AMERICAN MATHEMATICAL SOCIETY
COLLOQUIUM PUBLICATIONS, VOLUME XI

THE THEORY OF APPROXIMATION

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NEW YORK
PUBLISHED BY THE
AMERICAN MATHEMATICAL SOCIETY
501 WEST 116TH STREET
1930

Photographically
reproduced *1951*
EDWARDS BROTHERS, INC.
Lithoprinters
ANN ARBOR, MICHIGAN

PREFACE

The title of this volume is an abbreviation for the more properly descriptive one: "Topics in the theory of approximation". It is a brief essay in a field on which an encyclopedia might be written. On the personal side, it is an account of certain aspects and ramifications of a problem to which I was introduced at an early stage, and which has given direction to my reading and study ever since.

One day about twenty years ago I was admitted to the study of Professor Landau, seeking advice as to a subject for a thesis. After some preliminary inquiries as to my experience and preferences, he handed me a long sheet of paper, and directed me to take notes as he enumerated some dozen or fifteen topics in various fields of analysis and number theory, with a few words of explanation of each. He told me to think about them for a few days, and to select one of them, or any other problem of my own choosing, with the single reservation that I should *not* prove Fermat's theorem, an injunction which I have observed faithfully. Guided partly by natural inclination, perhaps, and partly by recollection of a course on methods of approximation which I had taken with Professor Bôcher a few years earlier, I committed myself to one of the topics which Landau had proposed, an investigation of the degree of approximation with which a given continuous function can be represented by a polynomial of given degree. When I reported my choice, he said meditatively, in words which I remember vividly in substance, if not perfectly as to idiom: "Das ist ein schönes Thema, ich beneide Sie um das Thema . . . Nein, ich beneide Sie nicht, aber es ist ein wunderschönes Thema!" It is in fact a problem which admits a surprising variety of interesting developments on its own account, and offers a natural avenue of approach to a number of fields of still broader importance.

Although delayed in its completion by the conflict of other duties, the following exposition is substantially in the form in which it was projected at the time of the Colloquium lectures in 1925, and presented in abstract in the lectures themselves. One section, on the vector analysis of function space, originally designed for inclusion in the Colloquium, has meanwhile been published separately instead. The sections which had been written at full length in September, 1925 — practically the whole of the first chapter, parts of the second, and most of the third — have been left unchanged, except in minor details. The elementary account of Legendre series in Chapter I, for example, was written before the appearance of the admirable article on the subject by M. H. Stone in vol. 27 of the Annals of Mathematics. A few other articles published since 1925 are mentioned in the text.

For the most part, however, citations of the literature have been omitted. The preparation of a really adequate bibliography would have been a task of such magnitude as to delay the publication indefinitely. References to some of the most important papers of not too recent date are contained in my thesis (Göttingen, 1911) and in my report on *The general theory of approximation by polynomials and trigonometric sums* in vol. 27 of the Bulletin of the American Mathematical Society. Among publications in book form supplementing the material given here, mention should be made of Borel's *Leçons sur les fonctions de variables réelles et les développements en séries de polynomes,* de la Vallée Poussin's *Leçons sur l'approximation des fonctions d'une variable réelle,* and S. Bernstein's *Leçons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variable réelle,* all appearing in the Borel series. As to the content of these lectures themselves, there are many points where it would be difficult now to recall the original sources either of specific results and proofs or of suggestions as to method. To the extent that the work is my own, some parts have been published previously, in my thesis, in various articles in the Transactions of the American Mathematical Society, and elsewhere; other parts are now offered in print for the first time. Numerous detailed acknowledgments, not repeated here, have been made in the pages of the earlier publications. In connection with Chapter IV, reference should still be made to the work of Faber on trigonometric interpolation in his memoir *Über stetige Funktionen (zweite Abhandlung)* in vol. 69 of the Mathematische Annalen. My acquaintance with the statistical formulas discussed in Chapter V, which might have come from any of a variety of sources, was in fact mostly obtained from Yule's *Introduction to the Theory of Statistics.* The lemma on which the method of Chapter III depends is derived from the most important single memoir in the literature on degree of approximation, S. Bernstein's epoch-making prize essay of 1912, with which the present work also has other points of contact. And in conclusion it should be said that my study of the problem has been dominated from the beginning not only by the influence of my own teachers, but also by the writings of Lebesgue and de la Vallée Poussin.

