SCHAUM'S OUTLINE OF

THEORY AND PROBLEMS

OF

COMPLEX VARIABLES

BY

MURRAY R. SPIEGEL, Ph.D.

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THEORY and PROBLEMS of

COMPLEX VARIABLES

with an introduction to

CONFORMAL MAPPING and its applications

by

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Preface

The theory of functions of a complex variable, also called for brevity complex variables or complex analysis, is one of the most beautiful as well as useful branches of mathematics. Although originating in an atmosphere of mystery, suspicion and distrust, as evidenced by the terms "imaginary" and "complex" present in the literature, it was finally placed on a sound foundation in the 19th century through the efforts of Cauchy, Riemann, Weierstrass, Gauss and other great mathematicians.

Today the subject is recognized as an essential part of the mathematical background of engineers, physicists, mathematicians and other scientists. From the theoretical viewpoint this is because many mathematical concepts become clarified and unified when examined in the light of complex variable theory. From the applied viewpoint the theory is of tremendous value in the solution of problems of heat flow, potential theory, fluid mechanics, electromagnetic theory, aerodynamics, elasticity and many other fields of science and engineering.

This book is designed for use as a supplement to all current standard texts or as a textbook for a formal course in complex variable theory and applications. It should also be of considerable value to those taking courses in mathematics, physics, aerodynamics, elasticity or any of the numerous other fields in which complex variable methods are employed.

Each chapter begins with a clear statement of pertinent definitions, principles and theorems together with illustrative and other descriptive material. This is followed by graded sets of solved and supplementary problems. The solved problems serve to illustrate and amplify the theory, bring into sharp focus those fine points without which the student continually feels himself on unsafe ground, and provide the repetition of basic principles so vital to effective learning. Numerous proofs of theorems and derivations of formulas are included among the solved problems. The large number of supplementary problems with answers serve as a complete review of the material in each chapter.

Topics covered include the algebra and geometry of complex numbers, complex differential and integral calculus, infinite series including Taylor and Laurent series, the theory of residues with applications to the evaluation of integrals and series, and conformal mapping with applications drawn from various fields. An added feature is the chapter on special topics which should prove useful as an introduction to some more advanced topics.

Considerably more material has been included here than can be covered in most first courses. This has been done to make the book more flexible, to provide a more useful book of reference and to stimulate further interest in the topics.

I wish to take this opportunity to thank the staff of the Schaum Publishing Company for their splendid cooperation.

M. R. SPIEGEL

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Chapter 1

Complex Numbers

THE REAL NUMBER SYSTEM

The number system as we know it today is a result of gradual development as indicated in the following list.

1. **Natural numbers** $1, 2, 3, 4, \ldots,$ also called *positive integers,* were first used in counting. The symbols varied with the times, e.g. the Romans used I, II, III, IV, ... If a and b are natural numbers, the *sum* $a+b$ and *product* $a \cdot b$, $(a)(b)$ or ab are also natural numbers. For this reason the set of natural numbers is said to be *closed* under the operations of *addition* and *multiplication* or to satisfy the *closure property* with respect to these operations.

2. **Negative integers and zero,** denoted by $-1, -2, -3, \ldots$ and 0 respectively, arose to permit solutions of equations such as $x + b = a$ where a and b are any natural numbers. This leads to the operation of *subtraction,* or *inverse of addition,* and we write $x = a - b$.

   The set of positive and negative integers and zero is called the set of *integers* and is closed under the operations of addition, multiplication and subtraction.

3. **Rational numbers** or *fractions* such as $\frac{3}{4}, -\frac{8}{3}, \ldots$ arose to permit solutions of equations such as $bx = a$ for all integers a and b where $b \neq 0$. This leads to the operation of *division* or *inverse of multiplication,* and we write $x = a/b$ or $a \div b$ [called the *quotient* of a and b] where a is the *numerator* and b is the *denominator.*

   The set of integers is a part or *subset* of the rational numbers, since integers correspond to rational numbers a/b where $b = 1$.

   The set of rational numbers is closed under the operations of addition, subtraction, multiplication and division, so long as division by zero is excluded.

4. **Irrational numbers** such as $\sqrt{2} = 1.41423\cdots$ and $\pi = 3.14159\cdots$ are numbers which are not rational, i.e. cannot be expressed as a/b where a and b are integers and $b \neq 0$.

The set of rational and irrational numbers is called the set of *real* numbers. It is assumed that the student is already familiar with the various operations on real numbers.

GRAPHICAL REPRESENTATION OF REAL NUMBERS

Real numbers can be represented by points on a line called the *real axis,* as indicated in Fig. 1-1. The point corresponding to zero is called the *origin.*

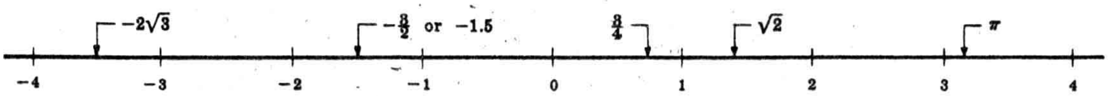

Fig. 1-1

Conversely, to each point on the line there is one and only one real number. If a point A corresponding to a real number a lies to the right of a point B corresponding to a real number b, we say that a is *greater than* b or b is *less than* a and write respectively $a > b$ or $b < a$.

The set of all values of x such that $a < x < b$ is called an *open interval* on the real axis while $a \leqq x \leqq b$, which also includes the endpoints a and b, is called a *closed interval.* The symbol x, which can stand for any of a set of real numbers, is called a *real variable.*

The *absolute value* of a real number a, denoted by $|a|$, is equal to a if $a > 0$, to $-a$ if $a < 0$ and to 0 if $a = 0$. The distance between two points a and b on the real axis is $|a - b|$.

