# SET THEORY

THOMAS JECH

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*Department of Mathematics*
*The Pennsylvania State University*
*University Park, Pennsylvania*

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# PREFACE

The main body of this book consists of 106 numbered theorems and a dozen of examples of models of set theory. A large number of additional results is given in the exercises, which are scattered throughout the text. Most exercises are provided with an outline of proof in square brackets [ ], and the more difficult ones are indicated by an asterisk.

I am greatly indebted to all those mathematicians, too numerous to mention by name, who in their letters, preprints, handwritten notes, lectures, seminars, and many conversations over the past decade shared with me their insight into this exciting subject.

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PART I

# Sets

CHAPTER 1

# AXIOMATIC SET THEORY

## 1. AXIOMS OF SET THEORY

### Axioms of Zermelo–Fraenkel:

*I. Axiom of Extensionality. If $X$ and $Y$ have the same elements, then $X = Y$.*

*II. Axiom of Pairing. For any $a$ and $b$ there exists a set $\{a, b\}$ that contains exactly $a$ and $b$.*

*III. Axiom Schema of Separation. If $\varphi$ is a property (with parameter $p$), then for any $X$ and $p$ there exists a set $Y = \{u \in X : \varphi(u, p)\}$ that contains all those $u \in X$ that have the property $\varphi$.*

*IV. Axiom of Union. For any $X$ there exists a set $Y = \bigcup X$, the union of all elements of $X$.*

*V. Axiom of Power Set. For any $X$ there exists a set $Y = P(X)$, the set of all subsets of $X$.*

*VI. Axiom of Infinity. There exists an infinite set.*

*VII. Axiom Schema of Replacement. If $\mathbf{F}$ is a function, then for any $X$ there exists a set $Y = \mathbf{F}[X] = \{\mathbf{F}(x) : x \in X\}$.*

*VIII. Axiom of Regularity. Every nonempty set has an $\in$-minimal element.*

*IX. Axiom of Choice. Every family of nonempty sets has a choice function.*

The theory with axioms I–VIII is ZF, Zermelo–Fraenkel axiomatic set theory; ZFC denotes the theory ZF with the axiom of choice.

### Why Axiomatic Set Theory?

Intuitively, a set is a collection of all elements that satisfy a certain given property. In other words, we might be tempted to postulate the following rule of formation for sets.

**Axiom Schema of Comprehension** (false). *If $\varphi$ is a property, then there exists a set $Y = \{x : \varphi(x)\}$.*

This principle, however, is false:

**Russell's Paradox.** Consider the set $S$ whose elements are all those (and only those) sets that are not members of themselves: $S = \{X : X \notin X\}$. Question: Does $S$ belong to $S$? If $S$ belongs to $S$, then $S$ is not a member of itself, and so $S \notin S$. On the other hand, if $S \notin S$, then $S$ belongs to $S$. In either case, we have a contradiction.

Thus we must conclude that

$$\{X : X \notin X\}$$

is not a set, and we must somewhat revise the intuitive notion of a set.

The safe way to eliminate paradoxes of this type is to abandon the schema of comprehension and keep its weak version, the *schema of separation*:

*If $\varphi$ is a property, then for any $X$ there exists a set $Y = \{x \in X : \varphi(x)\}$.*

Once we give up the full comprehension schema, Russell's paradox is no longer a threat; moreover, it provides useful information: The set of all sets does not exist. (Otherwise, apply the separation schema to the property $x \notin x$.)

In other words, it is the concept of the set of all sets that is paradoxical, not the idea of comprehension itself.

Replacing full comprehension by separation presents us with a new problem. The separation axioms are too weak to develop set theory with its usual operations and constructions. Notably, these axioms are not sufficient to prove that, e.g., the union $X \cup Y$ of two sets exists, or to define the notion of a real number.

Thus we have to add further construction principles that postulate the existence of sets obtained from other sets by means of certain operations.

The axioms of ZFC are generally accepted as a correct formalization of those principles that mathematicians apply when dealing with sets.

### Language of Set Theory, Formulas

The axiom schema of separation as formulated above uses the vague notion of a *property*. To give the axioms a precise form, we develop axiomatic set theory in the framework of the first order predicate calculus. Apart from the equality predicate $=$, the language of set theory consists of the binary predicate $\in$, the *membership relation.*

The *formulas* of set theory are built up from the *atomic formulas*

$$x \in y, \qquad x = y$$

by means of connectives

$$\varphi \wedge \psi, \qquad \varphi \vee \psi, \qquad \neg\varphi, \qquad \varphi \rightarrow \psi, \qquad \varphi \leftrightarrow \psi$$

(conjunction, disjunction, negation, implication, equivalence), and *quantifiers*

$$\forall x\ \varphi, \qquad \exists x\ \varphi$$

In practice, we shall use in formulas other symbols, namely defined predicates, operations, and constants, and even use formulas informally; but it will be tacitly understood that each such formula can be written in a form that only involves $\in$ and $=$ as nonlogical symbols.

