Lecture Notes in Mathematics

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Topics in Orbit Equivalence

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Preface

(A) These notes provide an introduction to some topics in orbit equivalence theory, a branch of ergodic theory. One of the main concerns of ergodic theory is the structure and classification of measure preserving (or more generally measure-class preserving) actions of groups. By contrast, in orbit equivalence theory one focuses on the equivalence relation induced by such an action, i.e., the equivalence relation whose classes are the orbits of the action. This point of view originated in the pioneering work of Dye in the late 1950's, in connection with the theory of operator algebras. Since that time orbit equivalence theory has been a very active area of research in which a number of remarkable results have been obtained.

Roughly speaking, two main and opposing phenomena have been discovered, which we will refer to as *elasticity* (not a standard terminology) and *rigidity*. To explain them, we will need to introduce first the basic concepts of orbit equivalence theory.

In these notes we will only consider countable, discrete groups Γ. If such a group Γ acts in a Borel way on a standard Borel space X, we denote by E^X_Γ the corresponding equivalence relation on X:

$$xE^X_\Gamma y \Leftrightarrow \exists \gamma \in \Gamma(\gamma \cdot x = y).$$

If μ is a probability (Borel) measure on X, the action *preserves* μ if $\mu(\gamma \cdot A) = \mu(A)$, for any Borel set $A \subseteq X$ and $\gamma \in \Gamma$. The action (or the measure) is *ergodic* if every Γ-invariant Borel set is null or conull.

Suppose now Γ acts in a Borel way on X with invariant probability measure μ and Δ acts in a Borel way on Y with invariant probability measure ν. Then these actions are *orbit equivalent* if there are conull invariant Borel sets $A \subseteq X$, $B \subseteq Y$ and a Borel isomorphism $\pi : A \to B$ which sends μ to ν (i.e., $\pi_* \mu = \nu$) and for $x, y \in A$:

$$xE^X_\Gamma y \Leftrightarrow \pi(x) E^Y_\Delta \pi(y).$$

We can now describe these two competing phenomena:

(I) Elasticity: For amenable groups there is exactly one orbit equivalence type of non-atomic probability measure preserving ergodic actions. More precisely, if Γ, Δ are amenable groups acting in a Borel way on X, Y with non-atomic, invariant, ergodic probability measures μ, ν, respectively, then these two actions are orbit equivalent. This follows from a combination of Dye's work with subsequent work of Ornstein-Weiss in the 1980's. Thus the equivalence relation induced by such an action of an amenable group does not "encode" or "remember" anything about the group (beyond the fact that it is amenable). For example, any two free, measure preserving ergodic actions of the free abelian groups $\mathbb{Z}^m, \mathbb{Z}^n$ $(m \neq n)$ are orbit equivalent.

(II) Rigidity: As originally discovered by Zimmer in the 1980's, for many non-amenable groups Γ we have the opposite situation: The equivalence relation induced by a probability measure preserving action of Γ "encodes" or "remembers" a lot about the group (and the inducing action). For example, a recent result of Furman, strengthening an earlier theorem of Zimmer, asserts that if the canonical action of $\mathrm{SL}_n(\mathbb{Z})$ on $\mathbb{T}^n$ $(n \geq 3)$ is orbit equivalent to a free, non-atomic probability measure preserving, ergodic action of a countable group Γ, then Γ is isomorphic to $\mathrm{SL}_n(\mathbb{Z})$ and under this isomorphism the actions are also Borel isomorphic (modulo null sets). Another recent result, due to Gaboriau, states that if the free groups F_m, F_n $(1 \leq m, n \leq \aleph_0)$ have orbit equivalent free probability measure preserving Borel actions, then $m = n$. (This should be contrasted with the result mentioned in (I) above about $\mathbb{Z}^m, \mathbb{Z}^n$.)

(B) These notes are divided into three chapters. The first, very short, chapter contains a quick introduction to some basic concepts of ergodic theory.

The second chapter is primarily an exposition of the "elasticity" phenomenon described above. Some topics included here are: amenability of groups, the concept of hyperfiniteness for equivalence relations, Dye's Theorem to the effect that hyperfinite equivalence relations with non-atomic, invariant ergodic probability measures are Borel isomorphic (modulo null sets), quasi-invariant measures and amenable equivalence relations, and the Connes-Feldman-Weiss Theorem that amenable equivalence relations are hyperfinite a.e. We also include topics concerning amenability and hyperfiniteness in the Borel and generic (Baire category) contexts, like the result that finitely generated groups of polynomial growth always give rise to hyperfinite equivalence relations (without neglecting null sets), that generically (i.e., on a comeager set) every countable Borel equivalence relation is hyperfinite, and finally that, also generically, a countable Borel equivalence relation admits no invariant Borel probability measure, and therefore all generically aperiodic, non-smooth countable Borel equivalence relations are Borel isomorphic modulo meager sets.

