap\Pi^1_n$. Free variables may represent either natural numbers or reals: we may therefore also define Σ^1_n sets of reals and Σ^1_n reals. All notation is light-faced. Note especially that if ϕ is, say, Σ^1_1 and we write $\underline{A}\models\phi$ this means that there is a real in A satisfying the appropriate condition; in other words, ϕ is not treated as a second-order formula. Our coverage of descriptive set theory is very skimpy and the reader should consult Jech [25] for details.

NOTATION

Other than the above most notation can be found in the index of definitions. For the rest the following is a list of standard symbols that may not be immediately recognisable as such.

(i) $\smallsetminus$ is complement, but - is negation.

(ii) $\cup$ and $\cap$ are used both for the binary and for the unary union and intersection.

(iii) $|$ denotes functional restriction, $f|X=\{\langle x,y\rangle\in f : x\in X\}$; f"X denotes the range of $f|X$.

(iv) $\leftrightarrow$ indicates a bijection, $\cong$ an isomorphism.

(v) ∞ and On are used interchangeably.

(vi) $\emptyset$ is the null set; ϕ is phi.

(vii) $\square$ marks the end of a proof.

(viii) $\mathcal{P}$ is power set.

(ix) # is a sharp, † is a dagger and ¶ is a pistol.

(x) $H_\kappa=\{x:\overline{\overline{TC(x)}}<\kappa\}$.

The core model arises from the mixture of two techniques that had once seemed incompatible: fine-structure and iterated ultrapowers. Fine structure was designed for use in Gödel's L; but the major application of iterated ultrapowers was to measurable cardinals: and there are no measurable cardinals in L. First let us examine the two sources separately.

1: FINE STRUCTURE

Gödel's constructible universe L may briefly be defined as follows:

$L_0=\phi;$

$L_{\alpha+1}=\mathrm{Def}(L_\alpha);$

$L_\lambda=\bigcup_{\alpha<\lambda} L_\alpha$ (λ a limit).

Then $L=\bigcup_{\alpha\in On} L_\alpha$. By Def(X) is meant the collection of sets first-order definable over X: that is, all sets $x\subseteq X$ such that for some formula ϕ of $\mathcal{L}$ with n+1 free variables and for $p_1\ldots p_n\in X$

$x=\{t\in X:\langle X,\in\rangle\models\phi(t,p_1\ldots p_n)\}$.

Gödel's construction is to be thought of as going on within some model V of ZF: then $L\models$ZFC. So Consis(ZF) $\Rightarrow$ Consis(ZFC).

L also has the important condensation property. Suppose κ is a cardinal and $X\prec L_\kappa$. Let $\pi:\underline{M}\cong\langle X,\in\rangle$ where M is transitive. Such an $\underline{M}$ exists by the Mostowski collapsing lemma. Then $M=L_\lambda$ for some λ. We could get by with conditions on κ and X much weaker than these.

Now consider any $a\subseteq\omega$, $a\in L$. For some cardinal κ $a\in L_\kappa$. Let $X\prec L_\kappa$ with $\overline{\overline{X}}=\aleph_0$, $a\in X$. $\omega\subseteq X$, of course. Let $\pi:\underline{M}\cong\langle X,\in\rangle$ with M transitive. Say $M=L_\lambda$. Then say $\bar{a}=\pi^{-1}(a)$. $\bar{a}\subseteq\omega$ and for all $n\in\omega$ $\pi(n)=n$ so $n\in\bar{a}\Leftrightarrow n\in a$. Hence $\bar{a}=a$: thus $a\in L_\lambda$. But it is easily seen that for infinite λ $\overline{\overline{L_\lambda}}=\overline{\overline{\lambda}}$. And $\overline{\overline{L_\lambda}}=\overline{\overline{X}}=\aleph_0$, so $\overline{\overline{\lambda}}=\aleph_0$: that is, $\lambda<\omega_1$. Hence $a\in L_{\omega_1}$. We have proved $P^L(\omega)\subseteq L_{\omega_1}$. If we carry out the argument in L we deduce that $2^{\aleph_0}\leq\overline{\overline{L_\omega}}=\aleph_1$. By Cantor's theorem $2^{\aleph_0}\geq\aleph_1$ so the continuum hypothesis holds in L. In fact an almost identical proof shows that for all infinite cardinals λ $P^L(\lambda)\subseteq L_{\lambda^+}$, so that GCH holds in L.