October 1, 1929

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CHAPTER I

CONTINUOUS FUNCTIONS

Introduction

Weierstrass first enunciated the theorem that an arbitrary continuous function can be approximately represented by a polynomial with any assigned degree of accuracy. The theorem may be stated with precision in the following form:

If $f(x)$ is a given function, continuous for $a \leqq x \leqq b$, and if ε is a given positive quantity, it is always possible to define a polynomial $P(x)$ such that

$$|f(x) - P(x)| < \varepsilon$$

for $a \leqq x \leqq b$.

To Weierstrass is due also the corresponding theorem on approximation by means of trigonometric sums:

If $f(x)$ is a given function of period 2π, continuous for all real values of x, and if ε is a given positive quantity, it is always possible to define a trigonometric sum $T(x)$ such that

$$|f(x) - T(x)| < \varepsilon$$

for all real values of x.

By a *polynomial* is meant an expression of the form

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$

This expression will be said to represent a polynomial *of the nth degree,* not only when a_n is different from zero, but, in distinction from the usage which prevails in some parts of algebra, also when $a_n = 0$. That is to say, the words "polynomial of the nth degree" will be used in place of the longer expression "polynomial of the nth degree at most". Even the case of identical vanishing is not excluded. A *trig-*

onometric sum, or more specifically a trigonometric sum *of the nth order,* is an expression of the form

$$a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx$$
$$+ b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx.$$

The definition is inclusive once more; the simultaneous vanishing of a_n and b_n is not ruled out.

These two types of approximating function show a persistent and fundamental similarity in their behavior, on which differences of more or less significance are from time to time superimposed. Simplicity of statement and proof will favor sometimes one and sometimes the other.

It is readily seen that the number of terms required to yield a specified degree of approximation, or, under the converse aspect, the degree of approximation attainable with a specified number of terms, will be related to the properties of continuity of $f(x)$. It is the purpose of the next pages to trace out this relationship in some detail.

1. Approximation by trigonometric sums

For a considerable body of results, the following theorem may be regarded as fundamental:

THEOREM I. *If $f(x)$ is a function of period 2π, such that*

$$|f(x_2) - f(x_1)| \leq \lambda |x_2 - x_1|$$

for all real values of x_1 and x_2, λ being a constant, there will exist for every positive integral value of n a trigonometric sum $T_n(x)$, of the nth order, such that, for all real values of x,

$$|f(x) - T_n(x)| \leq \frac{K\lambda}{n},$$

where K is an absolute constant, depending neither on x, nor on n, nor on λ, nor on any further specification with regard to the function $f(x)$.

In the proof of the theorem, use will be made of the following

LEMMA. *If m is a positive integer, the expression*

$$\frac{\sin^4(mx/2)}{\sin^4(x/2)}$$

is a trigonometric sum in x, of order $2m-2$.

