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Olga Krupková

The Geometry of Ordinary Variational Equations

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To my daughters
Olga and Sonja

Preface

Ordinary differential equations which are *variational*, i.e., come from Lagrangians as their Euler-Lagrange equations, represent a class of ODE interesting both for mathematicians and physicists. From the physical point of view these equations are *equations of motion* of important mechanical systems. Mathematics deals with such equations within the range of the *calculus of variations*. On the one hand, there are various geometric structures connected with these equations and their solutions, studied by means of differential geometry and global variational analysis. On the other hand, these equations represent an interesting object for the mathematical and global analysis, namely for the theory of ordinary differential equations, since for certain classical families of variational equations there have been invented powerful *integration methods* based on the theory of canonical transformations, symmetries, Hamilton-Jacobi theory, etc.

There exist many excellent textbooks and monographs the subject of which are in fact variational ODE. A large number of them, however, deal only with very particular classes of variational equations. First of all, many restrict to the so called *regular autonomous first-order Lagrangians*, i.e., to Lagrangians depending on "positions and velocities", $L(q^\sigma, \dot{q}^\sigma)$, satisfying the classical regularity condition

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}^\sigma \partial \dot{q}^\nu}\right) \neq 0 ;$$

this concerns, in particular, all the texts using methods of *symplectic geometry* (cf., e.g., R. Abraham and J. E. Marsden [1], C. Godbillon [1], P. Libermann and Ch.-M. Marle [1], J.-M. Souriau [1], S. Sternberg [1], A. Weinstein [1]), and works on *Finsler geometry* (M. Matsumoto [1], etc.). From the textbooks dealing also with some of the *singular first-order Lagrangians* in more detail, let us mention e.g. M. de León and P. R. Rodrigues [3], E. C. G. Sudarshan and N. Mukunda [1], and K. Sundermeyer [1]. Other books are usually general in the variational foundations of the theory but do not cover some interesting aspects, such as e.g. Hamilton equations and different integration methods for singular Lagrangians, or geometric structures connected with variational equations and their solutions (cf. C. Carathéodory [1], Th. De Donder [1], M. Giaquinta and S. Hildebrandt [1], P. A. Griffiths [1], R. Hermann [1], M. de León and P. R. Rodrigues [1], E. T. Whittaker [1], and others).

Generalizations of various structures and methods of classical mechanics to a wider class of Lagrangians are subject of an intensive research, and there exists a plenty of papers dealing with particular aspects of this problem. Much effort has been done to generalize the concept of *regularity*, and consequently of the Legendre transformation, Hamilton equations, Hamilton-Jacobi equation, etc., to *higher-order* Lagrangians on different geometric structures (tangent spaces, fibered manifolds $R \times M$ over R and Y over R, general fibered manifolds). Similarly, some of the questions of the Dirac's theory of constrained systems have been studied in higher-order situations. This seems to be important not only for the classical mechanics itself, but namely for the quantum mechanics, where many questions on quantization of Lagrangians which are not regular, or are of higher order, still remain open. Other directions of research have been oriented to study of *second-order ODE* which are explicitly solved with respect to the second derivatives (geometric structures, symmetries, existence of a Lagrangian, etc.). Bibliography included in this work can cover only a small part of existing papers on the above mentioned subjects; in fact, it is not possible to mention and discuss all contributions.

The aim of this work is to provide a *general, comprehensive and self-contained geometric theory* of ordinary differential equations (of any finite order) which are variational. We consider variational equations on *fibered manifolds over one-dimensional bases*. This underlying structure is very appropriate, since it is sufficiently general to cover all interesting physical applications, and to carry the possibility of straightforward or even obvious generalizations of many tools and constructions to partial differential equations ("field theory"). We put *no a priori restrictions* on the equations under consideration, i.e., we put no a priori restrictions on Lagrangians. This means that the presented theory is *universal*, and applies to equations explicitly solved with respect to the highest derivatives as well as to ODE which cannot be expressed in this form; from the viewpoint of Lagrangians, it covers "regular", as well as all kinds of "degenerate" Lagrangians of all finite orders, "autonomous" as well as "time-dependent" Lagrangians, variational problems such that a global Lagrangian exists, as well as (global) variational problems which do not posses this property (i.e., such that there exists only a family of local Lagrangians giving rise to a global Euler-Lagrange form), etc. We show that these equations can be represented by *distributions* (of not necessarily constant rank); we study symmetries and first integrals of these equations, their structure of solutions, introduce various integration methods for these equations, and clarify geometric structures connected with them. In this way we obtain a theory which includes Lagrangian and Hamiltonian dynamics, symmetries and first integrals, Hamilton-Jacobi theory, canonical transformations, Liouville integration theory, fields of extremals, and other aspects of the classical mechanics, generalized to *any* Lagrangian of any order. The general point of view leads to a new look at "standard" concepts (such as Lagrangean system, phase space, regularity, Hamiltonian, momenta, Legendre transformation, singular system, and many others) relating them with the *class of equivalent Lagrangians* (not with a particular Lagrangian). Consequently, this approach provides geometrically natural generalizations, which are "better adapted" to the geometry of variational systems and to the study of their dynamics.