Gödel deduced that Consis(ZF) $\Rightarrow$ Consis(ZF+GCH). In fact the condensation property can be used to deduce more powerful properties such as $\diamondsuit$ (see [6]).

A little close attention to the condensation property reveals many more details about the corollary that $P^L(\omega)\subseteq L_{\omega_1}$. We have already observed that we may replace ω_1 by ω_1^L: and this must be the best possible, since $L\models P(\omega)$ is uncountable. But that does not mean that every $\alpha<\omega_1$ must yield a new subset of ω - we may have $\omega<\alpha<\omega_1$ and $P(\omega)\cap L_\alpha=P(\omega)\cap L_{\alpha+1}$. This is called a gap. Indeed there can be gaps much longer than 1 - see [39].

The existence of a gap is equivalent to some form of comprehension: for it says that all subsets of ω definable over L_λ are already in L_λ. We could (and shall) define a measure of the failure of comprehension over L_λ as the least γ such that $L_\lambda\cap P(\gamma)\neq L_{\lambda+1}\cap P(\gamma)$. Such measures are clearly related to admissible set theory (KP).

In fact it pays to be even more precise. Maybe there are no new subsets of ω Σ_1-definable over L_λ, but there are such Σ_2-definable definable subsets? "We find such questions both interesting and important in their own right. Admittedly, however, the questions - and the methods used to solve them - are somewhat remote from the normal concerns of the set theorist. One might refer to "micro set theory" in contradistinction to the usual "macro set theory". Happily, micro set theory turns out to have non-trivial applications in macro set theory"(Jensen [26]). We shall return to the question of applications at the end of this section.

Jensen's paper just quoted contains a full fine-structural analysis of L. This involves putting "coarse" results into forms that involve "fine" definability distinctions. As an example consider the result of Gödel that if $a\subseteq\gamma$, $a\in L_{\lambda+1}\setminus L_\lambda$ then $\bar{\bar{\lambda}}=\bar{\bar{\gamma}}$. It turns out that if $a\in\Sigma_1(L_\lambda)\setminus L_\lambda$ then there is a function f Σ_1-definable over L_λ of a subset of γ onto λ. (We have to say "a subset of" because if we tried to add trivial values for $\beta\in\gamma\setminus\mathrm{dom}(f)$ we might end up with a Σ_2 function.)

It turns out to be convenient to replace the L hierarchy by a modified form, the J hierarchy. This leaves the total model the same but redistributes sets among the levels. One reason for doing this is that the pairing axiom fails in arbitrary L_λ, which makes for difficulties. As the text uses an odd definition it is worth giving the original here.

A function is called rudimentary if and only if it is finitely generated by the following schemata:

(a) $f(\vec{x})=x_i$;

(b) $f(\vec{x})=x_i\setminus x_j$;

(c) $f(\vec{x})=\{x_i,x_j\}$;

(d) $f(\vec{x})=h(\vec{g}(\vec{x}))$;

(e) $f(y,\vec{x}) = \bigcup_{z\in y} g(z,\vec{x})$.

Rudimentary functions were invented as part of a programme to generalise the notion of primitive recursive to arbitrary sets (see [17] and [32]).

Now the J hierarchy is defined as follows:

$J_0=\phi$;

$J_{\alpha+1}=R(J_\alpha \cup \{J_\alpha\})$;

$J_\lambda = \bigcup_{\alpha<\lambda} J_\alpha$ (λ a limit).

$R(X)$ denotes $\{f(\vec{x}): f$ rudimentary, $\vec{x}\in X\}$. Then

(i) $L= \bigcup_{\alpha\in On} J_\alpha$;

(ii) $On\cap J_\alpha=\omega\alpha$ (whereas $On\cap L_\alpha=\alpha$);

(iii) $J_{\alpha+1}\cap P(J_\alpha)=Def(J_\alpha)$.

$J_\alpha=L_\alpha$ if and only if $\omega\alpha=\alpha$. (iii) shows that R is very like Def; (ii) shows that in general it is longer.