THE COMPLEX NUMBER SYSTEM

There is no real number x which satisfies the polynomial equation $x^2 + 1 = 0$. To permit solutions of this and similar equations, the set of *complex numbers* is introduced.

We can consider a *complex number* as having the form $a + bi$ where a and b are real numbers and i, which is called the *imaginary unit,* has the property that $i^2 = -1$. If $z = a + bi$, then a is called the *real part* of z and b is called the *imaginary part* of z and are denoted by $\text{Re}\{z\}$ and $\text{Im}\{z\}$ respectively. The symbol z, which can stand for any of a set of complex numbers, is called a *complex variable.*

Two complex numbers $a + bi$ and $c + di$ are *equal* if and only if $a = c$ and $b = d$. We can consider real numbers as a subset of the set of complex numbers with $b = 0$. Thus the complex numbers $0 + 0i$ and $-3 + 0i$ represent the real numbers 0 and -3 respectively. If $a = 0$, the complex number $0 + bi$ or bi is called a *pure imaginary number.*

The *complex conjugate,* or briefly *conjugate,* of a complex number $a + bi$ is $a - bi$. The complex conjugate of a complex number z is often indicated by $\bar{z}$ or z^*.

FUNDAMENTAL OPERATIONS WITH COMPLEX NUMBERS

In performing operations with complex numbers we can proceed as in the algebra of real numbers, replacing i^2 by -1 when it occurs.

1. *Addition*
$$(a + bi) + (c + di) = a + bi + c + di = (a + c) + (b + d)i$$
2. *Subtraction*
$$(a + bi) - (c + di) = a + bi - c - di = (a - c) + (b - d)i$$
3. *Multiplication*
$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$$
4. *Division*
$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{ac - adi + bci - bdi^2}{c^2 - d^2 i^2}$$
$$= \frac{ac + bd + (bc - ad)i}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}\, i$$

ABSOLUTE VALUE

The *absolute value* or *modulus* of a complex number $a + bi$ is defined as $|a + bi| = \sqrt{a^2 + b^2}$.

Example: $|-4 + 2i| = \sqrt{(-4)^2 + (2)^2} = \sqrt{20} = 2\sqrt{5}$

If $z_1, z_2, z_3, \ldots, z_m$ are complex numbers, the following properties hold.

1. $|z_1 z_2| = |z_1||z_2|$ or $|z_1 z_2 \cdots z_m| = |z_1||z_2| \cdots |z_m|$
2. $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ if $z_2 \neq 0$
3. $|z_1 + z_2| \leqq |z_1| + |z_2|$ or $|z_1 + z_2 + \cdots + z_m| \leqq |z_1| + |z_2| + \cdots + |z_m|$
4. $|z_1 + z_2| \geqq |z_1| - |z_2|$ or $|z_1 - z_2| \geqq |z_1| - |z_2|$

AXIOMATIC FOUNDATIONS OF THE COMPLEX NUMBER SYSTEM

From a strictly logical point of view it is desirable to define a complex number as an ordered pair (a, b) of real numbers a and b subject to certain operational definitions which turn out to be equivalent to those above. These definitions are as follows, where all letters represent real numbers.

A. **Equality** $(a, b) = (c, d)$ if and only if $a = c,\ b = d$

B. **Sum** $(a, b) + (c, d) = (a + c,\ b + d)$

C. **Product** $(a, b) \cdot (c, d) = (ac - bd,\ ad + bc)$

$m(a, b) = (ma, mb)$

From these we can show [Problem 14] that $(a, b) = a(1, 0) + b(0, 1)$ and we associate this with $a + bi$ where i is the symbol for $(0, 1)$ and has the property that $i^2 = (0, 1)(0, 1) = (-1, 0)$ [which can be considered equivalent to the real number -1] and $(1, 0)$ can be considered equivalent to the real number 1. The ordered pair $(0, 0)$ corresponds to the real number 0.

From the above we can prove that if z_1, z_2, z_3 belong to the set S of complex numbers, then

1. $z_1 + z_2$ and $z_1 z_2$ belong to S — Closure law
2. $z_1 + z_2 = z_2 + z_1$ — Commutative law of addition
3. $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ — Associative law of addition
4. $z_1 z_2 = z_2 z_1$ — Commutative law of multiplication
5. $z_1(z_2 z_3) = (z_1 z_2) z_3$ — Associative law of multiplication
6. $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ — Distributive law
7. $z_1 + 0 = 0 + z_1 = z_1,\quad 1 \cdot z_1 = z_1 \cdot 1 = z_1$, 0 is called the *identity with respect to addition*, 1 is called the *identity with respect to multiplication*.
8. For any complex number z_1 there is a unique number z in S such that $z + z_1 = 0$; z is called the *inverse of z_1 with respect to addition* and is denoted by $-z_1$.
9. For any $z_1 \neq 0$ there is a unique number z in S such that $z_1 z = z z_1 = 1$; z is called the *inverse of z_1 with respect to multiplication* and is denoted by z_1^{-1} or $1/z_1$.

In general any set, such as S, whose members satisfy the above is called a *field*.

GRAPHICAL REPRESENTATION OF COMPLEX NUMBERS

If real scales are chosen on two mutually perpendicular axes $X'OX$ and $Y'OY$ [called the *x and y axes* respectively] as in Fig. 1-2, we can locate any point, in the plane determined by these lines, by the ordered pair of real numbers (x, y) called *rectangular coordinates* of the point. Examples of the location of such points are indicated by P, Q, R, S and T in Fig. 1-2.

