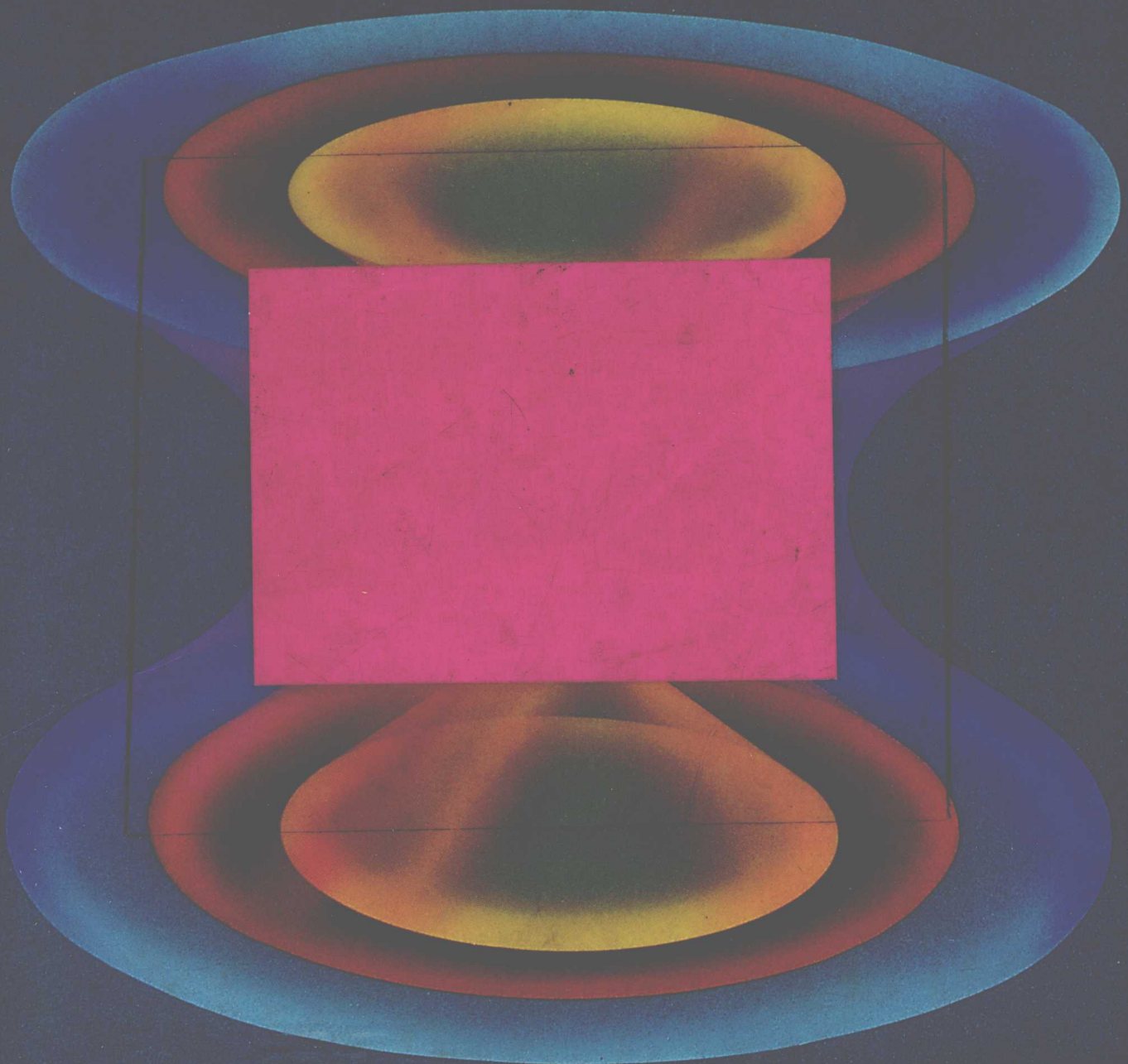


CALCULUS

ROBERT SEELEY



CALCULUS

Robert Seeley

University of Massachusetts at Boston



HARCOURT BRACE JOVANOVICH, PUBLISHERS

and its subsidiary, Academic Press

San Diego New York Chicago Austin Washington, D.C.
London Sydney Tokyo Toronto

Copyright © 1990 by Harcourt Brace Jovanovich, Inc.

All rights reserved. No part of this publication may be reproduced or transmitted in any form or by any means, electronic or mechanical, including photocopy, recording, or any information storage and retrieval system, without permission in writing from the publisher.