One of the most important technical devices of [26] is the "master code". One way of looking at this is to say that if there is some subset of γ in $\Sigma_1(J_\lambda)\setminus J_\lambda$ then there is a "universal" one. To put this more precisely, let us define the projectum. The Σ_n projectum of λ, ρ^n_λ, is the least γ such that $P(\omega\gamma)\cap\Sigma_n(J_\lambda)\not\subseteq J_\lambda$. We shall restrict ourselves to Σ_1 for a bit. As we have already said there is a Σ_1 function of a subset of $\omega\rho^1_\lambda$ onto J_λ. Indeed there is such a function uniformly, called the Σ_1 Skolem function. This is a partial Σ_1 function h with the property that given a fixed enumeration $\langle\phi_i : i<\omega\rangle$ of Σ_1 formulae with two free variables

$\exists y\phi_i(y,x) \Rightarrow \phi_i(h(i,x),x)$.

It may help if we sketch a proof of the assertion that h is a surjection when restricted to $\omega\times\omega\rho^1_\lambda$ (in fact this is not quite exact because h may need a parameter argument as well, but we shall avoid this difficulty). Suppose $a\in P(\omega\rho^1_\lambda)\cap(\Sigma_1(J_\lambda)\setminus J_\lambda)$. Assume for simplicity that a has a parameter-free Σ_1 definition over J_λ:

$\beta\in a \Leftrightarrow J_\lambda\models\phi(\beta)$.

Now let $X=h''(\omega\times\omega\rho^1_\lambda)$. It is easily seen that $X \prec_{\Sigma_1} J_\lambda$. Let $\pi:\underline{M}\cong\langle X,\in\rangle$ with M transitive. Now the condensation property holds in the J hierarchy for all levels - indeed there is a Π_2 sentence saying, in any transitive set, "I am a J_α" - so $M=J_\beta$ for some β. $\pi:J_\beta\to_{\Sigma_1} J_\lambda$. Now for $\gamma\in\omega\rho^1_\lambda$ $\pi(\gamma)=\gamma$, and $\underline{M}\models\phi(\gamma) \Leftrightarrow J_\lambda\models\phi(\pi(\gamma)) \Leftrightarrow J_\lambda\models\phi(\gamma)$, so $a\in\Sigma_1(J_\beta)$ If $\beta<\lambda$ we should have $a\in J_{\beta+1}\subseteq J_\lambda$, but $a\notin J_\lambda$, so $\beta=\lambda$. So $\pi:J_\lambda\to_{\Sigma_1} J_\lambda$. Furthermore given $x\in J_\lambda$ $\pi(x)\in X$ so $\pi(x)=h(i,\gamma)$ say, with $i\in\omega$ and $\gamma<\omega\rho^1_\lambda$. But then $x=h(i,\gamma)$ so $x\in X$; thus $X=J_\lambda$.

h is not total, of course. Now for our master code. Since there is a Σ_1 map of $\omega\rho^1_\lambda$ onto $J_{\rho^1_\lambda}$ it is possible to make it a subset of

$J_{\rho_\lambda^1}$ - we can then code it in $\omega\rho_\lambda^1$ if necessary. In general we only know that for some parameter p $h''(\omega\times(\omega\rho_\lambda^1\times\{p\}))\models J_\lambda$: let p_λ be the least such p in the well-order of J_λ used to prove $L\models AC$. Then let $A=\{\langle i,x\rangle: i\in\omega \wedge x\in J_{\rho_\lambda^1} \wedge J_\lambda\models\phi_i(x,p_\lambda)\}$. For every $B\in\Sigma_1(J_\lambda)\cap P(J_{\rho_\lambda^1})$ there is a rudimentary function f such that $x\in B \Leftrightarrow f(x)\in A$. A is a Σ_1 master code.

Having obtained this result we find that the means are more interesting that the end. For A does not merely code subsets of $\omega\rho_\lambda^1$: it codes the whole of J_λ. For example, if $\pi:\underline{M}\to_{\Sigma_1}\langle J_{\rho_\lambda^1},A\rangle$ then there is a unique μ and a unique $\tilde{\pi}\supseteq\pi$ such that $\tilde{\pi}:J_\mu\to_{\Sigma_1}J_\lambda$ and, letting $\underline{M}=\langle M,B\rangle$, B is a Σ_1 master code for J_μ, $M=J_{\rho_\mu^1}$ and $\tilde{\pi}(p_\mu)=p_\lambda$. Also $\tilde{\pi}:J_\mu\to_{\Sigma_2}J_\lambda$. This is proved by reconstructing J_μ from B (see lemma 3.26). To generalise this coding to ρ_λ^n it is necessary to define a relativised projectum of a structure $\langle J_\rho,A\rangle$ and to show that $\rho_\lambda^2=\rho^1_{\rho_\lambda^1,A}$. Since relativised projecta will need detailed consideration in a more general context we shall say nothing more about them here.