Since a complex number $x + iy$ can be considered as an ordered pair of real numbers, we can represent such numbers by points in an xy plane called the *complex plane* or *Argand diagram*. The complex number represented by P, for example, could then be read as either $(3, 4)$ or $3 + 4i$. To each complex number there corresponds one and only one point in the plane, and conversely to each point in the plane there corresponds one and only one complex number. Because of this we often refer to the complex number z as the *point* z.

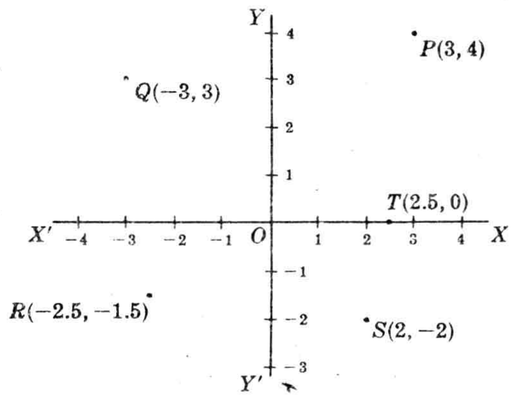

Fig. 1-2

Sometimes we refer to the x and y axes as the *real* and *imaginary* axes respectively and to the complex plane as the *z plane*. The distance between two points $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ in the complex plane is given by $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

POLAR FORM OF COMPLEX NUMBERS

If P is a point in the complex plane corresponding to the complex number (x, y) or $x + iy$, then we see from Fig. 1-3 that

$$x = r \cos \theta, \qquad y = r \sin \theta$$

where $r = \sqrt{x^2 + y^2} = |x + iy|$ is called the *modulus* or *absolute value* of $z = x + iy$ [denoted by *mod* z or $|z|$]; and θ, called the *amplitude* or *argument* of $z = x + iy$ [denoted by *arg* z], is the angle which line OP makes with the positive x axis.

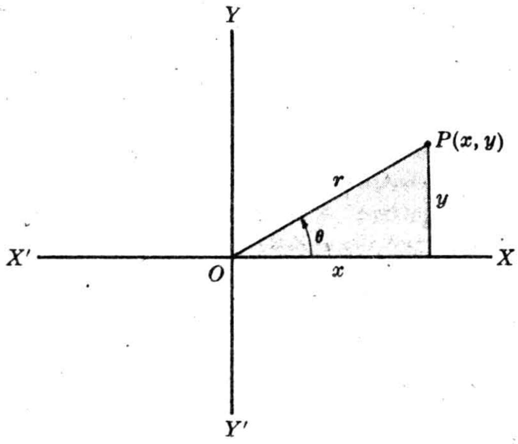

Fig. 1-3

It follows that

$$z = x + iy = r(\cos \theta + i \sin \theta) \qquad (1)$$

which is called the *polar form* of the complex number, and r and θ are called *polar coordinates*. It is sometimes convenient to write the abbreviation cis θ for $\cos \theta + i \sin \theta$.

For any complex number $z \neq 0$ there corresponds only one value of θ in $0 \leqq \theta < 2\pi$. However, any other interval of length 2π, for example $-\pi < \theta \leqq \pi$, can be used. Any particular choice, decided upon in advance, is called the *principal range*, and the value of θ is called its *principal value*.

DE MOIVRE'S THEOREM

If $z_1 = x_1 + iy_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$ and $z_2 = x_2 + iy_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$, we can show that [see Problem 19]

$$z_1 z_2 = r_1 r_2 \{\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)\} \qquad (2)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \{\cos (\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2)\} \qquad (3)$$

A generalization of (2) leads to

$$z_1 z_2 \cdots z_n = r_1 r_2 \cdots r_n \{\cos (\theta_1 + \theta_2 + \cdots + \theta_n) + i \sin (\theta_1 + \theta_2 + \cdots + \theta_n)\} \qquad (4)$$

and if $z_1 = z_2 = \cdots = z_n = z$ this becomes

$$z^n = \{r(\cos \theta + i \sin \theta)\}^n = r^n (\cos n\theta + i \sin n\theta) \qquad (5)$$

which is often called *De Moivre's theorem*.

ROOTS OF COMPLEX NUMBERS

A number w is called an *nth root* of a complex number z if $w^n = z$, and we write $w = z^{1/n}$. From De Moivre's theorem we can show that if n is a positive integer,

$$\begin{aligned} z^{1/n} &= \{r(\cos \theta + i \sin \theta)\}^{1/n} \\ &= r^{1/n} \left\{\cos \left(\frac{\theta + 2k\pi}{n}\right) + i \sin \left(\frac{\theta + 2k\pi}{n}\right)\right\} \qquad k = 0, 1, 2, \ldots, n-1 \end{aligned} \qquad (6)$$

from which it follows that there are n different values for $z^{1/n}$, i.e. n different nth roots of z, provided $z \neq 0$.

EULER'S FORMULA

By assuming that the infinite series expansion $e^x = 1 + x + x^2/2! + x^3/3! + \cdots$ of elementary calculus holds when $x = i\theta$, we can arrive at the result

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad e = 2.71828\ldots \tag{7}$$

which is called *Euler's formula*. It is more convenient, however, simply to take (7) as a definition of $e^{i\theta}$. In general, we define

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y) \tag{8}$$

In the special case where $y = 0$ this reduces to e^x.

Note that in terms of (7) De Moivre's theorem essentially reduces to $(e^{i\theta})^n = e^{in\theta}$.