Requests for permission to make copies of any part of the work should be mailed to: Copyrights and Permissions Department, Harcourt Brace Jovanovich, Publishers, Orlando, Florida 32887.

ISBN: 0-15-505681-6

Library of Congress Catalog Card Number: 89-84683

Printed in the United States of America

PREFACE

Mathematics, and particularly calculus, provides an essential language and intellectual framework for science. With the explosive growth of science and technology comes a growing need for mastery of calculus, at least as a language, and better yet, as a way of thinking.

The wide variation in backgrounds and talents of calculus students presents a real challenge both to the instructor and to the textbook used. This book responds by exposing the main ideas of calculus as quickly as possible, before elaborating all the details. For example, a chapter on the applications of calculus to problems of graphing, optimization, and motion precedes any real discussion of limits. The idea is to exploit the students' geometric intuition, which is generally much better developed than their facility in calculation.

In the course of class testing over the years, I have found only good consequences of this approach. Students easily grasp the basic ideas of the applications of calculus and, with the understanding of this background, develop the maturity and motivation necessary for confronting the technical questions of limits. The same philosophy determines the treatment of infinite series and other topics. The textbook presents infinite series as a means of calculation *before* developing the abstract convergence tests.

Such minor reordering of traditional topics does not imply a lax treatment of the subject. Theoretical questions are treated with the prevailing standards of rigor; they are just treated somewhat later than usual. The difficult parts of the theory—such as the proofs of the Intermediate Value Theorem and Extreme Value Theorem and the justification of term-by-term operations on infinite series—are postponed to an appendix. The $\varepsilon - \delta$ definition of function limits is also in the appendix because very few students are ready for it early in the course. But the $\varepsilon - N$ definition of sequence limits is in the main body of the text in Chapter 11; by that point the definition seems natural.

The transcendental functions are presented in the order that is currently standard. The trigonometric functions are discussed early and provide a variety of examples for limits and derivatives. The exponential and logarithmic functions follow integrals, but could actually be covered at any time after Chapter 4. (Limits involving these functions are treated by l'Hopital's Rule in Chapter 4.)

Differential equations, discussed as slope fields, are introduced as soon as possible, beginning with antiderivatives in Chapter 2. They provide significant examples and problems throughout the text. The final chapter presents complex numbers and their use in solving constant coefficient equations, particularly the damped oscillator equation.

The textbook uses several pedagogical features to reinforce topic presentations:

- *An extensive art program* illustrates concepts and problems with two- or full-color figures.
- *Examples*, essential to illustrate and clarify general principles, are used extensively. They are clearly set off from the text discussions and are worked in great detail. Students should be encouraged to do their own work in similar detail to clarify thinking and avoid careless errors.
- *Section summaries* precede the problem sets for that section so that students can easily review as they work the problems.
- The *problems* are grouped in "A," "B," and "C" sets in order of increasing maturity. "A" problems are straightforward applications of the text material, "B" problems require more initiative or originality to work, and "C" problems are for students who show greater mathematical sophistication.
- *Review problems* end each chapter.
- *Answers* to problems marked with an asterisk (*) are given at the end of the book. (Answers to all the problems are given in the Instructor's Manual.)

ACKNOWLEDGMENTS

A book such as this is not created single-handedly. I owe a great debt to my students for their patience in using class notes and for their suggestions. My family, too, has contributed a good share of patience. My son Karl did much of the word processing, while Joe provided answers to many problems. I am grateful to my colleagues, particularly to Matt Gaffney for a multitude of suggestions and conversations about the book, to Steve Parrott for reading all of one version, and to Paul Salomaa for providing many answers; but most of all I am grateful to Dennis Wortman, who undertook the major task of writing the Instructor's Manual.

The book would not have been published without the support of several people. John Parker of Harcourt Brace Jovanovich kept an interest as the book developed. It has had essential support from a sequence of editors. Shelley Langman, Wesley Lawton, Ted Buchholz, and Richard Wallis presided over its development and provided support for the class testing, and Mike Johnson brought it into the world. The conscientious and cooperative book team included Robert Watrous, manuscript editor; Karen Denhams, production editor; Martha Gilman, designer; Vicki Kelly, art editor; Lesley Lenox, production manager; and Pamela Whiting, associate editor.

Comments from reviewers have, of course, been essential. They all have my grateful thanks: Charles C. Alexander, University of Mississippi; Glen D. Anderson, Michigan State University; Thomas F. Banchoff, Brown University; Douglas B. Crawford, College of San Mateo; Daniel S. Drucker, Wayne State University; Roger W. Hansell, University of Connecticut; James E. Hodge, Angelo State University; Kendell Hyde, Weber State College; Eleanor L. Kendrick, San Jose City College; Eleanor Killam, University of Massachusetts,

Amherst; Matthew Liu, University of Wisconsin, Stevens Point; John Montgomery, University of Rhode Island; David Price, Tarrant County Junior College; Robert G. Russell, West Valley College; Donald Schmidt, University of Northern Colorado; and Keith L. Wilson, Oklahoma City Community College.