In [26] the applications of these principles are all combinatorial. It is difficult to give a brief account of these that reveals the role played by fine structure, and we shall not try, for the applications that concern us are not combinatorial. Essentially the point is this: many nice theorems can be proved about Σ_1 maps, sets and formulae but do not generalise to Σ_n: ultraproducts are a source of such resuls, as we shall see. By fine structural analysis, Σ_{n+1} properties of J_λ are reduced to Σ_1 properties of $J_{\rho_\lambda^n}$, which are then handled nicely and returned to Σ_{n+1} form by a futher bit of fine structure. For example, for every $\Sigma_1(J_\lambda)$ relation $R(x,y)$ there is a Σ_1 function $r(x)$ such that

$\exists yR(x,y) \Rightarrow R(x,r(x))$.

(Σ_1 relations are said to be "Σ_1 uniformisable"). This is not so clear for Σ_{n+1} relations. We may prove it as follows, by induction on n. n=0 is given. Suppose R is Σ_{n+1}; recall that there is a Σ_n f mapping a subset of $J_{\rho_\lambda^n}$ onto J_λ. Say $\tilde{R}(x,y) \Leftrightarrow R(f(x),f(y))$: then $\tilde{R}\subseteq J_{\rho_\lambda^n}$ is $\Sigma_{n+1}(J_\lambda)$, hence $\Sigma_1(J_{\rho_\lambda^n},A^n)$ where A^n is the nth master code formed by the inductive process we have just described. Uniformise $\tilde{R}$ by $\tilde{r}$, so that

$\exists y\tilde{R}(x,y) \Rightarrow \tilde{R}(x,\tilde{r}(x))$.

Finally f^{-1} is a $\Sigma_n(J_\lambda)$ relation so by induction hypothesis has a uniformising function g; in other words g is an inverse function for f. Let $r(x)=f(\tilde{r}(g(x)))$. r is the required Σ_{n+1} function. For details of other combinatorial applications the reader should consult [26] or [6]. In section 5 of this introduction we shall see how the covering lemma uses fine structure.

In this book we are dealing all the time with relative constructibility. That is, given a set A we define a function as "rudimentary in A" provided that it is generated by (a)-(e) above and

(f) $f(\vec{x})=x_i \cap A$.

We define

$J_0^A=\phi$;

$J_{\alpha+1}^A=R_A(J_\alpha^A \cup \{J_\alpha^A\})$;

$J_\lambda^A=\bigcup_{\alpha<\lambda} J_\alpha^A$ (λ a limit)

where R_A is like R with "rudimentary in A" in place of "rudimentary". It is also possible to define an L_α^A hierarchy by using an operation Def_A replacing L by L_1 and $\langle X,\in\rangle$ by $\langle X,\in,X\cap A\rangle$. Or, indeed, for any number of $A_1 \ldots A_N$. $\bigcup_{\alpha\in On} J_\alpha^A$ is always called L[A]. The most striking fact about the J_α^A hierarchy is that it does not have the condensation property: if $X \prec J_\alpha^A$ and $\pi:\underline{M}\cong\langle X,\in,A\cap X\rangle$ with M transitive and $\underline{M}=\langle M,\in,\bar{A}\rangle$ then certainly for some β $M=J_\beta^{\bar{A}}$. But $\bar{A}=A\cap J_\beta^A$ does not necessarily hold. But in the main application cited, the proof that there is a $\Sigma_1(J_\lambda)$ map of a subset of $J_{\rho_\lambda^1}$ onto J_λ it was essential that the transitive collapse should leave us in the same hierarchy. Of course if $\omega\rho_\lambda^1>$ $>\sup A$ this is possible so ordinary fine structure holds above sup A; this is not much consolation. In fact we speedily see that the Σ_1 map property fails. Suppose $a\subseteq\omega$ but $a\notin L$. Let $A=\{\omega_1+n:n\in a\}$. Then for $\alpha<\omega_1$ $A\cap J_\alpha=\phi$ so $J_\alpha^A=J_\alpha$: hence $a\notin J_{\omega_1}^A$. Also for all $\beta<\omega_1+n$ $A\cap\beta\in L$ as it is finite, so $J_{\omega_1+1}^A=J_{\omega_1+1}$ and $a\notin J_{\omega_1+1}^A$. But $a\in\Sigma_1(J_{\omega_1+1}^A)$, for $a=\{n:\omega_1+n\in A\}$. Yet it is impossible that there should be any map of ω onto $J_{\omega_1+1}^A$, whether $\Sigma_1(\underline{J}_{\omega_1+1}^A)$ or not.