POLYNOMIAL EQUATIONS

Often in practice we require solutions of polynomial equations having the form

$$a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n = 0 \tag{9}$$

where $a_0 \neq 0, a_1, \ldots, a_n$ are given complex numbers and n is a positive integer called the *degree* of the equation. Such solutions are also called *zeros* of the polynomial on the left of (9) or *roots of the equation.*

A very important theorem called the *fundamental theorem of algebra* [to be proved in Chapter 5] states that every polynomial equation of the form (9) has at least one root which is complex. From this we can show that it has in fact n complex roots, some or all of which may be identical.

If $z_1, z_2, \ldots, z_n$ are the n roots, (9) can be written

$$a_0(z - z_1)(z - z_2)\cdots(z - z_n) = 0 \tag{10}$$

which is called the *factored form* of the polynomial equation. Conversely if we can write (9) in the form (10), we can easily determine the roots.

THE *n*th ROOTS OF UNITY

The solutions of the equation $z^n = 1$ where n is a positive integer are called the nth *roots of unity* and are given by

$$z = \cos 2k\pi/n + i\sin 2k\pi/n = e^{2k\pi i/n} \qquad k = 0, 1, 2, \ldots, n-1 \tag{11}$$

If we let $\omega = \cos 2\pi/n + i\sin 2\pi/n = e^{2\pi i/n}$, the n roots are $1, \omega, \omega^2, \ldots, \omega^{n-1}$. Geometrically they represent the n vertices of a regular polygon of n sides inscribed in a circle of radius one with center at the origin. This circle has the equation $|z| = 1$ and is often called the *unit circle.*

VECTOR INTERPRETATION OF COMPLEX NUMBERS

A complex number $z = x + iy$ can be considered as a vector OP whose *initial point* is the origin O and whose *terminal point* P is the point (x, y) as in Fig. 1-4. We sometimes call $OP = x + iy$ the *position vector* of P. Two vectors having the same *length* or *magnitude* and *direction* but different initial points, such as OP and AB in Fig. 1-4, are considered equal. Hence we write $OP = AB = x + iy$.

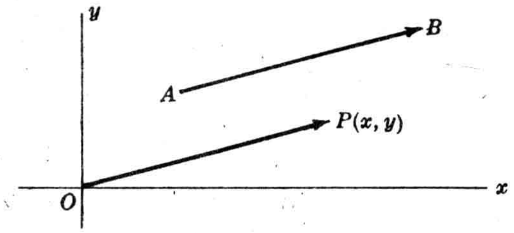

Fig. 1-4

Addition of complex numbers corresponds to the *parallelogram law* for addition of vectors [see Fig. 1-5]. Thus to add the complex numbers z_1 and z_2, we complete the parallelogram $OABC$ whose sides OA and OC correspond to z_1 and z_2. The diagonal OB of this parallelogram corresponds to $z_1 + z_2$. See Problem 5.

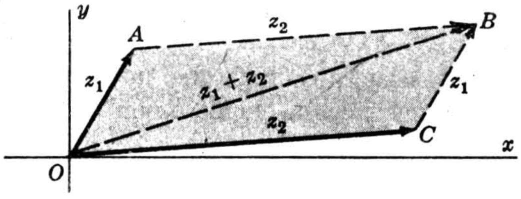

Fig. 1-5

SPHERICAL REPRESENTATION OF COMPLEX NUMBERS. STEREOGRAPHIC PROJECTION

Let $\mathcal{P}$ [Fig. 1-6] be the complex plane and consider a unit sphere $\mathcal{S}$ [radius one] tangent to $\mathcal{P}$ at $z = 0$. The diameter NS is perpendicular to $\mathcal{P}$ and we call points N and S the *north* and *south poles* of $\mathcal{S}$. Corresponding to any point A on $\mathcal{P}$ we can construct line NA intersecting $\mathcal{S}$ at point A'. Thus to each point of the complex plane $\mathcal{P}$ there corresponds one and only one point of the sphere $\mathcal{S}$, and we can represent any complex number by a point on the sphere. For completeness we say that the point N itself corresponds to the "point at infinity" of the plane. The set of all points of the complex plane including the point at infinity is called the *entire complex plane*, the *entire z plane*, or the *extended complex plane.*

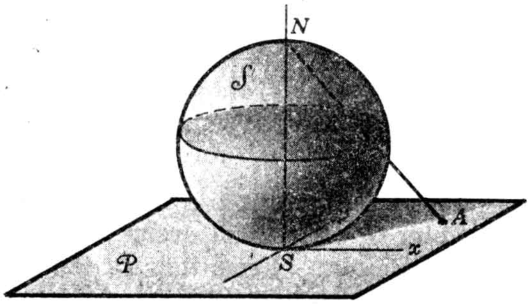

Fig. 1-6

The above method for mapping the plane on to the sphere is called *stereographic projection.* The sphere is sometimes called the *Riemann sphere.*

DOT AND CROSS PRODUCT

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers [vectors]. The *dot product* [also called the *scalar product*] of z_1 and z_2 is defined by

$$z_1 \circ z_2 = |z_1|\,|z_2| \cos\theta = x_1x_2 + y_1y_2 = \text{Re}\{\bar{z}_1 z_2\} = \tfrac{1}{2}\{\bar{z}_1 z_2 + z_1\bar{z}_2\} \qquad (12)$$

where θ is the angle between z_1 and z_2 which lies between 0 and π.