The work is based mainly on our papers [1–12], and it is an enlarged version of [7].

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Olga Krupková
Opava, September 1997

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Chapter 1.

INTRODUCTION

Problems to find an optimal solution have been already considered by the antique science. One of the best-known ones is the classical *isoperimetric problem*, i.e., the problem *to find in the plane a closed curve whose interior domain has maximal area among all interiors of closed curves of the same length.* Its formulation and solution were well known to Aristotle, Archimedes and others. Great interest in extremal problems motivated by mathematics, mechanics and philosophy has lead in the 17th, 18th and 19th centuries to rush developments of methods handling with variational functionals and their extremals. A new discipline—the classical *calculus of variations* came into being, having grown from the work of Isaac Newton, Leonhard Euler, Jacob Bernoulli, Johann Bernoulli, Jean le Rond D'Alembert, Joseph Louis Lagrange, Adrien Marie Legendre, Johann Friedrich Carl Gauss, William Rowan Hamilton, Carl Gustav Jacob Jacobi, Joseph Liouville, Amalie Emmy Noether, to mention at least some of the great founders of the theory.

Euler-Lagrange equations. The classical calculus of variations concerned in study of the so called *variational integrals*. For a first insight into the problem, consider the integral

$$I = \int_a^b L(x, y(x), y'(x))\, dx, \tag{1.1}$$

where the boudary points a, b are fixed, and the integrand L depends on x, a function $y(x)$ and its first derivative $y'(x)$. The function L is referred to as the *Lagrange function* . The problem is *to select the function $y(x)$ such that the above integral is extremized.* We shall recall a solution of this problem due to Lagrange (cf. e.g. B. Tabarrok and F. P. J. Rimrott [1]). Suppose that the required extremizing function is $y_0(x)$. Denote

$$I_0 = \int_a^b L(x, y_0(x), y_0'(x))\, dx.$$

For an admissible function $y(x)$ we have

$$I = I_0 + \Delta I.$$

We may consider admissible functions of the form

$$y(\varepsilon, x) = y_0(x) + \varepsilon u(x)$$

and

$$y'(\varepsilon, x) = y_0' + \varepsilon u'(x),$$

where ε is a small parameter which does not depend upon x, and the function $u(x)$ is an arbitrary function independent of ε vanishing at $x = a$ and $x = b$. The integrand in (1.1) can then be regarded as the function L when two of its independent variables, y_0 and y_0', are changed by amounts εu and $\varepsilon u'$, respectively. For a given x we may expand L by a Taylor series about y_0 and y_0'. Thus we write

$$L(x, y_0 + \varepsilon u, y_0' + \varepsilon u') = L(x, y_0, y_0') + \frac{\partial L}{\partial y}\bigg|_{(y_0, y_0')} \varepsilon u + \frac{\partial L}{\partial y'}\bigg|_{(y_0, y_0')} \varepsilon u'$$
$$+ \frac{1}{2!}\frac{\partial^2 L}{\partial y^2}\bigg|_{(y_0, y_0')} (\varepsilon u)^2 + \frac{1}{2!}\frac{\partial^2 L}{\partial y'^2}\bigg|_{(y_0, y_0')} (\varepsilon u')^2 + \frac{\partial^2 L}{\partial y \partial y'}\bigg|_{(y_0, y_0')} \varepsilon^2 u u' + \dots.$$

Now,

$$\Delta I = \varepsilon \int_a^b \left(u \frac{\partial L}{\partial y}\bigg|_{(y_0, y_0')} + u' \frac{\partial L}{\partial y'}\bigg|_{(y_0, y_0')} \right) dx$$
$$+ \frac{\varepsilon^2}{2} \int_a^b \left(u^2 \frac{\partial^2 L}{\partial y^2}\bigg|_{(y_0, y_0')} + 2uu' \frac{\partial^2 L}{\partial y \partial y'}\bigg|_{(y_0, y_0')} + u'^2 \frac{\partial^2 L}{\partial y'^2}\bigg|_{(y_0, y_0')} \right) dx + \dots,$$