Worse is to come. Suppose we had let $A=\{\langle\omega_1,n\rangle:n\in a\}$. Then we should have had $a\in J_{\omega_1+1}^A$: but since $A\cap J_{\omega_1}^A=\phi$, $\Sigma_\omega(\langle J_{\omega_1}^A,A\rangle)=\Sigma_\omega(J_{\omega_1}^A)\subseteq L$ so $a\notin\Sigma_\omega(J_{\omega_1}^A)$. Thus there may be undefinable new subsets of ω in $J_{\omega_1+1}^A$: this means that the projectum is not a satisfactory index of formation of new subsets. And we no longer have $R_A(J_\lambda^A\cup\{J_\lambda^A\})\cap P(J_\lambda^A)=$ $=Def_A(J_\lambda^A)$.

We may still define a master code as before; but, for example, the Σ_1 master code may no longer code all the Σ_1 subsets of $\omega\rho_{\lambda,A}^1$ over J_λ. For example in the first example above we may assume $\aleph_1^L=\aleph_1$; but no subset of ω could code $\aleph_1$ reals in the sort of coding we have used. We said, though, that the major interest of the master code was that it coded the whole of J_λ. What structure do these "inadequate" master codes code?

A fact about L that we did not state is that if A is the Σ_1 master code of J_λ then $\langle J_{\rho^1_\lambda}, A\rangle$ is amenable, i.e. for all $x\in J_{\rho^1_\lambda}$ $x\cap A\in J_{\rho^1_\lambda}$. The coding would have made no sense otherwise, for important arguments in $\langle J_{\rho^1_\lambda}, A\rangle$ would have failed. For example the Σ_1 master code of $\langle J_{\rho^1_\lambda}, A\rangle$ would not be Σ_1 definable over $\langle J_{\rho^1_\lambda}, A\rangle$. So we must know that for all $x\in J^A_{\rho^1_{\lambda,A}}$ $x\cap A\in J^A_{\rho^1_{\lambda,A}}$. For a start it is clear from the definition of $\rho^1_{\lambda,A}$ that $x\cap A\in J^A_\lambda$. But we cannot get any further than that. For example, take $a\notin L$ and $A=\{\omega_1+n : n\in a\}$ again. Take some $\alpha>\omega_1$ with $\rho^1_{\alpha,A}=\omega_1$. Let B be the Σ_1 master code of J^A_α. Then a is rudimentary in B so if $\langle J^A_{\rho^1_{\alpha,A}}, B\rangle$ is amenable $a\in J^A_{\omega_1}=J_{\omega_1}\subseteq L$.

$J^A_{\rho^1_{\lambda,A}}$ is not a convenient structure to work with; it is too small. All the sets we were trying to capture in $J^A_{\rho^1_{\lambda,A}}$ were in J^A_λ and bounded subsets of $\omega\rho^1_{\lambda,A}$; and $\omega\rho^1_{\lambda,A}$ is a cardinal in J^A_λ so we could get by with $H=(H_{\omega\rho^1_{\lambda,A}})^{J^A_\lambda}$, the collection of sets in J^A_λ whose transitive closure in J^A_λ is of cardinality less than $\omega\rho^1_{\lambda,A}$ in J^A_λ; certainly $\langle H, B\rangle$ will be amenable, where B is the Σ_1 master code of J^A_λ.

On the other hand H is not a very tidy structure. It turns out, though, that H is $J^B_{\rho^1_{\lambda,A}}$, to which fine structural techniques can be applied. At least, they are equal subject to the acceptability constraint. For example, if H is to be included in $J^B_{\rho^1_{\lambda,A}}$ there cannot be more than $\omega\rho^1_{\lambda,A}$ bounded subsets of $\omega\rho^1_{\lambda,A}$ in J^A_λ. We should have no difficulties if GCH held in J^A_λ.