The *cross product* of z_1 and z_2 is defined by

$$z_1 \times z_2 = |z_1|\,|z_2| \sin\theta = x_1y_2 - y_1x_2 = \text{Im}\{\bar{z}_1 z_2\} = \frac{1}{2i}\{\bar{z}_1 z_2 - z_1\bar{z}_2\} \qquad (13)$$

Clearly,

$$\bar{z}_1 z_2 = (z_1 \circ z_2) + i(z_1 \times z_2) = |z_1|\,|z_2|\, e^{i\theta} \qquad (14)$$

If z_1 and z_2 are non-zero, then

1. A necessary and sufficient condition that z_1 and z_2 be perpendicular is that $z_1 \circ z_2 = 0$.
2. A necessary and sufficient condition that z_1 and z_2 be parallel is that $z_1 \times z_2 = 0$.
3. The magnitude of the projection of z_1 on z_2 is $|z_1 \circ z_2|/|z_2|$.
4. The area of a parallelogram having sides z_1 and z_2 is $|z_1 \times z_2|$.

COMPLEX CONJUGATE COORDINATES

A point in the complex plane can be located by rectangular coordinates (x, y) or polar coordinates (r, θ). Many other possibilities exist. One such possibility uses the fact that $x = \frac{1}{2}(z + \bar{z})$, $y = \frac{1}{2i}(z - \bar{z})$ where $z = x + iy$. The coordinates $(z, \bar{z})$ which locate a point are called *complex conjugate coordinates* or briefly *conjugate coordinates* of the point [see Problems 43 and 44].

POINT SETS

Any collection of points in the complex plane is called a *(two-dimensional)* point set, and each point is called a *member* or *element* of the set. The following fundamental definitions are given here for reference.

1. **Neighborhoods.** A *delta, or δ, neighborhood* of a point z_0 is the set of all points z such that $|z - z_0| < \delta$ where δ is any given positive number. A *deleted δ neighborhood* of z_0 is a neighborhood of z_0 in which the point z_0 is omitted, i.e. $0 < |z - z_0| < \delta$.

2. **Limit Points.** A point z_0 is called a *limit point, cluster point,* or *point of accumulation* of a point set S if every deleted δ neighborhood of z_0 contains points of S.

   Since δ can be any positive number, it follows that S must have infinitely many points. Note that z_0 may or may not belong to the set S.

3. **Closed Sets.** A set S is said to be *closed* if every limit point of S belongs to S, i.e. if S contains all its limit points. For example, the set of all points z such that $|z| \leqq 1$ is a closed set.

4. **Bounded Sets.** A set S is called *bounded* if we can find a constant M such that $|z| < M$ for every point z in S. An *unbounded set* is one which is not bounded. A set which is both bounded and closed is sometimes called *compact.*

5. **Interior, Exterior and Boundary Points.** A point z_0 is called an *interior point* of a set S if we can find a δ neighborhood of z_0 all of whose points belong to S. If every δ neighborhood of z_0 contains points belonging to S and also points not belonging to S, then z_0 is called a *boundary point.* If a point is not an interior or boundary point of a set S, it is an *exterior point* of S.

6. **Open Sets.** An *open set* is a set which consists only of interior points. For example, the set of points z such that $|z| < 1$ is an open set.

7. **Connected Sets.** An open set S is said to be *connected* if any two points of the set can be joined by a path consisting of straight line segments (i.e. a *polygonal path*) all points of which are in S.

8. **Open Regions or Domains.** An open connected set is called an *open region* or *domain.*

9. **Closure of a Set.** If to a set S we add all the limit points of S, the new set is called the *closure* of S and is a closed set.

10. **Closed Regions.** The closure of an open region or domain is called a *closed region.*

11. **Regions.** If to an open region or domain we add some, all or none of its limit points, we obtain a set called a *region.* If all the limit points are added, the region is *closed*; if none are added, the region is *open.* In this book whenever we use the word *region* without qualifying it, we shall mean *open region* or *domain.*

12. **Union and Intersection of Sets.** A set consisting of all points belonging to set S_1 or set S_2 or to both sets S_1 and S_2 is called the *union* of S_1 and S_2 and is denoted by $S_1 + S_2$ or $S_1 \cup S_2$.

    A set consisting of all points belonging to both sets S_1 and S_2 is called the *intersection* of S_1 and S_2 and is denoted by S_1S_2 or $S_1 \cap S_2$.

13. **Complement of a Set.** A set consisting of all points which do not belong to S is called the *complement* of S and is denoted by $\tilde{S}$.

14. **Null Sets and Subsets.** It is convenient to consider a set consisting of no points at all. This set is called the *null set* and is denoted by $\emptyset$. If two sets S_1 and S_2 have no points in common (in which case they are called *disjoint* or *mutually exclusive sets*), we can indicate this by writing $S_1 \cap S_2 = \emptyset$.

    Any set formed by choosing some, all or none of the points of a set S is called a *subset* of S. If we exclude the case where all points of S are chosen, the set is called a *proper subset* of S.

15. **Countability of a Set.** If the members or elements of a set can be placed into a one to one correspondence with the natural numbers $1, 2, 3, \ldots$, the set is called *countable* or *denumerable*; otherwise it is *non-countable* or *non-denumerable.*

The following are two important theorems on point sets.

1. **Weierstrass-Bolzano Theorem.** Every bounded infinite set has at least one limit point.

2. **Heine-Borel Theorem.** Let S be a compact set each point of which is contained in one or more of the open sets $A_1, A_2, \ldots$ [which are then said to *cover* S]. Then there exists a finite number of the sets $A_1, A_2, \ldots$ which will cover S.