Because of the identity

$$\cos px \cos qx = \frac{1}{2}[\cos(p+q)x + \cos(p-q)x]$$

and the others of similar type, it is seen at once that the product of two trigonometric sums, of orders n_1 and n_2 respectively, is a trigonometric sum of order n_1+n_2. It is sufficient for the purpose in hand, therefore, to recall any one of the numerous proofs of the well-known fact that

$$\frac{\sin^2(mx/2)}{\sin^2(x/2)} = \frac{1-\cos mx}{1-\cos x}$$

is a trigonometric sum of order $m-1$; its square will then be a sum of order $2m-2$. The fact that $1-\cos mx$ is equal to the product of $1-\cos x$ by a trigonometric sum of order $m-1$ appears, for example, from the identities

$$1-\cos mx = \sum_{p=0}^{m-1}[\cos px - \cos(p+1)x],$$

$$\cos px - \cos(p+1)x$$
$$= (1-\cos x) - \sum_{q=1}^{p}[\cos(q-1)x - 2\cos qx + \cos(q+1)x],$$

$$\cos(q-1)x - 2\cos qx + \cos(q+1)x$$
$$= [\cos(q-1)x + \cos(q+1)x] - 2\cos qx$$
$$= 2\cos qx \cos x - 2\cos qx$$
$$= -2\cos qx(1-\cos x).$$

To proceed with the proof of the theorem, let

$$F_m(u) = \left[\frac{\sin mu}{m\sin u}\right]^4, \quad I_m(x) = h_m\int_{-\pi/2}^{\pi/2} f(x+2u)\,F_m(u)\,du,$$

where m is any positive integer, and h_m is defined by the equation

$$\frac{1}{h_m} = \int_{-\pi/2}^{\pi/2} F_m(u)\,du.$$

By means of the substitution $x+2u=v$, the expression for $I_m(x)$ is transformed into

$$\frac{1}{2}h_m\int_{x-\pi}^{x+\pi} f(v)\,F_m\left[\frac{1}{2}(v-x)\right]dv.$$

Both factors in the last integrand have the period 2π with regard to v, so that the value of the integral is unchanged if the interval of integration is replaced by any other interval of length 2π. In particular,

$$I_m(x) = \frac{1}{2}h_m\int_{-\pi}^{\pi} f(v)\,F_m\left[\frac{1}{2}(v-x)\right]dv.$$

The expression $F_m[\frac{1}{2}(v-x)]$, by the Lemma above, is a trigonometric sum of order $2m-2$ in $(v-x)$, and may be regarded as a trigonometric sum of the same order in x, with coefficients which are trigonometric functions of v. The whole integrand is a trigonometric sum in x with coefficients which are continuous functions of v, and $I_m(x)$ therefore is a trigonometric sum of order $2m-2$ in x, with constant coefficients. The proof that this sum is an approximate representation of $f(x)$, when m is large, will be based on the original representation of $I_m(x)$.

Let the equation defining h_m be multiplied by $h_m f(x)$. Since $f(x)$ is a constant as far as u is concerned, it may be placed under the sign of integration, so that

$$f(x) = h_m\int_{-\pi/2}^{\pi/2} f(x)\,F_m(u)\,du.$$

Consequently

$$I_m(x)-f(x) = h_m\int_{-\pi/2}^{\pi/2}[f(x+2u)-f(x)]\,F_m(u)\,du.$$

By the hypothesis imposed on $f(x)$,

$$|f(x+2u)-f(x)| \leqq 2\lambda\,|u|.$$

Hence

$$|I_m(x)-f(x)| \leqq 2\lambda h_m \int_{-\pi/2}^{\pi/2} |u|\, F_m(u)\, du,$$

or, since $F_m(u)$ and $|u|\, F_m(u)$ are even functions of u,

$$|I_m(x)-f(x)| \leqq 4\lambda h_m \int_0^{\pi/2} u\, F_m(u)\, du = 2\lambda \frac{\int_0^{\pi/2} u\, F_m(u)\, du}{\int_0^{\pi/2} F_m(u)\, du}.$$

To anticipate the conclusion of the proof, let

$$c_1 = \int_0^{\pi/2} \frac{\sin^4 t}{t^4}\, dt, \qquad c_2 = \int_0^{\infty} \frac{\sin^4 t}{t^3}\, dt.$$

These quantities are merely numerical constants. It is clear that each integrand approaches a limit for $t = 0$, and that the improper integral defining c_2 is convergent.