or

$$\Delta I = \varepsilon I_1 + \frac{\varepsilon^2}{2} I_2 + \dots,$$

where εI_1 is called the *first variation*, and $(\varepsilon^2/2) I_2$ the *second variation* of the variational integral (1.1). Evidently, sufficient conditions for I_0 to be maximum are

$$I_1 = 0, \quad I_2 < 0.$$

For I_0 to be minimum we have as sufficient conditions

$$I_1 = 0, \quad I_2 > 0.$$

Since at extremum condition we have $\partial L/\partial y|_{(y_0, y_0')} = \partial L/\partial y$, from now on we need not retain the subscript (y_0, y_0'). Consider now the integral

$$I_1 = \int_a^b \left(u \frac{\partial L}{\partial y} + u' \frac{\partial L}{\partial y'} \right) dx.$$

Integrating by parts we get

$$I_1 = \int_a^b u \left(\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'} \right) dx + \left(\frac{\partial L}{\partial y'} u \right)_{x=a}^{x=b}. \tag{1.2}$$

The formula (1.2) is called the *first variation formula* . Now, since $u(x)$ is equal to zero at $x = a$ and $x = b$, the second term in (1.2) vanishes. Further, recognizing that $u(x)$ is an arbitrary function we conclude that for I_1 to vanish identically we must have

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'} = 0. \tag{1.3}$$

Equation (1.3) is called the *Euler-Lagrange equation*. This equation was originally derived also by Euler via a different scheme of reasoning.

Many variational problems involve functionals whose integrands contain also higher-order derivatives of $y(x)$. For $L(x, y, y', y^{(2)}, \ldots, y^{(r)})$, the corresponding equation for extremals, also called the Euler-Lagrange equation, takes the form

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial L}{\partial y^{(2)}} - \cdots + (-1)^r \frac{d^r}{dx^r}\frac{\partial L}{\partial y^{(r)}} = 0.$$

Higher order Lagrangians were first systematically studied by M. V. Ostrogradskii in the first half of the 19th century.

In case that L depends on one independent variable x and m dependent variables y^σ, $1 \leqslant \sigma \leqslant m$, the Euler-Lagrange equations represent a system of m second order ordinary differential equations for extremals $y(x) \equiv (y^\sigma(x))$ of the form

$$\frac{\partial L}{\partial y^\sigma} - \frac{d}{dx}\frac{\partial L}{\partial y^{\sigma\prime}} = 0. \tag{1.4}$$

The functions on the left-hand sides are called the *Euler-Lagrange expressions* of the Lagrange function L, and are denoted by $E_\sigma(L)$.

Brachystochrone. One of the best known classical problems in the calculus of variations is the *brachystochrone problem*. In fact, it was this problem which became a strong impulse for intensive theoretic studies of extremal problems.

In 1696 there appeared a treatise by Johann Bernoulli, "Problema novum, ad cujus solutionen mathematici invitantur" ("A new problem to the solutin of which mathematicians are invited") where the problem of quickest descent was posed:

Let two points A, B be given in the vertical plane. Find a line connecting them, on which a movable point M descends from A to B under the influence of gravitation in the quickest possible way.

A solution has been given by Johann Bernoulli, independent solutions have been found also by Jacob Bernoulli, Leibnitz, and an anonymous author (probably Newton).

We shall recall here the original solution provided by Johann Bernoulli, based on an analogy with the propagation of light in an optically heterogeneous medium.

In the vertical plane consider coordinates (x, y) at the point A, such that the x-axis is horizontal and the y-axis is vertical, down-oriented. Denote $A = (0, 0)$, $B = (b_1, b_2)$. Assume that the point M moves without friction, and the initial velocity is zero. This means we have the boundary conditions $y(0) = 0$, $y(b_1) = b_2$ and $v(0) = 0$. The energy equation reads

$$\tfrac{1}{2}mv^2 - mgy = \text{const},$$

where g is the gravity acceleration. At the initial point it holds $y = 0$ and $v = 0$, i.e., the above constant equals zero. Consequently, the velocity at a point $(x, y(x))$ depends only on the coordinate $y(x)$ and equals $\sqrt{2gy(x)}$. Since one needs to find the shortest time to get from A to B, one has to minimize the integral

$$T = \int \frac{ds}{v} = \int \frac{ds}{\sqrt{2gy(x)}} \tag{1.5}$$

over the arc AB, where ds is an element of the path.