Solved Problems

FUNDAMENTAL OPERATIONS WITH COMPLEX NUMBERS

1. Perform each of the indicated operations.

(a) $(3+2i) + (-7-i) = 3 - 7 + 2i - i = -4 + i$

(b) $(-7-i) + (3+2i) = -7 + 3 - i + 2i = -4 + i$

The results (a) and (b) illustrate the *commutative law of addition.*

(c) $(8-6i) - (2i-7) = 8 - 6i - 2i + 7 = 15 - 8i$

(d) $(5+3i) + \{(-1+2i) + (7-5i)\} = (5+3i) + \{-1+2i+7-5i\} = (5+3i) + (6-3i) = 11$

(e) $\{(5+3i) + (-1+2i)\} + (7-5i) = \{5+3i-1+2i\} + (7-5i) = (4+5i) + (7-5i) = 11$

The results (d) and (e) illustrate the *associative law of addition.*

(f) $(2-3i)(4+2i) = 2(4+2i) - 3i(4+2i) = 8 + 4i - 12i - 6i^2 = 8 + 4i - 12i + 6 = 14 - 8i$

(g) $(4+2i)(2-3i) = 4(2-3i) + 2i(2-3i) = 8 - 12i + 4i - 6i^2 = 8 - 12i + 4i + 6 = 14 - 8i$

The results (f) and (g) illustrate the *commutative law of multiplication.*

(h) $(2-i)\{(-3+2i)(5-4i)\} = (2-i)\{-15+12i+10i-8i^2\}$
$= (2-i)(-7+22i) = -14 + 44i + 7i - 22i^2 = 8 + 51i$

(i) $\{(2-i)(-3+2i)\}(5-4i) = \{-6+4i+3i-2i^2\}(5-4i)$
$= (-4+7i)(5-4i) = -20 + 16i + 35i - 28i^2 = 8 + 51i$

The results (h) and (i) illustrate the *associative law of multiplication.*

(j) $(-1+2i)\{(7-5i)+(-3+4i)\} = (-1+2i)(4-i) = -4+i+8i-2i^2 = -2+9i$

Another method. $(-1+2i)\{(7-5i)+(-3+4i)\} = (-1+2i)(7-5i)+(-1+2i)(-3+4i)$

$$= \{-7+5i+14i-10i^2\}+\{3-4i-6i+8i^2\}$$

$$= (3+19i)+(-5-10i) = -2+9i$$

This illustrates the *distributive law.*

(k) $$\frac{3-2i}{-1+i} = \frac{3-2i}{-1+i}\cdot\frac{-1-i}{-1-i} = \frac{-3-3i+2i+2i^2}{1-i^2} = \frac{-5-i}{2} = -\frac{5}{2}-\frac{1}{2}i$$

Another method. By definition, $(3-2i)/(-1+i)$ is that number $a+bi$, where a and b are real, such that $(-1+i)(a+bi) = -a-b+(a-b)i = 3-2i$. Then $-a-b=3$, $a-b=-2$ and solving simultaneously, $a=-5/2$, $b=-1/2$ or $a+bi = -5/2-i/2$.

(l) $$\frac{5+5i}{3-4i}+\frac{20}{4+3i} = \frac{5+5i}{3-4i}\cdot\frac{3+4i}{3+4i}+\frac{20}{4+3i}\cdot\frac{4-3i}{4-3i}$$

$$= \frac{15+20i+15i+20i^2}{9-16i^2}+\frac{80-60i}{16-9i^2} = \frac{-5+35i}{25}+\frac{80-60i}{25} = 3-i$$

(m) $$\frac{3i^{30}-i^{19}}{2i-1} = \frac{3(i^2)^{15}-(i^2)^9 i}{2i-1} = \frac{3(-1)^{15}-(-1)^9 i}{-1+2i}$$

$$\frac{-3+i}{-1+2i}\cdot\frac{-1-2i}{-1-2i} = \frac{3+6i-i-2i^2}{1-4i^2} = \frac{5+5i}{5} = 1+i$$

2. If $z_1 = 2+i$, $z_2 = 3-2i$ and $z_3 = -\frac{1}{2}+\frac{\sqrt{3}}{2}i$, evaluate each of the following.

(a) $$|3z_1-4z_2| = |3(2+i)-4(3-2i)| = |6+3i-12+8i|$$
$$= |-6+11i| = \sqrt{(-6)^2+(11)^2} = \sqrt{157}$$

(b) $$z_1^3-3z_1^2+4z_1-8 = (2+i)^3-3(2+i)^2+4(2+i)-8$$
$$= \{(2)^3+3(2)^2(i)+3(2)(i)^2+i^3\}-3(4+4i+i^2)+8+4i-8$$
$$= 8+12i-6-i-12-12i+3+8+4i-8 = -7+3i$$

(c) $$(\bar{z}_3)^4 = \left(\overline{-\frac{1}{2}+\frac{\sqrt{3}}{2}i}\right)^4 = \left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^4 = \left[\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^2\right]^2$$
$$= \left[\frac{1}{4}+\frac{\sqrt{3}}{2}i+\frac{3}{4}i^2\right]^2 = \left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2 = \frac{1}{4}-\frac{\sqrt{3}}{2}i+\frac{3}{4}i^2 = -\frac{1}{2}-\frac{\sqrt{3}}{2}i$$

(d) $$\left|\frac{2z_2+z_1-5-i}{2z_1-z_2+3-i}\right|^2 = \left|\frac{2(3-2i)+(2+i)-5-i}{2(2+i)-(3-2i)+3-i}\right|^2$$
$$= \left|\frac{3-4i}{4+3i}\right|^2 = \frac{|3-4i|^2}{|4+3i|^2} = \frac{(\sqrt{(3)^2+(-4)^2})^2}{(\sqrt{(4)^2+(3)^2})^2} = 1$$

3. Find real numbers x and y such that $3x+2iy-ix+5y = 7+5i$.

The given equation can be written as $3x+5y+i(2y-x) = 7+5i$. Then equating real and imaginary parts, $3x+5y=7$, $2y-x=5$. Solving simultaneously, $x=-1$, $y=2$.