The third chapter contains an exposition of the theory of costs for equivalence relations and groups, originated by Levitt, and mainly developed by

Gaboriau, who used this theory to prove the rigidity results about free groups mentioned in (II) above.

(C) In order to make it easier for readers who are familiar with the material in Chapter II but would like to study the theory of costs, we have made Chapter III largely independent of Chapter II. This explains why several basic definitions and facts, introduced in Chapter II (or even in Chapter I), are again repeated in Chapter III. We apologize for this redundancy to the reader who starts from the beginning.

Chapter II grew out of a set of rough notes prepared by the first author in connection with teaching a course on orbit equivalence at Caltech in the Fall of 2001. It underwent substantial modification and improvement under the input of the second author, who prepared the current final version. Chapter III is based on a set of lecture notes written-up by the first author for a series of lectures at the joint Caltech-UCLA Logic Seminar during the Fall and Winter terms of the academic year 2000-2001. Various revised forms of these notes have been available on the web since that time.

(D) As the title of these notes indicates, this is by no means a comprehensive treatment of orbit equivalence theory. Our choice of topics was primarily dictated by the desire to keep these notes as elementary and self-contained as possible. In fact, the prerequisites for reading these notes are rather minimal: a basic understanding of measure theory, functional analysis, and classical descriptive set theory. Also helpful, but not necessary, would be some familiarity with the theory of countable Borel equivalence relations (see, e.g., Feldman-Moore [FM], Dougherty-Jackson-Kechris [DJK], and Jackson-Kechris-Louveau [JKL]).

Since this is basically a set of informal lecture notes, we have not attempted to present a detailed picture of the historical development of the subject nor a comprehensive list of references to the literature.

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Contents

I

Orbit Equivalence

1 Group Actions and Equivalence Relations

Suppose X is a standard Borel space and Γ is a countable (discrete) group. A *Borel action* of Γ on X is a Borel map $(\gamma, x) \mapsto \gamma \cdot x$ such that

1. $\forall x \in X \, (1 \cdot x = x)$, and
2. $\forall x \in X \forall \gamma_1, \gamma_2 \in \Gamma \, (\gamma_1 \cdot (\gamma_2 \cdot x) = (\gamma_1 \gamma_2) \cdot x)$.

We also say that X is a *Borel Γ-space*. Given $x \in X$, the *orbit* of x is

$$\Gamma \cdot x = \{\gamma \cdot x : \gamma \in \Gamma\},$$

and the *stabilizer* of x is given by

$$\Gamma_x = \{\gamma \in \Gamma : \gamma \cdot x = x\}.$$

We say that the action of Γ on X is *free* if

$$\forall x \in X \, (\Gamma_x = \{1\}),$$

that is, if $\gamma \cdot x \neq x$ whenever $\gamma \neq 1$. The *equivalence relation induced by the action of Γ* is given by

$$x E_\Gamma^X y \Leftrightarrow \exists \gamma \in \Gamma \, (\gamma \cdot x = y),$$

and the *quotient space* associated with the action of Γ is

$$X/\Gamma = X/E_\Gamma^X = \{\Gamma \cdot x : x \in X\}.$$

Example 1.1. The (left) *shift action* of Γ on X^Γ is given by

$$\gamma \cdot p(\delta) = p(\gamma^{-1}\delta).$$

For example, when $\Gamma = \mathbb{Z}$ and $X = 2 = \{0, 1\}$, we have

$$m \cdot p(n) = p(n - m),$$

so that

$$pE_{\mathbb{Z}}^{2^{\mathbb{Z}}} q \Leftrightarrow \exists m \in \mathbb{Z} \forall k \in \mathbb{Z} \, (p(k) = q(m + k)) \, .$$

Similarly, the *right shift* of Γ on X^Γ is given by

$$\gamma \cdot p(\delta) = p(\delta\gamma),$$

and the *conjugation action* of Γ on X^Γ is given by

$$\gamma \cdot p(\delta) = p(\gamma^{-1}\delta\gamma).$$

Example 1.2. If $X = G$ is a standard Borel group and $\Gamma \subseteq G$, then Γ acts on G by left translation, that is,

$$\gamma \cdot g = \gamma g.$$

Here the orbit of $g \in G$ is the right coset Γg, so that

$$gE_\Gamma^G h \Leftrightarrow \Gamma g = \Gamma h$$

and the quotient space G/Γ coincides with the space of right cosets of Γ. For example, take $G = \mathrm{SO}(3)$, the compact metrizable group of 3-dimensional rotations, and Γ the copy of F_2 (the free group on 2 generators) sitting inside of $\mathrm{SO}(3)$, whose two generators are given by rotation by $\cos^{-1}(1/3)$ around $\mathbf{k}$ and rotation by $\cos^{-1}(1/3)$ around $\mathbf{i}$. Then the left translation action gives a Borel action of F_2 on $\mathrm{SO}(3)$. Also, F_2 acts on S^2, the unit sphere in $\mathbb{R}^3$ (this action is related to the geometrical paradoxes of Hausdorff-Banach-Tarski; see [W]).