The above problem is completely equivalent to the problem of finding the trajectory of the light in a two-dimensional non-homogeneous medium where the velocity at a point (x, y) equals to $\sqrt{2gy}$. The medium can be divided into parallel slices, within each of which the velocity can be considered a constant equal v_i, $i = 1, 2, \ldots$. By the Snell Law one gets

$$\frac{\sin\alpha_1}{v_1} = \frac{\sin\alpha_2}{v_2} = \ldots,$$

i.e.,

$$\frac{\sin\alpha_i}{v_i} = \text{const},$$

where α_i are angles of incidence of the light rays. In the limit

$$\frac{\sin\alpha(x)}{v(x)} = \text{const},$$

where $v(x) = \sqrt{2gy(x)}$, and $\alpha(x)$ is the angle between the tangent to the curve $y(\cdot)$ at the point $(x, y(x))$ and the axis Oy, i.e., $\sin\alpha(x) = (1 + (y'(x))^2)^{-1/2}$. Hence, the equation of the extremal curve, called the *brachystochrone*, is

$$\sqrt{1 + (y')^2}\,\sqrt{y} = C$$

which is equivalent to

$$y' = \sqrt{\frac{C - y}{y}}, \tag{1.6}$$

i.e., to

$$\frac{\sqrt{y}}{\sqrt{C - y}}\,dy = dx.$$

Integrating this equation with help of the substitution $y = C\sin^2 t/2$, hence $dx = (C\sin^2 t/2)\,dt$ we get the following *equation of the cycloide*

$$x = C_1 + \tfrac{1}{2}C(t - \sin t), \quad y = \tfrac{1}{2}C(1 - \cos t).$$

The above equations represent a family of cycloids. A unique solution of the brachystochrone problem is obtained if the boundary conditions are considered.

The brachystochrone problem can also be solved with help of the variational techniques (see e.g. M. Giaquinta and S. Hildebrand [1]). It is the problem of minimizing the action functional (1.5). We have $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2}\,dx$, hence the Larange function is

$$L = \frac{\sqrt{1 + y'^2}}{\sqrt{2gy}}.$$

From the Euler-Lagrange equations of L we obtain a first integral

$$y(1 + y'^2) = C,$$

which is equivalent to the equation (1.6).

The inverse problem of the calculus of variations. Due to intensive studies of different extremal problems during the 17th–19th centuries it turned out that *variational equations*, i.e., equations which can be expressed in form of the Euler-Lagrange equations of a Lagrange function have a fundamental meaning in theoretical physics and engineering. Consequently, an essential problem appeared:

A system of differential equations being given, one needs to find out whether they are variational or not, and in the affirmative case to construct a Lagrange function for these equations.

This problem, referred to as the *inverse problem of the calculus of variations* was first posed by H. von Helmholtz in 1887. In his paper [1] Helmholtz studied a system of m second order differential equations of the form

$$B_{\sigma\nu}(t, q^\rho, \dot{q}^\rho)\ddot{q}^\nu + A_\sigma(t, q^\rho, \dot{q}^\rho) = 0, \tag{1.7}$$

where $1 \leqslant \sigma, \nu, \rho \leqslant m$. He found necessary conditions for the existence of a Lagrange function L such that

$$B_{\sigma\nu}\ddot{q}^\nu + A_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\sigma}.$$

These conditions, referred to as *Helmholtz conditions*, are of the form

$$\begin{aligned} &B_{\sigma\nu} = B_{\nu\sigma}, \quad \frac{\partial B_{\sigma\nu}}{\partial \dot{q}^\rho} = \frac{\partial B_{\sigma\rho}}{\partial \dot{q}^\nu}, \\ &\frac{\partial A_\sigma}{\partial \dot{q}^\nu} + \frac{\partial A_\nu}{\partial \dot{q}^\sigma} = 2\frac{\bar{d} B_{\sigma\nu}}{dt} \\ &\frac{\partial A_\sigma}{\partial q^\nu} - \frac{\partial A_\nu}{\partial q^\sigma} = \frac{1}{2}\frac{\bar{d}}{dt}\Big(\frac{\partial A_\sigma}{\partial \dot{q}^\nu} - \frac{\partial A_\nu}{\partial \dot{q}^\sigma}\Big), \end{aligned} \tag{1.8}$$

where $1 \leqslant \sigma, \nu \leqslant m$, and the operator $\bar{d}/dt$ is defined by

$$\frac{\bar{d}}{dt} = \frac{\partial}{\partial t} + \dot{q}^\rho \frac{\partial}{\partial q^\rho}.$$

Helmholtz's result was completed in 1896 by Mayer who showed that the Helmholtz conditions (1.8) are also sufficient for (1.7) be variational (A. Mayer [1]).