Robert Seeley

SUPPLEMENTS

For Students

Student Solutions Manual by Dennis Wortman contains solutions for those problems that have answers in the text. (These are marked in the text with an asterisk.) The solutions are worked out with unusual care and completeness, and will be particularly instructive to those who want this extra help.

For Instructors

Instructor's Manual with Solutions by Dennis Wortman contains the worked-out solutions for *all* the problems in the text.

Computerized Testbank (Micro-Pac Genie) by Microsystems Software, Ltd., is the most complete test-generating and author system with graphics on the market. This system is available for the IBM PC, XT or compatible system, and the Macintosh.

Testbank is also available in a printed version.

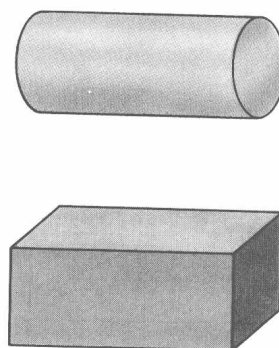
COMPUTER SUPPLEMENTS FOR INSTRUCTORS AND STUDENTS

CALCULUS 3.0 from True Basic is intended for users who wish to expand their basic understanding. Easy-to-use features include a simple, menu-driven interface and context-sensitive pop-up HELP and GLOSSARY screens. A dialog box is used to enter numbers and functions and to execute problems; a graph box displays plotted functions.

Calculus encourages free exploration of topics including the concept of limits, tangents, l'Hopital's Rule, as well as exploring and visualizing parametric equations and differential equations. It is available for Apple Macintosh, IBM PC, and IBM PS/2.

INTERACTIVE CALCULUS from Math Lab allows students to experience complex mathematical ideas in a graphic and intuitive setting. Concepts such as limits, differentiation, and integration are numerically and graphically performed in a single keystroke. It is available for Apple II, IBM PC, and IBM PS/2.

CALCAIDE, a computer software program by Elizabeth Chang, Hood College, is a microcomputer disk for IBM PC and IBM PS/2 systems with interactive color graphing and numerical computation programs for the mainstream calculus course.



CONTENTS

CHAPTER 1 BACKGROUND 1

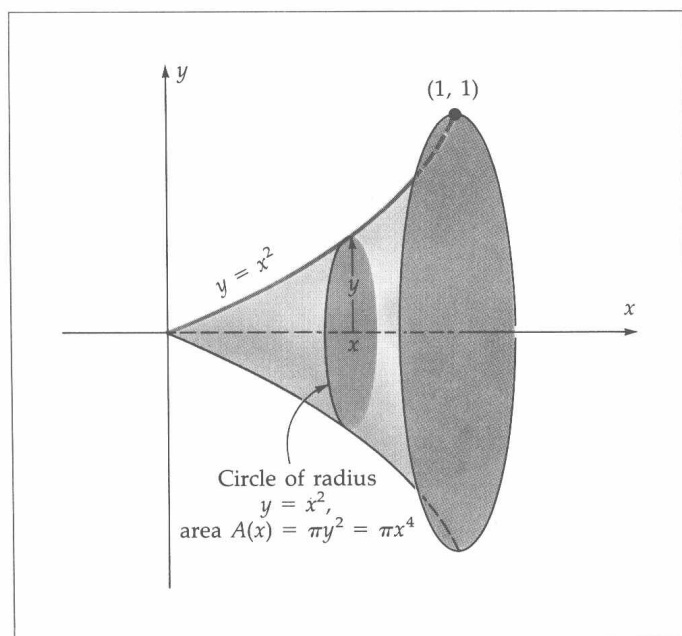
- 1.1 Coordinates 2
- 1.2 Straight Lines and Slope 9
- 1.3 Functions 18
- 1.4 Optimization 25
- Review Problems 28

CHAPTER 2 THE DERIVATIVE OF A POLYNOMIAL 31

- 2.1 The Derivative and the Tangent Line 32
- 2.2 Quadratic Functions 40
- 2.3 The Derivative of a Polynomial 45
- Appendix: The Derivative of x^n . Factoring Polynomials 51
- 2.4 Graphing with the Derivative 53
- 2.5 The Second Derivative. Concavity 61
- 2.6 Newton's Method (optional) 66
- 2.7 Velocity and Acceleration 70
- 2.8 Antiderivatives 77
- Review Problems 86