An equivalence relation E on X is (*finite*) *countable* if its equivalence classes are (finite) countable. We use $[E]$ to denote the group of Borel automorphisms of X whose graphs are contained in E. We use $[[E]]$ to denote the set of partial Borel automorphisms of X whose graphs are contained in E. (A *partial Borel automorphism* is a Borel bijection $\phi : A \to B$, where $A = \mathrm{dom}(\phi), B = \mathrm{rng}(\phi)$ are Borel subsets of X. As usual the *graph* of a function is the set $\mathrm{graph}(f) = \{(x, y) : f(x) = y\}$.)

Theorem 1.3 (Feldman-Moore [FM]). *Let E be a countable Borel equivalence relation on X. Then there is a countable group Γ and a Borel action of Γ on X such that $E = E_\Gamma^X$. Moreover, Γ and the action can be chosen so that*

$$xEy \Leftrightarrow \exists g \in \Gamma (g^2 = 1 \;\&\; g \cdot x = y).$$

Proof. As $E \subseteq X^2$ has countable sections, it follows from Theorem 18.10 of [K] that

$$E = \bigcup_{n \in \mathbb{N}} F_n,$$

for some sequence $\{F_n\}_{n\in\mathbb{N}}$ of Borel graphs. We can assume that $F_n \cap F_m = \emptyset$ if $n \neq m$. Let $F_{n,m} = F_n \cap F_m^{-1}$, where for $F \subseteq X^2$,

$$F^{-1} = \{(y,x) : (x,y) \in F\}.$$

Since X is Borel isomorphic to a subset of $\mathbb{R}$, it follows that $X^2 \setminus \Delta_X$, where $\Delta_X = \{(x,x) : x \in X\}$, is of the form

$$X^2 \setminus \Delta_X = \bigcup_{p\in\mathbb{N}} (A_p \times B_p),$$

where A_p, B_p are disjoint Borel subsets of X. It follows that $F_{n,m,p} = F_{n,m} \cap (A_p \times B_p)$ is of the form

$$F_{n,m,p} = \text{graph}(f_{n,m,p}),$$

for some Borel bijection $f_{n,m,p} : D_{n,m,p} \to R_{n,m,p}$, where

$$D_{n,m,p} \cap R_{n,m,p} = \emptyset.$$

Now define a sequence $\{g_{n,m,p}\}$ of Borel automorphisms of X by

$$g_{n,m,p}(x) = \begin{cases} f_{n,m,p}(x) & \text{if } x \in D_{n,m,p}, \\ f_{n,m,p}^{-1}(x) & \text{if } x \in R_{n,m,p}, \\ x & \text{otherwise.} \end{cases}$$

Note that $g_{n,m,p}$ is an involution and

$$E = \bigcup \text{graph}(g_{n,m,p}),$$

thus the induced action of $\Gamma = \langle g_{n,m,p} \rangle$ on X is as desired. $\dashv$

2 Invariant Measures

By a *measure* on a standard Borel space X we mean a non-zero σ-finite Borel measure on X. If $\mu(X) < \infty$ we call μ *finite*, and if $\mu(X) = 1$ we call μ a *probability measure*. Given a countable Borel equivalence relation E on X, we say that μ is *E-invariant* if

$$\forall f \in [E] \, (f_* \mu = \mu),$$

where $f_*\mu(A) = \mu(f^{-1}(A))$.

Proposition 2.1. *The following are equivalent:*

(a) μ *is* E*-invariant,*
(b) μ *is* Γ*-invariant, whenever* Γ *is a countable group acting in a Borel fashion on* X *such that* $E = E_\Gamma^X$*,*
(c) μ *is* Γ*-invariant for some countable group* Γ *acting in a Borel fashion on* X *such that* $E = E_\Gamma^X$*, and*
(d) $\forall \phi \in [[E]] \, (\mu(\text{dom}(\phi)) = \mu(\text{rng}(\phi)))$.

Proof. The proof of $(d) \Rightarrow (a) \Rightarrow (b) \Rightarrow (c)$ is straightforward. To see $(c) \Rightarrow (d)$, simply observe that, if $\mathrm{dom}(\phi) = A, \mathrm{rng}(\phi) = B$, there is a Borel partition $A = \bigcup A_n$ and a sequence $\{\gamma_n\} \subseteq \Gamma$ such that for $x \in A_n$, $\phi(x) = \gamma_n \cdot x$, so that $B = \bigcup \gamma_n \cdot A_n$. ⊣

3 Ergodicity

A measure μ is *E-ergodic* if every E-invariant Borel set is null or conull. A measure μ is *ergodic with respect to an action* of a group Γ if it is ergodic for the induced equivalence relation.