In 1913 Volterra discovered that if the Helmholtz conditions are satisfied then the left-hand sides of (1.7) are Euler-Lagrange expressions of the Lagrange function

$$L = q^\sigma \int_0^1 E_\sigma(t, uq^\nu, u\dot{q}^\nu, u\ddot{q}^\nu)\, du. \tag{1.9}$$

Formula (1.9) was subsequently generalized by M. M. Vainberg [1] and E. Tonti [1]. Notice that the function L is of order two. It is called *Vainberg-Tonti Lagrangian* related to the variational expressions E_σ.

One can find the necessary and sufficient conditions of variationality very easily using methods of differential geometry. Let us recall here a proof due to O. Štěpánková [1], and L. Klapka [2].

For a system of equations (1.7) put

$$E_\sigma = A_\sigma + B_{\sigma\nu}\dot{q}^\nu,$$

and define a differential two-form E by

$$E = E_\sigma dq^\sigma \wedge dt.$$

Let us study a question under what conditions there exists a two-form F,

$$F = F_{\sigma\nu}(dq^\sigma - \dot{q}^\sigma dt) \wedge (dq^\nu - \dot{q}^\nu dt) + G_{\sigma\nu}(dq^\sigma - \dot{q}^\sigma dt) \wedge (d\dot{q}^\nu - \ddot{q}^\nu dt) \quad (1.10)$$

such that the form

$$\alpha = E + F$$

is closed. We may suppose that the functions $F_{\sigma\nu}$ are antisymmetric in σ, ν, i.e., that $F_{\sigma\nu} = -F_{\nu\sigma}$. Denote by d/dt the total derivative operator,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{q}^\rho \frac{\partial}{\partial q^\rho} + \ddot{q}^\rho \frac{\partial}{\partial \dot{q}^\rho} + \dddot{q}^\rho \frac{\partial}{\partial \ddot{q}^\rho}.$$

Computing the condition $d\alpha = 0$ we get

$$\begin{aligned} &\frac{1}{2}\left(\frac{\partial E_\sigma}{\partial q^\nu} - \frac{\partial E_\nu}{\partial q^\sigma}\right) - \frac{dF_{\sigma\nu}}{dt} = 0, \\ &\frac{\partial E_\sigma}{\partial \dot{q}^\nu} - 2F_{\sigma\nu} - \frac{dG_{\sigma\nu}}{dt} = 0, \\ &\frac{\partial E_\sigma}{\partial \ddot{q}^\nu} - G_{\sigma\nu} = 0, \quad G_{\sigma\nu} = G_{\nu\sigma}, \end{aligned} \quad (1.11)$$

and since the identities (1.11) imply that

$$\frac{\partial F_{\sigma\nu}}{\partial \ddot{q}^\rho} = 0, \quad \frac{\partial G_{\sigma\nu}}{\partial \ddot{q}^\rho} = 0, \quad (1.12)$$

the remaining identities following from $d\alpha = 0$ reduce to

$$\begin{aligned} \frac{\partial F_{\sigma\nu}}{\partial q^\rho} + \frac{\partial F_{\rho\sigma}}{\partial q^\nu} + \frac{\partial F_{\nu\rho}}{\partial q^\sigma} &= 0, \\ \frac{\partial F_{\sigma\nu}}{\partial \dot{q}^\rho} + \frac{1}{2}\left(\frac{\partial G_{\nu\rho}}{\partial q^\sigma} - \frac{\partial G_{\sigma\rho}}{\partial q^\nu}\right) &= 0, \\ \frac{\partial G_{\sigma\nu}}{\partial \dot{q}^\rho} - \frac{\partial G_{\sigma\rho}}{\partial \dot{q}^\nu} &= 0 \end{aligned} \quad (1.13)$$

Now we have from (1.11)

$$G_{\sigma\nu} = \frac{1}{2}\left(\frac{\partial E_\sigma}{\partial \ddot{q}^\nu} + \frac{\partial E_\nu}{\partial \ddot{q}^\sigma}\right), \quad F_{\sigma\nu} = \frac{1}{4}\left(\frac{\partial E_\sigma}{\partial \dot{q}^\nu} - \frac{\partial E_\nu}{\partial \dot{q}^\sigma}\right), \quad (1.14)$$

and

$$\begin{aligned} \frac{\partial E_\sigma}{\partial \ddot{q}^\nu} - \frac{\partial E_\nu}{\partial \ddot{q}^\sigma} &= 0, \\ \frac{\partial E_\sigma}{\partial \dot{q}^\nu} + \frac{\partial E_\nu}{\partial \dot{q}^\sigma} - \frac{d}{dt}\left(\frac{\partial E_\sigma}{\partial \ddot{q}^\nu} + \frac{\partial E_\nu}{\partial \ddot{q}^\sigma}\right) &= 0, \\ \frac{\partial E_\sigma}{\partial q^\nu} - \frac{\partial E_\nu}{\partial q^\sigma} - \frac{1}{2}\frac{d}{dt}\left(\frac{\partial E_\sigma}{\partial \dot{q}^\nu} - \frac{\partial E_\nu}{\partial \dot{q}^\sigma}\right) &= 0. \end{aligned} \quad (1.15)$$

