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PREFACE

This volume contains the texts of the principal survey papers presented at *ALGORITHMS and ORDER*, held at Ottawa, Canada from June 1 to June 12, 1987. The conference was supported by grants from the N.A.T.O. Advanced Study Institute programme, the University of Ottawa, and the Natural Sciences and Engineering Research Council of Canada. We are grateful for this considerable support.

Over fifty years ago, the *Symposium on Lattice Theory*, in Charlottesville, U.S.A., proclaimed the vitality of ordered sets. Only twenty years later the Symposium on Partially Ordered Sets and Lattice Theory, held at Monterey, U.S.A., had solved many of the problems that had been originally posed.

In 1981, the *Symposium on Ordered Sets* held at Banff, Canada, continued this tradition. It was marked by a landmark volume containing twenty-three articles on almost all current topics in the theory of ordered sets and its applications. Three years after, *Graphs and Orders*, also held at Banff, Canada, aimed to document the role of graphs in the theory of ordered sets and its applications.

Because of its special place in the landscape of the mathematical sciences order is especially sensitive to new trends and developments. Today, the most important current in the theory and application of order springs from theoretical computer science.

Two themes of computer science lead the way.

The first is *data structure*. Order is common to data structures. The order may arise according to precedence relations, due either to technological constraints or even to social choice, on an underlying set of tasks. How should this order be represented? By a graph? By a diagram? By an incidence matrix? By geometrical figures? By time diagrams?

The second theme is *optimization*. Order is common in optimization problems. Scheduling, sorting and search problems are among the most common instances of order. Typically an order must be transformed to another, say a partial extension or a linear extension, which itself may represent a schedule or a sort.

It was the aim of *ALGORITHMS and ORDER*, the conference and this volume, to survey and monitor these aspects of order. The algorithmic approach is playing an ever-increasing role and we have good reason to expect continued growth and applications. The twelve articles in this volume cover the important ground of algorithms and data structures in ordered sets. They are based on the principal expository lectures presented during this two week conference. There were also frequent special seminars and informal sessions organized spontaneously and according to individual initiatives. Among these were "problem sessions", each occupying the better part of an evening. Many unsolved problems were recorded and are here transcribed in the "problem sessions" section. This volume also includes an index.

We are grateful to the many who helped in all aspects of this meeting. Among them C. Sinclair of the Scientific Affairs Division of N.A.T.O. was especially helpful in the design of the format for the scientific sessions. We lament too the passing away, recently, of his predecessor, M. Di Lullo, who assisted us during the earlier Advanced Study Institutes in Banff (1981, 1984). Several of the participants, too, assisted in many ways. I am especially grateful to R. Nowakowski and J. Urrutia. As ever, Hetje Rival encouraged us, gave enthusiasm and supplied support - always.

Ottawa, Canada, July 1988 Ivan Rival

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PART I

GRAPHICAL DATA STRUCTURES

GRAPHICAL DATA STRUCTURES FOR ORDERED SETS

by

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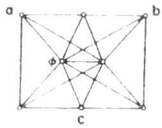
THE DIAGRAM

Ordered sets occur widely in computation, in scheduling, in sorting, in social choice, and even in geography. For some years research on these themes has focussed first on combinatorial optimization and then on "algorithmics". Important advances have been made both at practical and, at theoretical levels. There is little doubt that the modern mathematical theory of ordered sets owes much of its vitality to these recent developments. While some of the problems remain exceedingly difficult, such as the "three-machine scheduling problem", attention is shifting from the usual optimization themes to data structures; indeed, there is emerging a need for efficient data structures to code and store ordered sets. Among these data structures, graphical ones are coming to play a decisive role, for instance, in problems in which decisions must be made from among alternatives ranked according to precedence or preference relations.

There are numerous graphical schemes in common use to represent an ordered set, each highlighting some order-theoretical property, usually without determining it entirely. Some are fairly crude (e.g., resembling 'potatoes' or 'barrels') and are intended to serve as a blackboard shorthand for an unwritten mathematical polish. Other schemes (e.g., "time" or 'arrow' diagrams) are specific in delineating particular order-theoretical properties, for instance in scheduling. Still others (e.g., 'block' diagrams) are contrived as mnemonic aids to represent large ordered sets which might otherwise remain unexplored.