Concerning formulas with free variables, we adopt the notational convention that all free variables of a formula

$$\varphi(u_1, \ldots, u_n)$$

are among $u_1, \ldots, u_n$ (possibly some $u_i$ are not free, or even do not occur, in $\varphi$). A formula without free variables is called a *sentence.*

## Classes

Although we work in ZFC which, unlike alternative axiomatic set theories, has only one type of object, namely sets, we introduce the informal notion of a *class.* We do this for practical reasons: It is easier to manipulate classes than formulas.

If $\varphi(x, p_1, \ldots, p_n)$ is a formula, we call

$$\mathbf{C} = \{x : \varphi(x, p_1, \ldots, p_n)\}$$

a *class.* Members of the class $\mathbf{C}$ are all those sets $x$ that satisfy $\varphi(x, p_1, \ldots, p_n)$:

$$x \in \mathbf{C} \quad \text{iff} \quad \varphi(x, p_1, \ldots, p_n)$$

We say that $\mathbf{C}$ is *definable from* $p_1, \ldots, p_n$; if $\varphi$ has just one free variable, then the class $\mathbf{C}$ is *definable.*

We shall use boldface capital letters to denote classes. We shall, however, make departures from this general rule in cases when the standard notation is different (e.g., $V$, $L$, $Ord$, $\aleph$, etc.)

Two classes are considered equal if they have the same elements: If

$$\mathbf{C} = \{x : \varphi(x, p_1, \ldots, p_n)\}, \qquad \mathbf{D} = \{x : \psi(x, q_1, \ldots, q_m)\}$$

then $\mathbf{C} = \mathbf{D}$ iff for all $x$

$$\varphi(x, p_1, \ldots, p_n) \quad \leftrightarrow \quad \varphi(x, q_1, \ldots, q_m)$$

The *universal class*, or *universe*, is the class of all sets:

$$V = \{x : x = x\}$$

We define *inclusion* of classes

$$\mathbf{C} \subseteq \mathbf{D} \quad \leftrightarrow \quad \forall x(x \in \mathbf{C} \to x \in \mathbf{D})$$

and the following operations on classes:

$$\mathbf{C} \cap \mathbf{D} = \{x : x \in \mathbf{C} \wedge x \in \mathbf{D}\}$$

$$\mathbf{C} \cup \mathbf{D} = \{x : x \in \mathbf{C} \vee x \in \mathbf{D}\}$$

$$\mathbf{C} - \mathbf{D} = \{x : x \in \mathbf{C} \wedge x \notin \mathbf{D}\}$$

$$\bigcup \mathbf{C} = \{x : x \in S \text{ for some } S \in \mathbf{C}\} = \bigcup \{S : S \in \mathbf{C}\}$$

Every set can be considered a class. If $S$ is a set, consider the formula $x \in S$ and the class

$$\{x : x \in S\}$$

That the set $S$ is uniquely determined by its elements follows from the axiom of extensionality.

A class that is not a set is a *proper class.*

**Extensionality**

*If $X$ and $Y$ have the same elements, then $X = Y$:*

$$\forall u(u \in X \leftrightarrow u \in Y) \to X = Y$$

The converse, namely, if $X = Y$, then $u \in X \leftrightarrow u \in Y$, is an axiom of predicate calculus. Thus we have

$$X = Y \quad \leftrightarrow \quad \forall u(u \in X \leftrightarrow u \in X)$$

The axiom expresses the basic idea of a set: A set is determined by its elements.

**Pairing**

*For any $a$ and $b$ there exists a set $\{a, b\}$ that contains exactly $a$ and $b$:*

$$\forall a \, \forall b \, \exists c \, \forall x(x \in c \leftrightarrow x = a \vee x = b)$$

By Extensionality, the set $c$ is unique, and we can define the *pair*

$$\{a, b\} = \text{the unique } c \text{ such that } \forall x(x \in c \leftrightarrow x = a \vee x = b)$$

The *singleton* $\{a\}$ is the set

$$\{a\} = \{a, a\}$$

Since $\{a, b\} = \{b, a\}$, we further define an *ordered pair*

$$(a, b)$$

so as to satisfy the following condition:

$$(1.1) \qquad (a, b) = (c, d) \quad \text{iff} \quad a = c \quad \text{and} \quad b = d$$

For the formal definition of an ordered pair, we take

$$(a, b) = \{\{a\}, \{a, b\}\}$$

*Exercise 1.1.* Verify that the definition of an ordered pair satisfies (1.1).