By the use of the fact that $0 < \sin u < u$ for $0 < u \leqq \pi/2$, and the substitution $mu = t$, it is recognized that

$$\int_0^{\pi/2} F_m(u)\, du > \int_0^{\pi/2} \left[\frac{\sin mu}{mu}\right]^4 du = \frac{1}{m} \int_0^{m\pi/2} \frac{\sin^4 t}{t^4}\, dt$$
$$\geqq \frac{1}{m} \int_0^{\pi/2} \frac{\sin^4 t}{t^4}\, dt = \frac{c_1}{m}.$$

On the other hand, $(\sin u)/u$ decreases monotonically as u goes from 0 to $\pi/2$, so that

$$\frac{\sin u}{u} > \frac{\sin(\pi/2)}{(\pi/2)} = \frac{2}{\pi}, \qquad \frac{1}{\sin u} < \frac{\pi}{2} \cdot \frac{1}{u}$$

throughout the interior of this interval. Hence

$$\int_0^{\pi/2} u\, F_m(u)\, du \leqq \left(\frac{\pi}{2}\right)^4 \int_0^{\pi/2} u \left[\frac{\sin mu}{mu}\right]^4 du$$
$$= \frac{1}{m^2} \left(\frac{\pi}{2}\right)^4 \int_0^{m\pi/2} \frac{\sin^4 t}{t^3}\, dt$$
$$< \frac{1}{m^2} \left(\frac{\pi}{2}\right)^4 \int_0^{\infty} \frac{\sin^4 t}{t^3}\, dt = \left(\frac{\pi}{2}\right)^4 \frac{c_2}{m^2}.$$

From these relations it follows that

$$|I_m(x)-f(x)| \leqq \frac{\pi^4 c_2 \lambda}{8 c_1 m}.$$

Now let n be an arbitrary integer, and let m be taken equal to $\frac{1}{2}n+1$ or $\frac{1}{2}(n+1)$, according as n is even or odd. In either case, $2m-2 \leq n < 2m$. Let the corresponding expression $I_m(x)$ be denoted by $T_n(x)$. Then $T_n(x)$ is a trigonometric sum of the nth order (it will be remembered that this is understood to mean of the nth order *at most*, according to the more usual terminology), and, since $1/m < 2/n$,

$$|I_m(x) - f(x)| \leq \frac{\pi^4 c_2 \lambda}{4 c_1 n} = \frac{K\lambda}{n},$$

if K is taken equal to $\pi^4 c_2/(4c_1)$. Thus the proof of the theorem is completed.

So much has been conceded to simplicity of outline, in building up the above inequalities, that the final upper limits are quite unnecessarily large, giving little indication of the actual magnitude of the quantities that precede. It will add a little to the definiteness of the conclusion to point out that $c_1 > (2/\pi)^3$, since $(\sin t)/t > 2/\pi$ throughout the interior of the interval of integration, while

$$c_2 = \int_0^1 t\cdot\frac{\sin^4 t}{t^4}\,dt + \int_1^\infty \frac{\sin^4 t}{t^3}\,dt < \int_0^1 t\,dt + \int_1^\infty \frac{dt}{t^3} = 1,$$

so that

$$K = \frac{\pi^4 c_2}{4c_1} < \frac{\pi^7}{32} < 100.$$

With more attention to detail, however, the estimate can be cut very much closer. The theorem is actually true with $K = 3$, instead of the value adopted above, or even with a somewhat smaller value of K. On the other hand, it can be shown that the statement is *not* generally true with a value of K smaller than $\pi/2$.

To pass on to a more general theorem, let $f(x)$ be an arbitrary continuous function of period 2π, and let $\omega(\delta)$ be the maximum of $|f(x_2)-f(x_1)|$ for $|x_2 - x_1| \leq \delta$. The function $\omega(\delta)$ has been called by de la Vallée Poussin the *modulus of continuity* of $f(x)$. With the word *maximum* replaced by *least upper bound*, it can be defined for any bounded function, whether continuous or not. The characteristic property of a uniformly continuous function is that $\lim_{\delta=0} \omega(\delta) = 0$.