It is easy to see that relations (1.13) do not represent independent conditions on the functions $F_{\sigma\nu}$, $G_{\sigma\nu}$. Indeed, the third relation of (1.13) is obtained by differentiating the second relation of (1.11) by $\ddot{q}^\rho$, the second relation of (1.13) is obtained by differentiating the first relation of (1.11) by $\ddot{q}^\rho$. Finally, we show that the first relation of (1.13) is satisfied identically. Differentiating the last relation of (1.15) by $\dot{q}^\rho$, rotating the indices and summing up the resulting three identities we get

$$\frac{\partial^2 E_\sigma}{\partial q^\rho \partial \dot{q}^\nu} - \frac{\partial^2 E_\nu}{\partial q^\rho \partial \dot{q}^\sigma} + \frac{\partial^2 E_\rho}{\partial q^\nu \partial \dot{q}^\sigma} - \frac{\partial^2 E_\sigma}{\partial q^\nu \partial \dot{q}^\rho} + \frac{\partial^2 E_\nu}{\partial q^\sigma \partial \dot{q}^\rho} - \frac{\partial^2 E_\rho}{\partial q^\sigma \partial \dot{q}^\nu} = 0.$$

Applying (1.14) the first relation of (1.13) follows.

Summarizing the results, we have proved the following assertion:

If the E_σ satisfy the conditions (1.15) *then there exits a unique form F* (1.10) *such that the two-form $\alpha = E + F$ is closed. α is then expressed by*

$$\begin{aligned} \alpha = {} & E_\sigma dq^\sigma \wedge dt \\ & + \frac{1}{4}\Big(\frac{\partial E_\sigma}{\partial \dot{q}^\nu} - \frac{\partial E_\nu}{\partial \dot{q}^\sigma}\Big)(dq^\sigma - \dot{q}^\sigma dt) \wedge (dq^\nu - \dot{q}^\nu dt) \\ & + \frac{\partial E_\sigma}{\partial \ddot{q}^\nu}(dq^\sigma - \dot{q}^\sigma dt) \wedge (d\dot{q}^\nu - \ddot{q}^\nu dt). \end{aligned} \tag{1.16}$$

Conversely, if there exists a two-form F (1.10) *such that $E + F$ is closed then E_σ satisfy the conditions* (1.15).

We are ready to prove *the equivalence of the conditions* (1.15) *with variationality of E_σ*.

Suppose that E_σ satisfy the conditions (1.15). We can take the closed two-form α (1.16), and using the Poincaré Lemma we find a local one-form θ such that $\alpha = d\theta$. Denote by χ the mapping $(u, (t, q^\nu, \dot{q}^\nu, \ddot{q}^\nu)) \to (t, uq^\nu, u\dot{q}^\nu, u\ddot{q}^\nu)$ for $u \in [1, 0]$, and $(t, q^\nu, \dot{q}^\nu, \ddot{q}^\nu) \in V$, where V is a convex open set. We have

$$\begin{aligned} \theta = {} & \Big(q^\sigma \int_0^1 (E_\sigma \circ \chi)\, du\Big) dt \\ & + \Big(2q^\sigma \int_0^1 (F_{\sigma\nu} \circ \chi)\, u\, du + \dot{q}^\sigma \int_0^1 (G_{\sigma\nu} \circ \chi)\, u\, du\Big)(dq^\nu - \dot{q}^\nu dt), \end{aligned} \tag{1.17}$$

where $F_{\sigma\nu}$ and $G_{\sigma\nu}$ are given by (1.14). We put

$$L = q^\sigma \int_0^1 (E_\sigma \circ \chi)\, du. \tag{1.18}$$

Since L is a second order Lagrange function, the Euler-Lagrange expressions of L are

$$E_\sigma(L) = \frac{\partial L}{\partial q^\sigma} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\sigma} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}^\sigma}.$$