We further define ordered triples, quadruples, etc., as follows:

$$(a, b, c) = ((a, b), c)$$

$$(a, b, c, d) = ((a, b, c), d)$$

$$\vdots$$

$$(a_1, \ldots, a_{n+1}) = ((a_1, \ldots, a_n), a_{n+1})$$

It follows that two ordered $n$-tuples $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are equal iff $a_1 = b_1, \ldots, a_n = b_n$.

**Separation Schema**

*Let* $\varphi(u, p)$ *be a formula. For any* $X$ *and* $p$, *there exists a set* $Y = \{u \in X : \varphi(u, p)\}$:

$$(1.2) \qquad \forall X\, \forall p\, \exists Y\, \forall u(u \in Y \leftrightarrow u \in X \wedge \varphi(u, p))$$

For each formula $\varphi(u, p)$, the formula (1.2) is an axiom (of separation). The set $Y$ in (1.2) is unique by Extensionality.

Note that a more general version of separation axioms can be proved using ordered $n$-tuples: Let $\psi(u, p_1, \ldots, p_n)$ be a formula. Then

$$(1.3) \qquad \forall X\, \forall p_1 \cdots \forall p_n\, \exists Y\, \forall u(u \in Y \leftrightarrow u \in X \wedge \psi(u, p_1, \ldots, p_n))$$

Simply let $\varphi(u, p)$ be the formula

$$\exists p_1 \cdots \exists p_n(p = (p_1, \ldots, p_n) \text{ and } \psi(u, p_1, \ldots, p_n))$$

and then, given $X$ and $p_1, \ldots, p_n$, let

$$Y = \{u \in X : \varphi(u, (p_1, \ldots, p_n))\}$$

We can give the separation axioms the following form: Consider the class $\mathbf{C} = \{u : \varphi(u, p_1, \ldots, p_n)\}$; then by (1.3)

$$\forall X\, \exists Y(\mathbf{C} \cap X = Y)$$

Thus the intersection of a class $\mathbf{C}$ with any set is a set; or, we can say even more informally

*a subclass of a set is a set*

One consequence of the separation axioms is that the intersection and the difference of two sets is a set, and so we can define the operations

$$X \cap Y = \{u \in X : u \in Y\}, \qquad X - Y = \{u \in X : u \notin Y\}$$

Similarly, it follows that the empty class

$$\emptyset = \{u : u \neq u\}$$

is a set—the *empty set*; this, of course, only under the assumption that at least one set $X$ exists (because $\emptyset \subseteq X$):

(1.4) $$\exists X(X = X)$$

We have not included (1.4) among the axioms, but it follows from the axiom of infinity.

Two sets $X$, $Y$ are called *disjoint* if $X \cap Y = \emptyset$.

If $\mathbf{C}$ is a nonempty class of sets, we let

$$\bigcap \mathbf{C} = \bigcap \{X : X \in \mathbf{C}\} = \{u : u \in X \text{ for every } X \in \mathbf{C}\}$$

Note that $\bigcap \mathbf{C}$ is a set (it is a subset of any $X \in \mathbf{C}$). Also, $X \cap Y = \bigcap \{X, Y\}$.

Another consequence of the separation axioms is that the universal class $V$ is a proper class; otherwise,

$$S = \{x \in V : x \notin x\}$$

would be a set.

**Union**

*For any $X$ there exists a set $Y = \bigcup X$:*

(1.5) $$\forall X\, \exists Y\, \forall u(u \in Y \leftrightarrow \exists z(z \in X \wedge u \in z))$$

Let us introduce the abbreviations

$$(\exists z \in X)\, \varphi \qquad \text{for} \qquad \exists z(z \in X \wedge \varphi)$$

and

$$(\forall z \in X)\, \varphi \qquad \text{for} \qquad \forall z(z \in X \rightarrow \varphi)$$

By (1.5), for every $X$ there is a unique set

$$Y = \{u : (\exists z \in X)\, [u \in z]\} = \bigcup \{z : z \in X\} = \bigcup X$$

the *union* of $X$.