4. Prove: (a) $\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$, (b) $|z_1z_2| = |z_1|\,|z_2|$.

Let $z_1 = x_1+iy_1$, $z_2 = x_2+iy_2$. Then

(a) $$\overline{z_1+z_2} = \overline{x_1+iy_1+x_2+iy_2} = \overline{x_1+x_2+i(y_1+y_2)}$$
$$= x_1+x_2-i(y_1+y_2) = x_1-iy_1+x_2-iy_2 = \overline{x_1+iy_1}+\overline{x_2+iy_2} = \bar{z}_1+\bar{z}_2$$

(b) $$|z_1z_2| = |(x_1+iy_1)(x_2+iy_2)| = |x_1x_2-y_1y_2+i(x_1y_2+y_1x_2)|$$
$$= \sqrt{(x_1x_2-y_1y_2)^2+(x_1y_2+y_1x_2)^2} = \sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)} = \sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2} = |z_1|\,|z_2|$$

Another method.

$$|z_1z_2|^2 = (z_1z_2)(\overline{z_1z_2}) = z_1z_2\bar{z}_1\bar{z}_2 = (z_1\bar{z}_1)(z_2\bar{z}_2) = |z_1|^2|z_2|^2 \quad\text{or}\quad |z_1z_2| = |z_1|\,|z_2|$$

where we have used the fact that the conjugate of a product of two complex numbers is equal to the product of their conjugates (see Problem 55).

GRAPHICAL REPRESENTATION OF COMPLEX NUMBERS. VECTORS

5. Perform the indicated operations both analytically and graphically:

(a) $(3 + 4i) + (5 + 2i)$, (b) $(6 - 2i) - (2 - 5i)$, (c) $(-3 + 5i) + (4 + 2i) + (5 - 3i) + (-4 - 6i)$.

(a) *Analytically.* $(3 + 4i) + (5 + 2i) = 3 + 5 + 4i + 2i = 8 + 6i$

Graphically. Represent the two complex numbers by points P_1 and P_2 respectively as in Fig. 1-7 below. Complete the parallelogram with OP_1 and OP_2 as adjacent sides. Point P represents the sum, $8 + 6i$, of the two given complex numbers. Note the similarity with the parallelogram law for addition of vectors OP_1 and OP_2 to obtain vector OP. For this reason it is often convenient to consider a complex number $a + bi$ as a vector having *components* a and b in the directions of the positive x and y axes respectively.

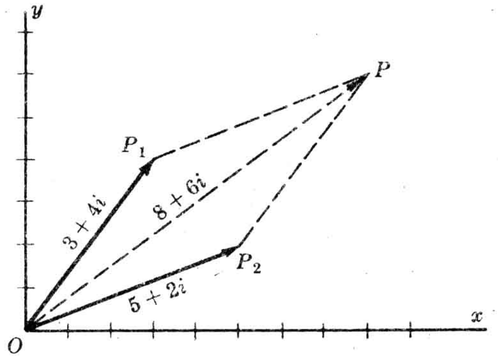

Fig. 1-7

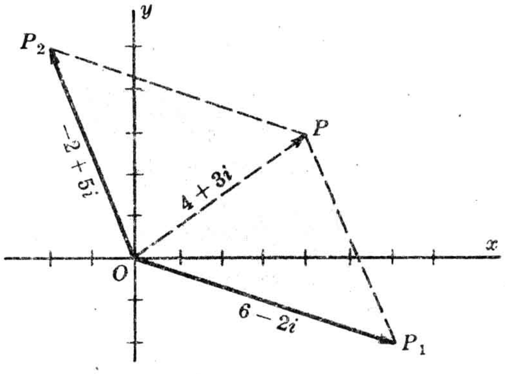

Fig. 1-8

(b) *Analytically.* $(6 - 2i) - (2 - 5i) = 6 - 2 - 2i + 5i = 4 + 3i$

Graphically. $(6 - 2i) - (2 - 5i) = 6 - 2i + (-2 + 5i)$. We now add $6 - 2i$ and $(-2 + 5i)$ as in part (a). The result is indicated by OP in Fig. 1-8 above.

(c) *Analytically.*

$$(-3 + 5i) + (4 + 2i) + (5 - 3i) + (-4 - 6i) = (-3 + 4 + 5 - 4) + (5i + 2i - 3i - 6i) = 2 - 2i$$

Graphically. Represent the numbers to be added by z_1, z_2, z_3, z_4 respectively. These are shown graphically in Fig. 1-9. To find the required sum proceed as shown in Fig. 1-10. At the terminal point of vector z_1 construct vector z_2. At the terminal point of z_2 construct vector z_3, and at the terminal point of z_3 construct vector z_4. The required sum, sometimes called the *resultant*, is obtained by constructing the vector OP from the initial point of z_1 to the terminal point of z_4, i.e. $OP = z_1 + z_2 + z_3 + z_4 = 2 - 2i$.