Now,

$$\begin{aligned} \frac{\partial L}{\partial q^\sigma} &= \int_0^1 (E_\sigma \circ \chi)\, du + q^\nu \int_0^1 \Big(\frac{\partial E_\nu}{\partial q^\sigma} \circ \chi\Big)\, u\, du, \\ \frac{\partial L}{\partial \dot{q}^\sigma} &= q^\nu \int_0^1 \Big(\frac{\partial E_\nu}{\partial \dot{q}^\sigma} \circ \chi\Big)\, u\, du, \\ \frac{\partial L}{\partial \ddot{q}^\sigma} &= q^\nu \int_0^1 \Big(\frac{\partial E_\nu}{\partial \ddot{q}^\sigma} \circ \chi\Big)\, u\, du. \end{aligned}$$

Using the formula

$$\begin{aligned} E_\sigma &= \int_0^1 d\big((E_\sigma \circ \chi)\,u\big) \\ &= \int_0^1 (E_\sigma \circ \chi)\,du + q^\nu \int_0^1 \Big(\frac{\partial E_\sigma}{\partial q^\nu} \circ \chi\Big)\,u\,du \\ &\quad + \dot{q}^\nu \int_0^1 \Big(\frac{\partial E_\sigma}{\partial \dot{q}^\nu} \circ \chi\Big)\,u\,du + \ddot{q}^\nu \int_0^1 \Big(\frac{\partial E_\sigma}{\partial \ddot{q}^\nu} \circ \chi\Big)\,u\,du, \end{aligned}$$

and conditions (1.15), we get by a direct computation that $E_\sigma - E_\sigma(L) = 0$, proving that the E_σ are Euler-Lagrange expressions of the Lagrange function (1.18).

Conversely, suppose that E_σ are variational, let L be a Lagrange function for E_σ. Put $E = E_\sigma(L)\,dq^\sigma \wedge dt$ and

$$\begin{aligned} F_{\sigma\nu} &= \frac{1}{4}\Big(\frac{\partial E_\sigma(L)}{\partial \dot{q}^\nu} - \frac{\partial E_\nu(L)}{\partial \dot{q}^\sigma}\Big) = \frac{1}{2}\Big(\frac{\partial^2 L}{\partial q^\sigma \partial \dot{q}^\nu} - \frac{\partial^2 L}{\partial q^\nu \partial \dot{q}^\sigma}\Big), \\ G_{\sigma\nu} &= \frac{1}{2}\Big(\frac{\partial E_\sigma}{\partial \ddot{q}^\nu} + \frac{\partial E_\nu}{\partial \ddot{q}^\sigma}\Big) = \frac{\partial^2 L}{\partial \dot{q}^\sigma \partial \dot{q}^\nu}. \end{aligned}$$

Now it is easy to show that $\alpha = E + F$ is closed. By the above proposition, E_σ satisfy (1.15).

Note that $\alpha = d\theta_L$ where

$$\theta_L = L\,dt + \frac{\partial L}{\partial \dot{q}^\sigma}(dq^\sigma - \dot{q}^\sigma\,dt)$$

is the famous 1-form, usually called *Cartan form*. This form was introduced to the calculus of variations independently by E. T. Whittaker [1] and É. Cartan [1].

Although not apparent at a first sight, conditions (1.15) represent an *equivalent* form of the *Helmholtz conditions* (1.8). To check this, it is sufficient to substitute into (1.15) the relation $E_\sigma = A_\sigma + B_{\sigma\nu}\ddot{q}^\nu$. Then one gets the identities (1.15) in the form (1.8) plus one additional identity,

$$\frac{\partial B_{\sigma\rho}}{\partial q^\nu} - \frac{\partial B_{\nu\rho}}{\partial q^\sigma} = \frac{1}{2}\frac{\partial}{\partial \dot{q}^\rho}\Big(\frac{\partial A_\sigma}{\partial \dot{q}^\nu} - \frac{\partial A_\nu}{\partial \dot{q}^\sigma}\Big), \tag{1.19}$$

for all σ, ν, ρ. The latter identity, however, is not independent. To see this, let us denote

$$\phi_{\sigma\nu} = \frac{1}{2}\Big(\frac{\partial A_\sigma}{\partial \dot{q}^\nu} - \frac{\partial A_\nu}{\partial \dot{q}^\sigma}\Big), \quad \psi_{\sigma\nu} = \frac{1}{2}\Big(\frac{\partial A_\sigma}{\partial \dot{q}^\nu} + \frac{\partial A_\nu}{\partial \dot{q}^\sigma}\Big).$$

Evidently, the functions $\phi_{\sigma\nu}$ and $\psi_{\sigma\nu}$ identically obey the relation

$$\frac{\partial \phi_{\sigma\nu}}{\partial \dot{q}^\rho} = \frac{\partial \psi_{\rho\sigma}}{\partial \dot{q}^\nu} - \frac{\partial \psi_{\rho\nu}}{\partial \dot{q}^\sigma}. \tag{1.20}$$