CHAPTER 3 LIMITS, CONTINUITY, AND DERIVATIVES 89

- 3.1 Limits 90
- 3.2 Some Trigonometric Limits. The Trapping Theorem 99
- 3.3 Limits as $x \rightarrow +\infty$ or $x \rightarrow -\infty$ 111
- 3.4 Continuity. The Intermediate Value Theorem 120
- 3.5 Derivatives. The Product and Quotient Rules 131
- 3.6 Derivatives of the Trigonometric Functions 142
- 3.7 Composite Functions. The Chain Rule 148
- 3.8 Implicit Differentiation. The Derivative of $x^{m/n}$ 157
- Review Problems 174



CHAPTER 4 MORE APPLICATIONS OF DERIVATIVES 177

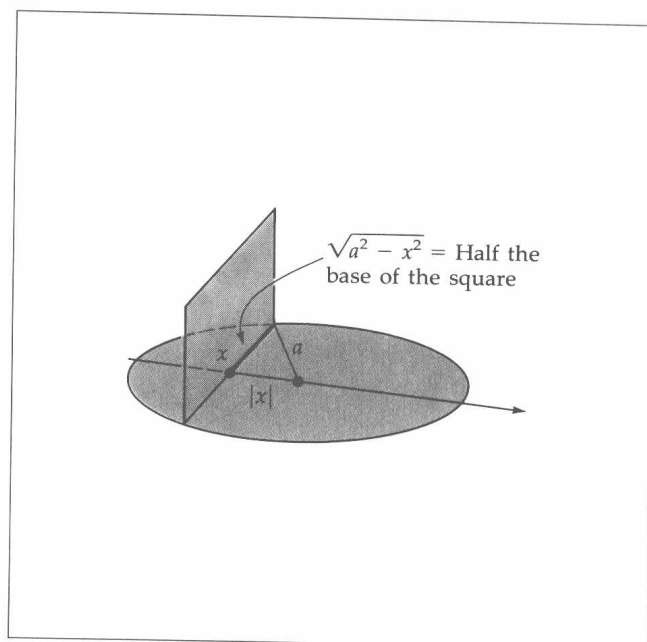
- 4.1 Rate of Change. The Linear Approximation 178
- 4.2 Related Rates 186
- 4.3 Parametric Curves and l'Hopital's Rule 194
- 4.4 Fermat's Principle and the Law of Refraction (optional) 202
- 4.5 The Lorenz Curve (optional) 208
- Review Problems 210

CHAPTER 5 THE MEAN VALUE THEOREM 213

- 5.1 The Extreme Value Theorem and the Critical Point Theorem 214
- 5.2 The Mean Value Theorem 220
- 5.3 Applications to Graphing and Optimization 226
- Review Problems 232

CHAPTER 6 INTEGRALS 235

- 6.1 The Integral as Signed Area 236
- 6.2 Summation Notation. Limits of Riemann Sums 245
- 6.3 The Integral of a Derivative. The First Fundamental Theorem of Calculus 252
- 6.4 The Integral of a Rate of Change. Trapezoid Sums 258
- 6.5 Properties of Integrals. Mean Value 265
- 6.6 The Second Fundamental Theorem of Calculus 275
- 6.7 Indefinite Integrals 280
- 6.8 Differentials and Substitution 285
- Review Problems 292