Let $\varphi(x)$ be the continuous function of period 2π which takes on the same values as $f(x)$ at the points

$$-\pi, \quad -\pi + \frac{2\pi}{n}, \quad -\pi + \frac{4\pi}{n}, \quad \ldots, \quad \pi - \frac{2\pi}{n}, \quad \pi,$$

and is linear from each point of this set to the next. The graph of $\varphi(x)$ is a broken line, no segment of which has a slope greater than $\omega(2\pi/n)/(2\pi/n)$ in absolute value. In analytical language, $\varphi(x)$ satisfies the hypothesis of Theorem I, with

$$\lambda = \frac{\omega(2\pi/n)}{2\pi/n}.$$

For every positive integral value of n, therefore, there is a trigonometric sum $T_n(x)$, of the nth order, such that

$$|\varphi(x) - T_n(x)| \leqq \frac{K}{2\pi} \omega\left(\frac{2\pi}{n}\right).$$

On the other hand, any specified value of x differs by less than $2\pi/n$ from one of those for which f and φ are by definition equal to each other; neither $f(x)$ nor $\varphi(x)$ can differ by more than $\omega(2\pi/n)$ from the corresponding common value; and hence

$$|f(x) - \varphi(x)| \leqq 2\omega\left(\frac{2\pi}{n}\right)$$

for all values of x. If the quantity $K/(2\pi) + 2$ is denoted by K', the last two inequalities may be combined to yield the following statement:

THEOREM II. *If $f(x)$ is a continuous function of period 2π, with modulus of continuity $\omega(\delta)$, there exists for every positive integral value of n a trigonometric sum $T_n(x)$, of the nth order, such that, for all real values of x,*

$$|f(x) - T_n(x)| \leqq K'\omega\left(\frac{2\pi}{n}\right),$$

where K' is an absolute constant.

While this theorem is applicable to any continuous function, it involves the modulus of continuity in the inequality which forms the essence of its conclusion. It can be shown that the assignment of an outer limit of error for an arbitrary continuous function, without some dependence on properties of the function beyond the mere fact of its continuity, is impossible.

Since $\lim_{n=\infty} \omega(2\pi/n) = 0$, it is to be noted that Theorem II includes one of the theorems of Weierstrass to which reference was made in the opening lines of the chapter.

In preparation for the next developments, there is occasion to examine more closely the proof that was given above for Theorem I. It will be recalled that to an arbitrary positive integer n a second positive integer m was assigned, in terms of which a function $F_m(u)$ was constructed; and a trigonometric sum $T_n(x)$, yielding an approximate representation of the given function $f(x)$, was defined as equal to an expression which could be reduced to the form

$$\frac{1}{2} h_m \int_{-\pi}^{\pi} f(v)\, F_m\left[\frac{1}{2}(v-x)\right] dv,$$

h_m being independent of x. A lemma stated essentially that $F_m(\frac{1}{2}u)$ is a trigonometric sum in u, of order $2m-2 \leq n$. It is possible therefore to write $F_m[\frac{1}{2}(v-x)]$ in the form

$$\frac{1}{2} A_{n0} + \sum_{k=1}^{n} [A_{nk} \cos k(v-x) + B_{nk} \sin k(v-x)].$$

When the above expression for $T_n(x)$ is expanded as a trigonometric sum in x, the constant term is

$$\frac{1}{4} h_m A_{n0} \int_{-\pi}^{\pi} f(v)\, dv,$$

and is zero if the last integral vanishes, an observation which will presently be important, for the reason that the indefinite integral of a trigonometric sum without constant term is itself a trigonometric sum, while this is not the case if the sum to be integrated has a constant term different from zero.