To summarize there are three recurrent themes that lie at the heart of the study of graphical data structures for ordered sets: *comparability*, *covering* and *diagram*. Each graphical scheme uses vertices (little circles in the plane) for the elements of the ordered set. The *comparability graph* is an undirected graph in which an edge joins two vertices a and b precisely if, either $a < b$ or $b < a$. Actually much is known

I. Rival (ed.), Algorithms and Order, 3–31.

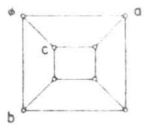


The comparability graph of 2^3, the ordered set of all subsets of {a,b,c,} ordered by inclusion.

Figure 1

about this graphical scheme (cf. Gallai (1967), Golumbic (1980), Kelly (1985), Möhring (1985)). Loosely speaking the comparability graph has so many edges that, while it is an undirected graph, the actual orientation ($a < b$ or $b < a$) can be determined, at least up to duality. Nevertheless, this abundance of edges is the source of its practical uselessness. The clutter of edges results in a disordered jumble; far from serving to aid readability it results in confusion.

What is an efficient graphical presentation of an ordered set? The profusion of edges in the comparability graph may be avoided by exploiting the 'transitivity' of an order. For elements a and b in an ordered set P say that a *covers* b or b is *covered by* a, if $a > b$ and, if, for each x in P, $a > x \geq b$ implies $x = b$. We also call a an *upper cover* of b, and b a *lower cover* of a. We write $a \succ b$, or $b \prec a$. The *covering graph* of P is an undirected graph whose vertices are the elements of P and in which an edge joins two vertices a and b precisely if a covers b or b covers a.

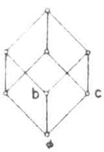


The covering graph of 2^3

Figure 2

The apparent sparsity of edges in the covering graph makes it a tidier graphical scheme. Indeed, sometimes, it may even be planar. The trade-off, however, is that the orientation of P is hardly ever determined from its covering graph alone. And that, of course, is a serious drawback for, after all, these pictures are meant to be read. The foremost practical feature is that, for elements a and b in P, we may readily decide whether or not $a < b$. Of course, $a < b$ just if there is a *covering chain* from a to b, that is, a sequence $a = a_0, a_1, a_2, \ldots, a_k = b$ such that a_{i+1} covers a_i, $i = 0,1,2,\ldots, k-1$. On the other hand, a path from a to b in the covering graph need not necessarily correspond to a covering chain and it may even be that a is noncomparable to b.

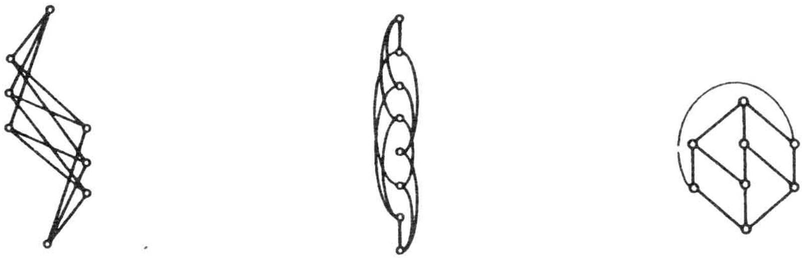
'Antisymmetry' of the order relation makes possible an orientation of the covering graph from which the comparability relations may be readily inferred. To this end we orient any edge a >- b of the covering graph so that it makes an angle Θ with the horizontal satisfying $0° < \Theta < 180°$. This is a *diagram* of P. Thus, the elements of P are represented by small circles on the plane so arranged that any



The diagram of 2^3

Figure 3

circle corresponding to an upper cover a of b is situated higher in the plane than the circle corresponding to b and is joined to it by a monotonic arc (that is, an arc with no repeated y-coordinates). Insofar as a diagram of P is a drawing there is, of course, considerable variation possible in its actual rendering. Still, any diagram of P determines it and it is common practice to identify P with a diagram of P itself. Despite its apparent simplicity and almost universal usage it is a graphical scheme



A diagram of 2^3 | Another diagram | Not a diagram

Figure 4

fraught with subtlety and, frequently, more artifice than method. Indeed, the diagram is so important and yet so little understood that recent years have witnessed an unprecedented growth in research devoted to it.

This survey is intended to illustrate several current directions and ideas useful in the study of graphical data structures for ordered sets.

HOW IS THE DIAGRAM USEFUL?

Here are three preliminary examples to illustrate the usefulness of the diagram.