CHAPTER 7 SOME APPLICATIONS OF INTEGRALS 295

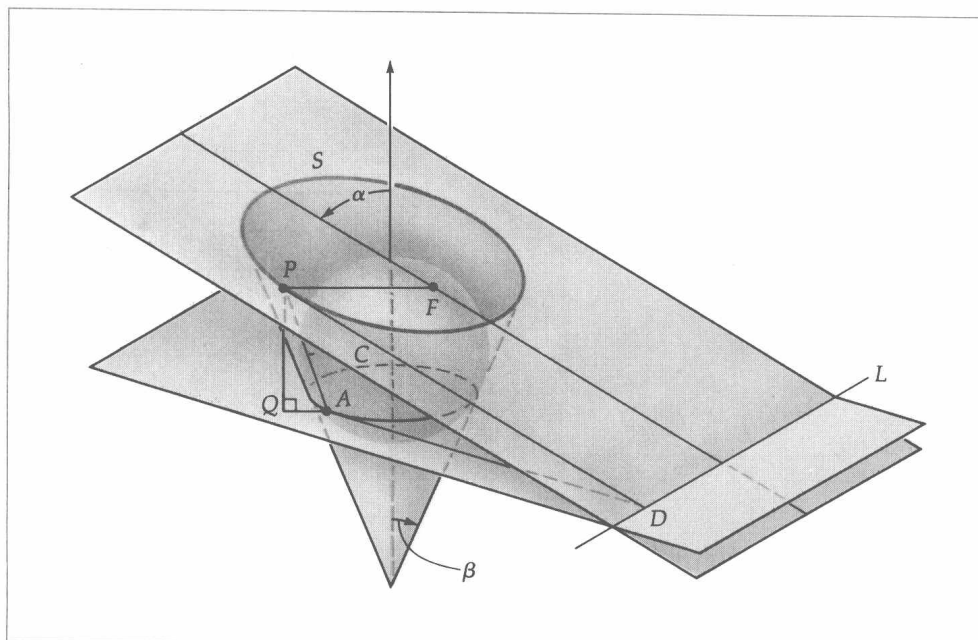
- 7.1 Areas by Integration 296
- 7.2 Volumes by Cross Sections 301
- 7.3 Volumes by Cylindrical Shells. Flow in Pipes 312
- 7.4 Energy and Work 318
- 7.5 Improper Integrals 325
- Review Problems 332

CHAPTER 8 EXPONENTIALS, LOGARITHMS, AND INVERSE FUNCTIONS 335

- 8.1 The Natural Exponential Function 336
- 8.2 Inverse Functions 346
- 8.3 Natural Logarithms 352
- 8.4 Other Bases. Logarithmic Differentiation. Indeterminate Forms 361
- 8.5 Exponential and Logarithmic Integrals 364
- 8.6 Proportional Growth and Decay 366
- 8.7 Separable Differential Equations 371
- 8.8 Inverse Trigonometric Functions 381
- 8.9 Oscillations (optional) 390
- 8.10 The Hyperbolic Functions 400
- Review Problems 404

CHAPTER 9 TECHNIQUES OF INTEGRATION 407

- 9.1 Integration by Parts 408
- Appendix: The Differential Equation $\frac{dy}{dt} + by = f(t)$ (optional) 415
- 9.2 Powers and Products of Sines and Cosines 417
- 9.3 Powers of Tangent and Secant 424
- 9.4 Trigonometric Substitution 429
- 9.5 Some Algebraic Methods 436
- 9.6 Integrating Rational Functions by Partial Fractions 442
- 9.7 Rationalizing Substitutions 452
- Review Problems 454



CHAPTER 10 MORE APPLICATIONS OF INTEGRALS 457

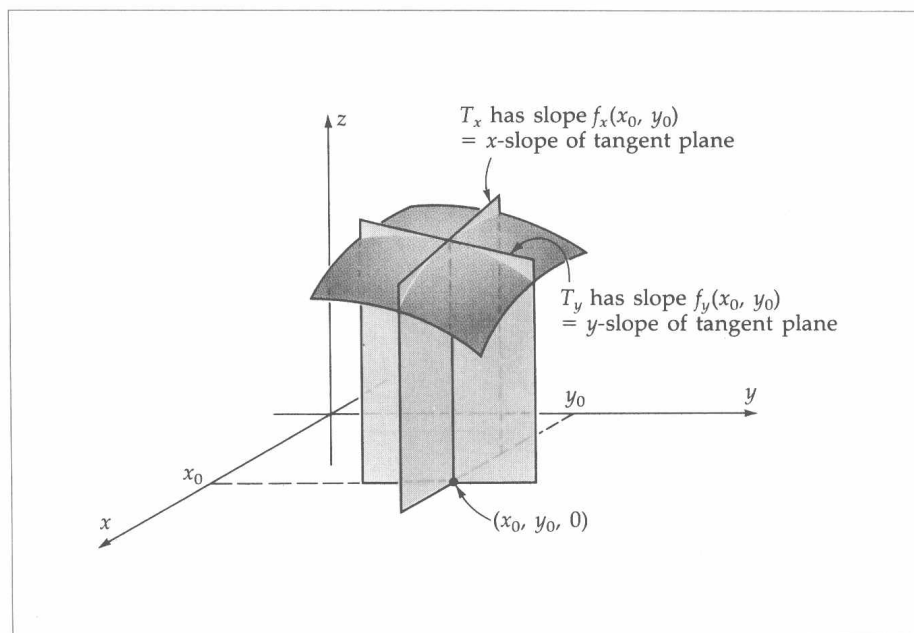
- 10.1 Arc Length 458
- 10.2 Area of a Surface of Revolution 463
- 10.3 Center of Mass 467
- 10.4 Probability Density Functions 472
- 10.5 Numerical Integration 480
- Appendix: Proving the Remainder Estimates 490
- Review Problems 494