Here is another difficulty: to get $J^B_{\rho^1_{\lambda,A}}\subseteq H$ we need not only a cardinality restriction (which is easy) but also we need to know that $J^B_{\rho^1_{\lambda,A}}\subseteq J^A_\lambda$. This may fail. (This is exercise 2 of chapter 3. Here is a hint: suppose $\aleph_1=\aleph_1^L$, $2^{\aleph_0}=\aleph_1$ but $V\neq L$. Let $\langle x_\alpha : \alpha<\omega_1\rangle$ enumerate $P(\omega)$ and let $A=\{\langle\omega_1,\alpha,n\rangle : n\in x_\alpha\}$.)

Again GCH will prevent this. GCH can only be formulated if we have the power set axiom, which in general we certainly do not, so we formulate a weaker constraint, called acceptability:

$$P(\gamma)\cap J^A_{\lambda+1}\not\subseteq J^A_\lambda \Rightarrow \forall u\in PP(\gamma)\cap J^A_{\lambda+1}\ J^A_{\lambda+1}\models \bar{\bar{u}}\leq\bar{\bar{\gamma}}.$$

This implies a weak form of GCH: if $P(\gamma)$ is a set then $\overline{\overline{P(\gamma)}}=\gamma^+$; otherwise if $u\subseteq P(\gamma)$ then $\bar{\bar{u}}\leq\bar{\bar{\gamma}}$. Acceptability plainly fails in the case in the hint. (Actually as stated the acceptability property is not sufficiently uniform for the coding property claimed).

"Generalised fine structure" is fine structure that applies to all acceptable J^A_λ, i.e. to all J^A_λ in which the acceptability axiom holds. It is also generalised in that it applies to non well-founded structures. The reason for this latter extension should become apparent as we go along: in order to avoid excessive indexing we present the theory internally as an axiomatic development of sentences that say roughly "I am an acceptable J_α^A". It is always easy to convert back to the terminology of this introduction when dealing with transitive models. Since our structures may have several added predicates, $\underline{M}=\langle M,\in,A_1\ldots A_N\rangle$, it is more convenient to write ρ_M than to list all the predicates by writing $\rho_{\lambda,A_1\ldots A_N}$. Anyway, non transitive models of the axioms are not of the form $J_\lambda^{A_1\ldots A_N}$ so the other form would not work.

Generalised fine-structure preserves the coding property of master codes - that if A_N is the Σ_1 master code of $\underline{N}$ then $\underline{N}$ can be coded in $\langle H_{\rho_N}1,A_N\rangle$. But for $\underline{N}$ to be recovered exactly from the code another condition is necessary. Recall our counterexample in which $\rho_N^1=1$ but $\overline{\overline{N}}=\aleph_1$; it would not be reasonable to expect A_N to code all of N. Since A_N was defined to be $\{\langle i,x\rangle : i\in\omega \wedge x\in H_{\rho_N}1 \wedge \underline{N}\models\phi_i(x,p)\}$ for some p $\underline{N}$ will only be coded by A_N if every x in N is Σ_1 definable from parameters in $H_{\rho_N}1\cup\{p\}$, that is, if $N=h_N$"$(\omega\times(\omega\rho_N^1\times\{p\}))$. Such $\underline{N}$ are called p-sound. All levels of the J_α hierarchy are p-sound for some p, but most of the structures that concern us are not.

Given an acceptable $\underline{N}$ we may form its master code even if it is not p-sound and then decode the master code to obtain a p-sound structure $\underline{N}'$.Apart from the fact that there is a Σ_1 map of $\underline{N}'$ into $\underline{N}$ there is little to be said about the relation between $\underline{N}$ and $\underline{N}'$ in general. Provisionally $\underline{N}'$ will be called the core of $\underline{N}$; later we shall give a rather different definition of this term. Although generalised fine structure has little to say about the relation of a structure to its core, the theory of iterated ultrapowers will reveal a very simple relation for the structures that we shall be examining.

Unfortunately we encounter considerable difficulties when we try to iterate this construction to the projectum ρ_N^n. The core in this case is defined differently. Another problem is that although we can form cores we cannot in general extend maps $\pi:\langle H_{\rho_N}1,A_N\rangle\to\underline{M}'$ to $\tilde{\pi}:\underline{N}\to\underline{M}$ with $\underline{M}'=\langle H_{\rho_M}1,A_M\rangle$; unless $\underline{N}$ is p-sound for some p the usual construction breaks down. This does not matter for the structures considered in this book; it is a problem when more general structures - with lots of measurable cardinals, for example - are taken into account. Generalised fine structure does provide an extension of embeddings lemma in this case but it involves a new hierarchy of formulae, called Σ_n^{*h}; this would take us