Example I. Chain Decomposition. What is the least number of planes needed in an airline fleet to carry out all of a set of trips with specified origin, destination, departure time and arrival time? Let P stand for the set of trips. Each trip x in P has a required *departure time* d(x) from its origin and a specified *arrival time* $a(x) > d(x)$ at its destination. For trips x and y there is a nonzero *transition time* t(x,y) needed to prepare for the trip y after the completion of x. The transition time may be due to the time it takes to prepare for a trip y; for example, instead of waiting inactive to start another trip at x's destination, it may be more efficient to incur extra cost by flying, perhaps even without passengers, to another airport, the origin of a trip y. We write

$$x < y$$

if, in time,

$$a(x) + t(x,y) < d(y).$$

This relation on P is an order provided that the transition times satisfy this triangle inequality

$$t(x,y) \leq t(x,z) + t(z,y).$$

The point is that a single plane can carry out a sequence x,y,z, ... of trips only if the sequence is a chain x<y<z<... in the ordered set of trips. Therefore, the least number of planes required is precisely the

A chain decomposition

A minimum chain decomposition

Figure 5

number of disjoint chains in a chain decomposition - the subject of the well-known Chain Decomposition Theorem according to which this number is actually the greatest number of pairwise noncomparable elements (cf. Dilworth (1950)). It is the conventional wisdom that, with respect to the diagram' 'geometry', chains are rising paths, perhaps with the fewest number of deviations, thus, as near to vertical as possible. While one rendering of the diagram may be quite misleading, another may yield a minimum chain decomposition by inspection.

Example II. Planarity testing. Diagrams to represent the precedence relations in an organization chart are drawn to be read. It follows, therefore, that the foremost feature of a diagram of an ordered set P is that, for elements a and b in P, we may readily decide whether or not a < b. The most obvious graphical criterion is 'planarity'. We say that P is *planar* if it has a diagram in which none of the lines corresponding to the covering pairs intersect, except possibly at an endpoint, where they may meet a small circle corresponding to an element of P. Such a rendering of P we call a *planar representation* of it. Planarity seems to enhance the understanding of the order represented by the diagram.

A (nonplanar) diagram of an ordered set

A planar representation of the same ordered set

Figure 6

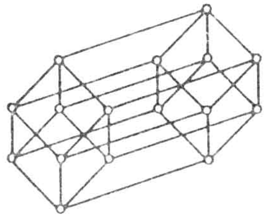
It may even be a physical constraint if the diagram stands, say, for a logic circuit whose wires are not to cross except at contact points.

Most of what is known about planarity is for lattices, that is, ordered sets in which, for every pair of elements there is supremum and infimum, both belonging to the ordered set. For instance, any planar ordered set with a top and a bottom must be a planar lattice (cf. [Kelly and Rival (1975)]). For lattices there is a linear time planarity-testing algorithm which derives from the reduction of planarity for lattices [Platt (1976)] to planarity for graphs [Hopcorft and Tarjan (1974)]. Perhaps the most important facts about planar lattices are these:

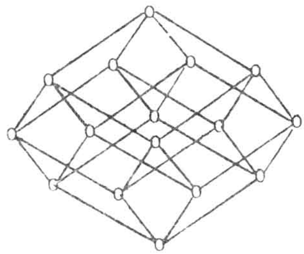
(i) the (order) dimension of an ordered set P is preserved by its 'completion (by cuts)' [Baker, Fishburn and Roberts (1971)]
(ii) a lattice has dimension at most two just if it is planar [Kelly and Rival (1975)]
(iii)there is a full theory and description of planarity for lattices [Kelly and Rival (1975)]

Together these facts lead to the characterization of all ordered sets of dimension at most three [Kelly (1975)].

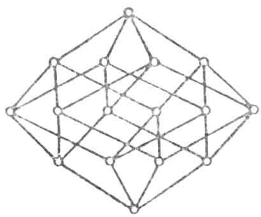
Example III. Structural Analysis. There are many other features, especially [Kelly (1977)] of a structural character, which may be highlighted by a particular diagram, that is, which may be read from a diagram appropriate to it. Thus, whether P has a decomposition either as a direct product, or as a linear sum, or a lexicographic sum, etc., may not be readily apparent just from a full listing of the comparabilities themselves. An interesting example is the ordered set 2^4 of all subsets of a four-element set ordered by set inclusion. In Figure 7 we have given four, quite different, diagrams each highlighting a particular structural feature of 2^4.



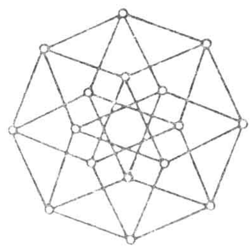
The direct product decomposition of 2^4 as 2×2^3



The direct product decomposition of 2^4 as $2^2 \times 2^2$



A 'symmetric' diagram of 2^4



Another 'symmetric' diagram of 2^4.

Figure 7