CHAPTER 11 INFINITE SEQUENCES AND SERIES 495

- 11.1 Taylor Polynomials 496
- Appendix: The Remainder for Taylor Polynomials 506
- 11.2 Limit of a Sequence Defined 508
- 11.3 More on Sequences 519
- 11.4 Infinite Series 528
- 11.5 Taylor Series 536
- 11.6 Operations on Power Series 542
- 11.7 Estimating Remainders (optional) 549
- 11.8 Convergence Tests. The Integral Test 553
- 11.9 The Comparison Test 562
- 11.10 Alternating Series 567
- 11.11 Absolute and Conditional Convergence 570
- 11.12 The Ratio Test. The Radius of Convergence 574
- Review Problems 581

CHAPTER 12 CONIC SECTIONS. POLAR COORDINATES 583

- 12.1 Parabolas. Completing the Square 584
- 12.2 Ellipses 591
- 12.3 Hyperbolas 598
- 12.4 General Quadratics. Rotation of Axes 606
- 12.5 Polar Coordinates 614
- 12.6 Conic Sections in Polar Coordinates. Eccentricity 620
- 12.7 Area in Polar Coordinates 628
- Review Problems 634



CHAPTER 13 PLANE VECTORS 635

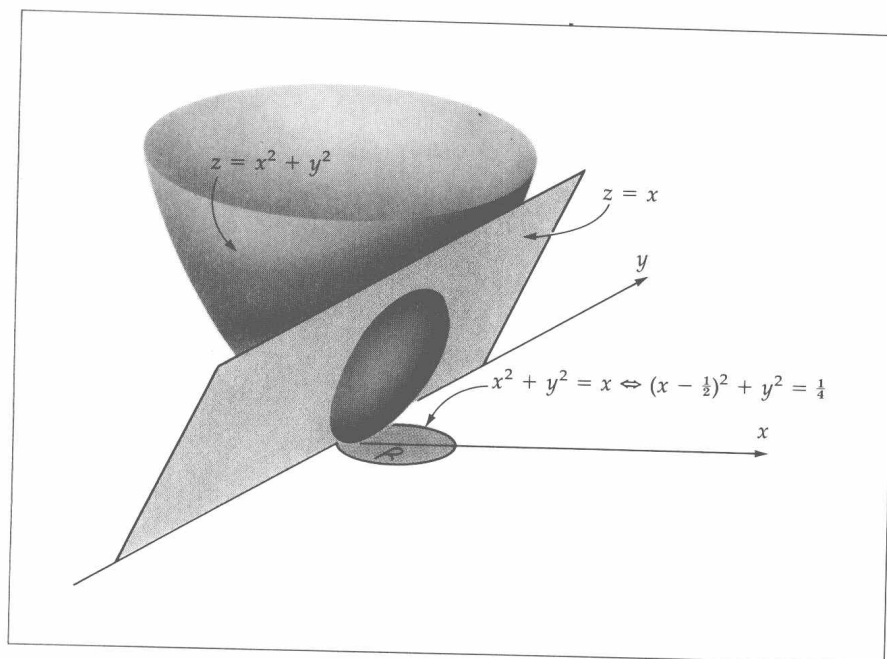
- 13.1 The Vector Space R^2 636
- 13.2 Parametric and Vector Equations 645
- 13.3 Limits and Derivatives. Velocity and Acceleration 651
- 13.4 Arc Length 659
- 13.5 Planetary Motion 665
- Review Problems 671

CHAPTER 14 SPACE COORDINATES AND VECTORS 675

- 14.1 Space Coordinates 676
- 14.2 The Vector Space R^3 . The Dot Product 680
- 14.3 The Cross Product. Determinants 689
- 14.4 Lines and Planes 698
- 14.5 Vector Functions and Space Curves 706
- 14.6 Quadric Surfaces. Cylinders 716
- Review Problems 724

CHAPTER 15 FUNCTIONS OF TWO VARIABLES 727

- 15.1 Graphs and Level Curves 728
- 15.2 Continuity and Limits 736
- 15.3 Partial Derivatives. The Tangent Plane and the Gradient 741
- 15.4 The Linear Approximation. Tangency 748
- 15.5 The Derivative along a Curve. The Directional Derivative 757
- 15.6 Higher Derivatives. Mixed Partials. C^k Functions 768
- 15.7 Critical Points. The Second Derivative Test 772
- 15.8 The Least Squares Line (optional) 782
- Review Problems 785