Now, the third condition of (1.8) in terms of $\psi_{\sigma\nu}$ reads

$$\psi_{\sigma\nu} = \frac{\bar{d}B_{\sigma\nu}}{dt}$$

which implies that

$$\frac{\partial \psi_{\sigma\nu}}{\partial \dot{q}^\rho} = \frac{\partial B_{\sigma\nu}}{\partial q^\rho} + \frac{\bar{d}}{dt}\frac{\partial B_{\sigma\nu}}{\partial \dot{q}^\rho}.$$

Using the first and second identity of (1.8) we get

$$\frac{\partial \psi_{\rho\sigma}}{\partial \dot{q}^{\nu}} - \frac{\partial \psi_{\rho\nu}}{\partial \dot{q}^{\sigma}} = \frac{\partial B_{\rho\sigma}}{\partial q^{\nu}} + \frac{\bar{d}}{dt}\frac{\partial B_{\rho\sigma}}{\partial \dot{q}^{\nu}} - \frac{\partial B_{\rho\nu}}{\partial q^{\sigma}} - \frac{\bar{d}}{dt}\frac{\partial B_{\rho\nu}}{\partial \dot{q}^{\sigma}} = \frac{\partial B_{\sigma\rho}}{\partial q^{\nu}} - \frac{\partial B_{\nu\rho}}{\partial q^{\sigma}}.$$

Hence, the relation (1.19) identifies with (1.20).

The inverse problem originally posed by Helmholtz is naturally generalized to the following question, which also is referred to as *inverse problem of the calculus of variations*: *Under what conditions equations* (1.7) *are equivalent with some Euler-Lagrange equations*? In other words, one asks whether there exist Euler-Lagrange equations the solutions of which coincide with the solutions of equations (1.7). In this formulation the problem is very general and its solution is yet not known. However, one can simplify this problem restricting the class of equivalent equations, namely, to equations of the form

$$f_{\rho}^{\sigma}(A_{\sigma} + B_{\sigma\nu}\ddot{q}^{\nu}) = 0, \tag{1.21}$$

where f_{ρ}^{σ} are some functions of $(t, q^{\mu}, \dot{q}^{\mu})$. Evidently, equations (1.7) and (1.21) will have the same solutions in case that the matrix (f_{ρ}^{σ}) is everywhere regular. The inverse problem then takes the following formulation:

Given equations (1.7), *find out whether there exists an everywhere regular matrix* (f_{ρ}^{σ}) *such that the left-hand sides of* (1.21) *indentify with the Euler-Lagrange expressions of a Lagrange function.*

Such a matrix (f_{ρ}^{σ}) is then called a *variational integrating factor*, or *variational multiplier* for the equations (1.7).

It is clear that to solve this problem one can apply Helmoltz conditions (1.8) resp. (1.15). Putting $E'_{\rho} = f_{\rho}^{\sigma}(A_{\sigma} + B_{\sigma\nu}\ddot{q}^{\nu})$ and applying Helmholtz conditions to E'_{ρ} one obtains a system of first order partial differential equations for the functions f_{ρ}^{σ}. Although some concrete examples can be solved in this way, for a general analysis these equations are very complicated. This is why the problem has been extensively studied by different methods (see e.g. N. Ya. Sonin [1], G. Darboux [1], J. Douglas [1], P. Havas [1], W. Sarlet [1,2,3], M. Henneaux [1,2], I. Anderson and G. Thompson [1], M. Crampin, W. Sarlet, E. Martínez, G. B. Byrnes, and G. E. Prince [1], and many others).

However, the situation extremely simplifies if $m = 1$, i.e., if variational integrating factors for *one* ordinary (nondegenerate) second-order differential equation are studied. In this case one can consider the equation in the form

$$\ddot{q} - g = 0, \tag{1.22}$$

and the Helmholtz conditions (1.15) applied to $f(\ddot{q} - g)$ reduce to a single equation

$$\frac{\partial f}{\partial \dot{q}}g + f\frac{\partial g}{\partial \dot{q}} + \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q}\dot{q} = 0$$

for $f(t, q, \dot{q}) \neq 0$, which is equivalent to

$$\frac{\partial \ln f}{\partial t} + \frac{\partial \ln f}{\partial q}\dot{q} + \frac{\partial \ln f}{\partial \dot{q}}g + \frac{\partial g}{\partial \dot{q}} = 0.$$

The general solution of such equations depends upon a single arbitrary function of any two specific solutions of the corresponding homogeneous equation. Consequently the