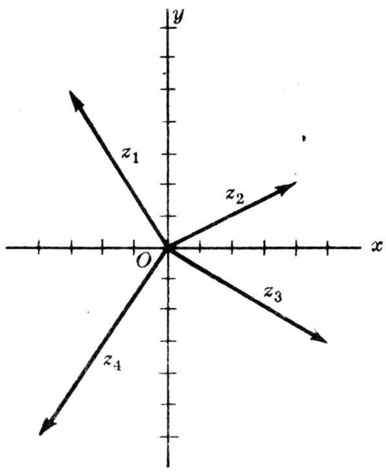

Fig. 1-9

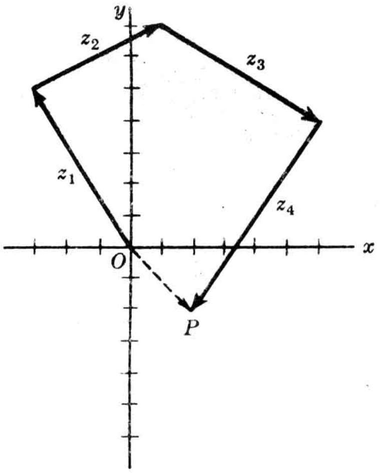

Fig. 1-10

6. If z_1 and z_2 are two given complex numbers (vectors) as in Fig. 1-11, construct graphically

(a) $3z_1 - 2z_2$ (b) $\frac{1}{2}z_2 + \frac{5}{3}z_1$

(a) In Fig. 1-12 below, $OA = 3z_1$ is a vector having length 3 times vector z_1 and the same direction.

$OB = -2z_2$ is a vector having length 2 times vector z_2 and the opposite direction.

Then vector $OC = OA + OB = 3z_1 - 2z_2$.

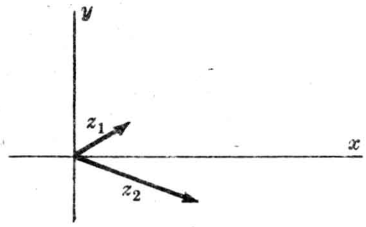

Fig. 1-11

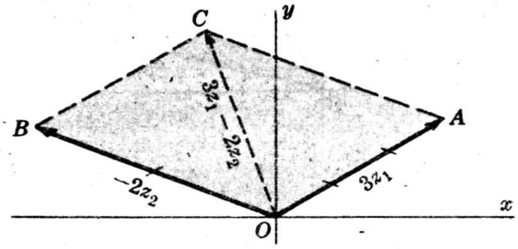

Fig. 1-12

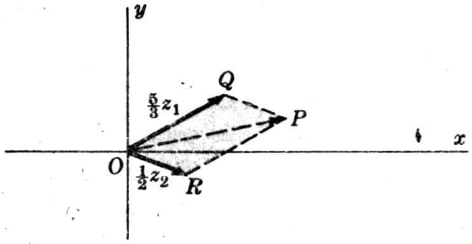

Fig. 1-13

(b) The required vector (complex number) is represented by OP in Fig. 1-13 above.

7. Prove (a) $|z_1 + z_2| \leqq |z_1| + |z_2|$, (b) $|z_1 + z_2 + z_3| \leqq |z_1| + |z_2| + |z_3|$, (c) $|z_1 - z_2| \geqq |z_1| - |z_2|$ and give a graphical interpretation.

(a) *Analytically.* Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. Then we must show that

$$\sqrt{(x_1+x_2)^2 + (y_1+y_2)^2} \leqq \sqrt{x_1^2+y_1^2} + \sqrt{x_2^2+y_2^2}$$

Squaring both sides, this will be true if

$$(x_1+x_2)^2 + (y_1+y_2)^2 \leqq x_1^2 + y_1^2 + 2\sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)} + x_2^2 + y_2^2$$

i.e. if

$$x_1x_2 + y_1y_2 \leqq \sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}$$

or if (squaring both sides again)

$$x_1^2x_2^2 + 2x_1x_2y_1y_2 + y_1^2y_2^2 \leqq x_1^2x_2^2 + x_1^2y_2^2 + y_1^2x_2^2 + y_1^2y_2^2$$

or

$$2x_1x_2y_1y_2 \leqq x_1^2y_2^2 + y_1^2x_2^2$$

But this is equivalent to $(x_1y_2 - x_2y_1)^2 \geqq 0$ which is true. Reversing the steps, which are reversible, proves the result.

Graphically. The result follows graphically from the fact that $|z_1|, |z_2|, |z_1 + z_2|$ represent the lengths of the sides of a triangle (see Fig. 1-14) and that the sum of the lengths of two sides of a triangle is greater than or equal to the length of the third side.

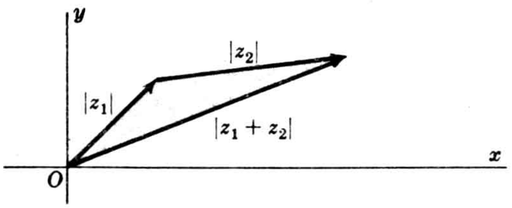

Fig. 1-14

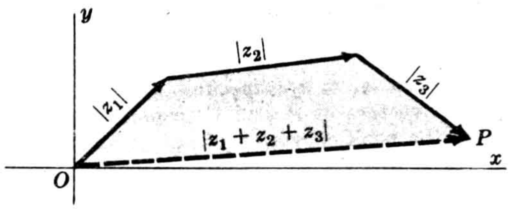

Fig. 1-15

(b) *Analytically.* By part (a),

$$|z_1 + z_2 + z_3| = |z_1 + (z_2 + z_3)| \leqq |z_1| + |z_2 + z_3| \leqq |z_1| + |z_2| + |z_3|$$

Graphically. The result is a consequence of the geometric fact that in a plane a straight line is the shortest distance between two points O and P (see Fig. 1-15).