CHAPTER 16 FUNCTIONS OF SEVERAL VARIABLES. FURTHER TOPICS 787

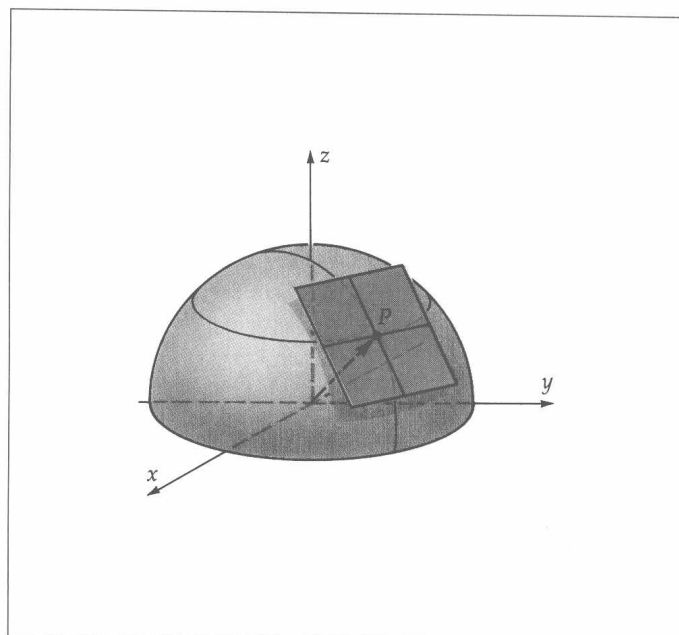
- 16.1 Functions of Three Variables 788
- 16.2 The General Chain Rule 795
- 16.3 Implicit Functions 801
- 16.4 Exact Differential Equations (optional) 807
- 16.5 Lagrange Multipliers (optional) 813
- Review Problems 824

CHAPTER 17 MULTIPLE INTEGRALS 825

- 17.1 Double Integrals 826
- 17.2 More General Regions 835
- 17.3 Mass, Center of Mass, Moment of Inertia 844
- 17.4 Double Integrals in Polar Coordinates 850
- 17.5 Bivariate Probability Densities (optional) 856
- 17.6 Surface Area 860
- 17.7 Triple Integrals 865
- 17.8 Cylindrical and Spherical Coordinates 872
- 17.9 General Coordinate Systems, The Jacobian (optional) 879
- Review Problems 886

CHAPTER 18 LINE AND SURFACE INTEGRALS 887

- 18.1 Work and Line Integrals 888
- 18.2 Gradient Fields, Curl 898
- 18.3 Green's Theorem 908
- 18.4 Surface Integrals, Gauss' Theorem 916
- 18.5 Stokes' Theorem 926
- Review Problems 932



CHAPTER 19
COMPLEX NUMBERS AND
DIFFERENTIAL EQUATIONS
935

- | | | |
|------|---|-----|
| 19.1 | Complex Numbers | 936 |
| 19.2 | Multiplication of Complex
Numbers | 940 |
| 19.3 | Complex Functions of a
Real Variable | 946 |
| 19.4 | Damped Oscillations | 951 |
| 19.5 | Forced Oscillations | 958 |

APPENDIX A
PROOFS OF THE HARD
THEOREMS 963

- | | | |
|-----|--|-----|
| A.1 | The Definition of Limit | 964 |
| A.2 | Some Limit Theorems | 971 |
| A.3 | Continuity | 975 |
| A.4 | The Intermediate Value Theorem | 978 |
| A.5 | The Extreme Value Theorem | 980 |
| A.6 | Uniform Convergence. Term-by-term Operations on Series | 983 |

APPENDIX B
REVIEW OF EXPONENTS
AND LOGARITHMS 991APPENDIX C
REVIEW OF
TRIGONOMETRY 999

SELECTED ANSWERS 1009

INDEX 1045

1

BACKGROUND

1.1

COORDINATES

We begin by reviewing basic ideas and notation concerning coordinate systems. (Some of you know all of this, and all of you know some of it; that's the nature of a review.) Then we use coordinates to investigate two "optimization" problems, typical of those to be solved by calculus later, in Chapters 2–4.

The Number Line

The humble ruler (Fig. 1) suggests the basic premise of analytic geometry: *The points of a straight line can be identified with numbers.* This appealing idea gave rise to some deep questions: What is a line? What is a number? Fortunately the questions have been answered¹, and we accept this premise as the basis for uniting algebra and geometry.