It may be pointed out in this connection, though it is not essential to the main argument, that the coefficients B_{nk} are all zero. This can be inferred from an elementary theorem on trigonometric sums, since $F_m(\frac{1}{2}u)$ is an even function of u, and is also directly apparent on inspection of the proof of the lemma. If a_k, b_k are the Fourier coefficients of $f(x)$:

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(v)\cos kv\,dv, \quad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(v)\sin kv\,dv,$$

and if $\frac{1}{2}\pi h_m A_{nk}$ is denoted by d_{nk}, it is seen that

$$T_n(x) = \frac{1}{2}d_{n0}\,a_0 + \sum_{k=1}^{n} d_{nk}(a_k\cos kx + b_k \sin kx).$$

As the d's are independent of the function to be represented, the calculation of the successive expressions $T_n(x)$ amounts to a method of summation of the Fourier series for $f(x)$.

To return from the digression of the last paragraph, let $f(x)$ be a function of period 2π, which has everywhere a continuous derivative $f'(x)$. For a particular value of n, let $t_n'(x)$ be a trigonometric sum of the nth order, without constant term:

$$t_n'(x) = \sum_{k=1}^{n}(\alpha_k \cos kx + \beta_k \sin kx),$$

and let ε_n be a constant such that

$$|f'(x) - t_n'(x)| \leqq \varepsilon_n$$

for all values of x. Let $t_n(x)$ be the trigonometric sum, without constant term, which has $t_n'(x)$ for its derivative:

$$t_n(x) = \sum_{k=1}^{n}\left(\frac{\alpha_k}{k}\sin kx - \frac{\beta_k}{k}\cos kx\right),$$

and let $r_n(x) = f(x) - t_n(x)$. *Then $r_n(x)$ has the period 2π, and, since $|r_n'(x)| \leqq \varepsilon_n$, satisfies the conditions imposed on $f(x)$ in the hypothesis of Theorem* I, *with $\lambda = \varepsilon_n$.* Hence there exists a trigonometric sum of the nth order, which may be denoted by $\tau_n(x)$, such that

$$|r_n(x) - \tau_n(x)| \leqq \frac{K\varepsilon_n}{n}.$$

If $T_n(x) = t_n(x) + \tau_n(x)$, then $f(x) - T_n(x) = r_n(x) - \tau_n(x)$, and

$$|f(x) - T_n(x)| \leqq \frac{K\varepsilon_n}{n}.$$

From the existence of an approximation for $f'(x)$ it has been possible to draw an important inference with regard to the approximation of $f(x)$. If $f(x)$ is itself the derivative of a function of period 2π, so that the integral of $f(x)$ over an interval of length 2π is zero, it follows that

$$\int_{-\pi}^{\pi} r_n(x)\,dx = \int_{-\pi}^{\pi} f(x)\,dx - \int_{-\pi}^{\pi} t_n(x)\,dx = 0,$$

whence, according to the second paragraph preceding, the sum $\tau_n(x)$ given by the proof of Theorem I as an approximation for $r_n(x)$ will have no constant term. So the constant term in the present $T_n(x)$, defined in terms of this $\tau_n(x)$, will be zero likewise.

The way has now been prepared for a demonstration of

THEOREM III. *If $f(x)$ is a function of period 2π, having a pth derivative $f^{(p)}(x)$ such that*

$$|f^{(p)}(x_2) - f^{(p)}(x_1)| \leqq \lambda\,|x_2 - x_1|$$

for all real values of x_1 and x_2, λ being a constant, there will exist for every positive integral value of n a trigonometric sum $T_n(x)$, of the nth order, such that, for all real values of x,

$$|f(x) - T_n(x)| \leqq \frac{K^{p+1}\lambda}{n^{p+1}},$$

where K is the absolute constant found in the proof of Theorem I.

It is to be noticed that the argument is based on the explicit construction of the approximating sum in Theorem I,