Now we can define

$$X \cup Y = \bigcup \{X, Y\}, \qquad X \cup Y \cup Z = (X \cup Y) \cup Z, \qquad \text{etc.}$$

and also

$$\{a, b, c\} = \{a, b\} \cup \{c\}$$

and in general

$$\{a_1, \ldots, a_n\} = \{a_1\} \cup \cdots \cup \{a_n\}$$

**Power Set**

*For any X there exists a set $Y = P(X)$:*

$$\forall X\ \exists Y\ \forall u(u \in Y \leftrightarrow u \subseteq X)$$

A set $U$ is a *subset* of $X$, $U \subseteq X$, if

$$\forall z(z \in U \rightarrow z \in X)$$

If $U \subseteq X$ and $U \neq X$, then $U$ is a *proper subset*, $U \subset X$.

The set of all subsets of $X$

$$P(X) = \{u : u \subseteq X\}$$

is called the *power set* of $X$.

*Exercise 1.2.* There is no set $X$ such that $P(X) \subseteq X$.
[Use Russell's paradox.]

Using the power set axiom we can define other basic notions of set theory.

The *Cartesian product* of $X$ and $Y$ is the set of all pairs $(x, y)$ such that $x \in X$ and $y \in Y$:

(1.6) $$X \times Y = \{(x, y) : x \in X \wedge y \in Y\}$$

The notation $\{(x, y) : \cdots\}$ in (1.6) is justified because

$$\{(x, y) : \varphi(x, y)\} = \{u : \exists x\ \exists y(u = (x, y) \wedge \varphi(x, y))\}$$

The product $X \times Y$ is a set because

$$X \times Y \subseteq PP(X \cup Y)$$

Further, we define

$$X \times Y \times Z = (X \times Y) \times Z$$

and in general

$$X_1 \times \cdots \times X_{n+1} = (X_1 \times \cdots \times X_n) \times X_{n+1}$$

Thus

$$X_1 \times \cdots \times X_n = \{(x_1, \ldots, x_n) : x_1 \in X_1 \wedge \cdots \wedge x_n \in X_n\}$$

We also let

$$X^n = X \times \cdots \times X \quad (n \text{ times})$$

An *n-ary relation* $R$ is a set of $n$-tuples. $R$ is a relation *over* $X$ if $R \subseteq X^n$. It is customary to write $R(x_1, \ldots, x_n)$ instead of

$$(x_1, \ldots, x_n) \in R$$

and in case that $R$ is binary, then we also use

$$x \; R \; y$$

for $(x, y) \in R$.

If $R$ is a binary relation, then the *domain of* $R$ is the set

$$\operatorname{dom}(R) = \{u : \exists v[(u, v) \in R]\}$$

and the *range* of $R$ is the set

$$\operatorname{ran}(R) = \{v : \exists u[(u, v) \in R]\}$$

Note that $\operatorname{dom}(R)$ and $\operatorname{ran}(R)$ are sets because

$$\operatorname{dom}(R) \subseteq \bigcup \bigcup R, \qquad \operatorname{ran}(R) \subseteq \bigcup \bigcup R.$$

The *field* of a relation $R$ is the set

$$\text{field } (R) = \operatorname{dom}(R) \cup \operatorname{ran}(R).$$

In general, we call a class $\mathbf{R}$ an $n$-ary relation if all its elements are $n$-tuples; in other words, if

$$\mathbf{R} \subseteq V^n = \text{the class of all } n\text{-tuples}$$

where $\mathbf{C}^n$ (and $\mathbf{C} \times \mathbf{D}$) is defined in the obvious way.

A binary relation $f$ is a *function* if

$$(x, y) \in f \quad \text{and} \quad (x, z) \in f \qquad \text{implies} \qquad y = z$$

The unique $y$ such that $(x, y) \in f$ is the *value* of $f$ at $x$; we use the standard notation

$$y = f(x)$$

or its variations

$$f\colon x \mapsto y, \qquad y = f_x, \qquad \text{etc.}$$

for $(x, y) \in f$.

$f$ is a function *on* $X$ if $X = \operatorname{dom}(f)$. If $\operatorname{dom}(f) = X^n$, then $f$ is an *n-ary function* on $X$.

$f$ is a function *from* $X$ *to* $Y$,

$$f\colon X \to Y$$

if $\operatorname{dom}(f) = X$ and $\operatorname{ran}(f) \subseteq Y$. The set of all functions from $X$ to $Y$ is denoted by ${}^XY$. Note that ${}^XY$ is a set:

$${}^XY \subseteq P(X \times Y)$$

If $Y = \operatorname{ran}(f)$, then $f$ is a function *onto* $Y$. A function $f$ is *one-to-one* if

$$f(x) = f(y) \quad \text{implies} \quad x = y$$

An *n-ary operation over* $X$ is a function $f: X^n \to X$.