To identify numbers with points on a horizontal line, choose a unit of distance and an origin. Then label each point according to its distance from the origin, with negative numbers to the left and positive ones to the right (Fig. 2). You now have a **coordinate axis**, or a **number line**. The axis is the line itself; the number associated with a point on the line is a coordinate.

On the coordinate axis, the distance from point x to the origin is the **absolute value of x** , denoted $|x|$ (Fig. 3):

$$|-2| = 2, \quad |2| = 2, \quad |-3| = 3, \quad |0| = 0.$$

The absolute value $|x|$ gives the "size" of x but not its sign; that is, both 2 and -2 have the same absolute value 2. In algebraic terms, the absolute value is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

That $|x| = x$ when $x \geq 0$ is clear; but to understand the definition when $x < 0$, try for example $x = -2$:

$$|-2| = 2 = -(-2)$$

so indeed $|x| = -x$ when x is the negative number -2 .

With a little thought and experimentation, you can understand two properties of the absolute value:

$$|xy| = |x| \cdot |y|$$

and

$$|x + y| \leq |x| + |y|.$$

¹ One of the best answers was given by Richard Dedekind in 1872; see James R. Newman *The World of Mathematics*, Vol. 1, pp 525–36.

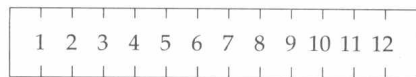


FIGURE 1
Humble ruler.

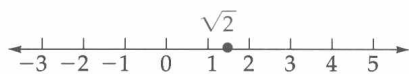


FIGURE 2
Number line.

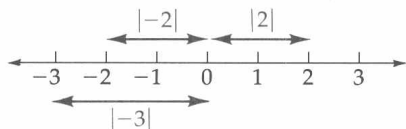


FIGURE 3
 $|x|$ = distance from x to 0.



FIGURE 4

$a < b$ means a is to the left of b .

Numbers are *ordered* according to their positions on the line, as shown in Figure 4:

$a < b$ means a is to the left of b .

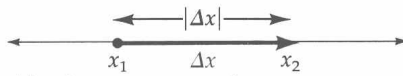
So $-100 < 2$; in words, -100 is less than 2 . To compare size without regard to sign, use the absolute value; the inequality

$$|2| < |-100|$$

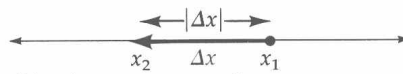
says that 2 is “smaller” than -100 .

The *difference* of two numbers x_1 and x_2 is denoted Δx :

$$\Delta x = x_2 - x_1.$$



(a) $\Delta x = x_2 - x_1 > 0$, so arrow points to the right.



(b) $\Delta x = x_2 - x_1 < 0$, so arrow points to the left.

(The symbol Δ is the Greek capital letter delta, which corresponds to our letter D , for difference. Thus Δx is the *difference* of two values of x , not the product of a number Δ times a number x .) To form the difference, you need to know which number is first and which is second, that is, you need an *ordered* pair: x_1 first and x_2 second. Then Δx is *positive* if $x_2 > x_1$, and *negative* if $x_2 < x_1$. As in Figure 5 we indicate Δx by an arrow from x_1 to x_2 ; it points to the right if $\Delta x > 0$, and to the left if $\Delta x < 0$. The length of the arrow is the absolute value $|\Delta x|$, which gives the *distance* between x_1 and x_2 :

$$\text{distance from } x_1 \text{ to } x_2 = |x_2 - x_1| = |\Delta x|.$$

Segments on the line are called **intervals** of numbers. It can be crucial to know whether the endpoints are included or not. Square brackets indicate an included endpoint, parentheses an excluded one (Fig. 6). For example,

$[a, b)$ = all numbers x between a and b , including a and excluding b .

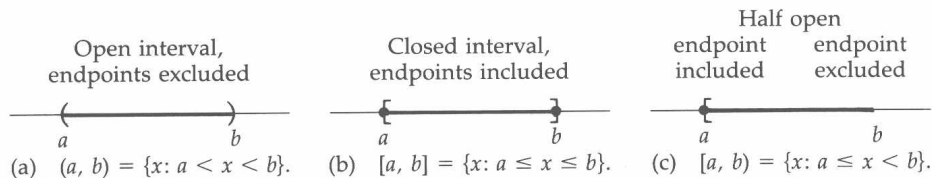


FIGURE 6

Intervals: Open, closed, half open.

If both endpoints are included, the interval is called **closed**; if both are excluded, it is **open**.

To describe intervals, and sets in general, we use the concise “set-builder notation.” With this, the interval $[a, b)$ is described as

$$\{x: \overbrace{a \leq x < b}^{\text{Condition determining whether } x \text{ is in the set.}}\}$$

Typical member of set.