The *restriction* of a function $f$ to a set $X$ (usually a subset of $\operatorname{dom}(f)$) is the function

$$f \mid X = \{(x, y) \in f : x \in X\}$$

A function $g$ is an *extension* of a function $f$ if $g \supseteq f$, i.e., $\operatorname{dom}(f) \subseteq \operatorname{dom}(g)$ and $g(x) = f(x)$ for all $x \in \operatorname{dom}(f)$.

We denote the *image of $X$ by $f$* by

$$f[X] = \{y : (\exists x \in X)\, [y = f(x)]\}$$

and the *inverse image* by

$$f_{-1}(X) = \{x : f(x) \in X\}$$

If $f$ is one-to-one, then $f^{-1}$ denotes the *inverse* of $f$:

$$f^{-1}(x) = y \quad \text{iff} \quad x = f(y)$$

These definitions also apply to functions that are classes, i.e., a relation $\mathbf{F}$ such that

$$(x, y) \in \mathbf{F} \quad \text{and} \quad (x, z) \in \mathbf{F} \quad \text{implies} \quad y = z$$

For instance, $\mathbf{F}[\mathbf{C}]$ denotes the image of the class $\mathbf{C}$ under the function $\mathbf{F}$.

It should be noted that a function is often called a *mapping* or a *correspondence* (and similarly, a set is called a *family* or a *collection*).

An *equivalence relation* over a set $X$ is a binary relation $\equiv$ which is *reflexive*, *symmetric*, and *transitive*:

$$x \equiv x \text{ for all } x \in X$$

$$x \equiv y \text{ implies } y \equiv x$$

$$x \equiv y \text{ and } y \equiv z \text{ implies } x \equiv z$$

A family of sets is *disjoint* if any two of its members are disjoint. A *partition* of a set $X$ is a disjoint family $P$ of nonempty sets such that

$$X = \bigcup \{Y : Y \in P\}$$

Let $\equiv$ be an equivalence relation over $X$. For every $x \in X$, let

$$[x] = \{y \in X : y \equiv x\}$$

(the *equivalence class* of $x$). The set

$$X/\!\equiv \;= \{[x] : x \in X\}$$

is a partition of $X$ (the *quotient* of $X$ by $\equiv$). Conversely, each partition $P$ of $X$ defines an equivalence relation over $X$:

$$x \equiv y \quad \leftrightarrow \quad (\exists Y \in P)\,[x \in Y \wedge y \in Y]$$

If an equivalence relation is a class, then its equivalence classes may be proper classes. In Section 9 we shall introduce a trick that enables us to handle equivalence classes as if they were sets.

## Infinity

*There exists an infinite set.*

To give a precise formulation of the axiom of infinity, we have to define first the notion of finiteness. The most obvious definition of finiteness uses the notion of a natural number, which is as yet undefined. We shall define natural numbers (as finite ordinals) in Section 2 and give only a quick treatment of natural numbers and finiteness in the exercises below.

In principle, it is possible to give a definition of finiteness that does not mention numbers, but such definitions necessarily look artificial. We give the most successful version below.

We therefore formulate the axiom of infinity differently:

$$\exists S[\emptyset \in S \wedge (\forall x \in S)\,[x \cup \{x\} \in S]]$$

We call a set $S$ with the above property *inductive*. Thus we have:

***Axiom of Infinity.*** *There exists an inductive set.*

The idea is that an inductive set is infinite. We note that using the replacement schema, one can show that an inductive set exists if there exists an infinite set (see Section 2).

A set $S$ is *T-finite* if every nonempty $X \subseteq P(S)$ has a $\subset$-maximal element, i.e., $u \in X$ such that there is no $v \in X$ with $u \subset v$.

Let

$$N = \bigcap \{X : X \text{ is inductive}\}$$

$N$ is the smallest inductive set. Let us use the following notation:

$$0 = \emptyset, \qquad 1 = \{0\}, \qquad 2 = \{0, 1\}, \qquad \ldots$$

If $n \in N$, let $n + 1 = n \cup \{n\}$. Let us define $<$ (over $N$) by

$$n < m \quad \leftrightarrow \quad n \in m$$

A set $T$ is *transitive* if $x \in T$ implies $x \subseteq T$.

***Exercise 1.3.*** If $X$ is inductive, then the set $\{x \in X : x \subseteq X\}$ is inductive. Hence $N$ is transitive, and for each $n$,

$$n = \{m \in N : m < n\}$$