Example 3.1. Let μ be the product measure on $X = 2^\Gamma$, where Γ is a countably infinite group, and 2 has the $(1/2, 1/2)$-measure. For each finite $S \subseteq \Gamma$ and $s : S \to 2$, put

$$\mathcal{N}_s = \{f : f|\mathrm{dom}(s) = s\},$$

noting that the $\mathcal{N}_s$'s form a clopen basis for 2^Γ and $\mu(\mathcal{N}_s) = 2^{-|\mathrm{dom}(s)|}$, where $|A| = \mathrm{card}(A)$. As

$$\gamma \cdot \mathcal{N}_s = \mathcal{N}_{\gamma \cdot s},$$

it follows that μ is shift-invariant. To see that μ is ergodic with respect to the shift, we will actually show the stronger fact that it is *mixing*, that is,

$$\lim_{\gamma \to \infty} \mu(\gamma \cdot A \cap B) = \mu(A)\mu(B),$$

for $A, B \subseteq X$ Borel. (That mixing implies ergodicity follows from the fact that, when $A = B$ is invariant, mixing implies $\mu(A) = \mu(A)^2$, thus $\mu(A) \in \{0, 1\}$.) Given $A, B \subseteq X$ and $\epsilon > 0$, we can find A', B', each a finite union of basic clopen sets, such that

$$\mu(A \Delta A'), \mu(B \Delta B') < \epsilon,$$

where Δ denotes *symmetric difference*. Thus, it suffices to show the mixing condition when A, B are finite unions of basic clopen sets. Say A is supported by S and B by T, where $S, T \subseteq \Gamma$ are finite. Then off a finite set of γ's, $\gamma \cdot S \cap T = \emptyset$, thus $\gamma \cdot A$, B are independent and

$$\begin{aligned}\mu(\gamma \cdot A \cap B) &= \mu(\gamma \cdot A)\mu(B) \\ &= \mu(A)\mu(B).\end{aligned}$$

Example 3.2. Suppose G is an infinite compact metrizable group, $\Delta \leq G$ is a dense subgroup, and μ is Haar measure on G. For example, we could take $G = \mathbb{Z}_2^{\mathbb{N}}$ and $\Delta = \mathbb{Z}_2^{<\mathbb{N}}$. Then Δ acts by left-translation and μ is clearly invariant. To see that μ is ergodic, consider $L^1(G)$ and its dual $L^\infty(G)$. G acts on $L^1(G)$ via left shift, i.e.,

$$g \cdot p(h) = p(g^{-1}h).$$

This is a continuous action, i.e., if $g_n \to g$ in G and $p_n \to p$ in $L^1(G)$, then $g_n \cdot p_n \to g \cdot p$ in $L^1(G)$. So, by duality, it induces a continuous action of G on the unit ball of $L^\infty(G)$, with the weak*-topology, given by

$$g \cdot \Lambda(p) = \Lambda(g^{-1} \cdot p).$$

Now, if $A \subseteq G$ is Δ-invariant, let $\chi_A = \Lambda \in L^\infty(G)$, so that

$$\forall \delta \in \Delta \, (\delta \cdot \Lambda = \Lambda) \, .$$

But since Δ is dense in G and the action is continuous, this means that

$$\forall g \in G \, (g \cdot \Lambda = \Lambda) \, ,$$

or equivalently, that

$$\forall g \forall^*_\mu h \left(\chi_A(g^{-1}h) = \chi_A(h)\right) ,$$

where "$\forall^*_\mu$" means "for μ-almost all." It follows from Fubini's Theorem that

$$\forall^*_\mu h \forall^*_\mu g \left(g^{-1}h \in A \Leftrightarrow h \in A\right) ,$$

so if $\mu(A) > 0$ we can find $h \in A$ such that

$$\forall^*_\mu g \left(g^{-1}h \in A \Leftrightarrow h \in A\right) ,$$

thus $\mu(A) = 1$.

It should be noted, however, that translation is not mixing! To see this, fix $\delta_n \in \Delta$ with $\delta_n \to g \neq 1$, and let N be a sufficiently small compact neighborhood of 1 such that $N \cap gN = \emptyset$. Then

$$\mu(N \cap \delta_n \cdot N) \to 0,$$

while $\mu(N)\mu(\delta_n \cdot N) = \mu(N)^2 > 0$.

For each countable Borel equivalence relation E on X, let $\mathcal{I}_E$ be the set of invariant probability measures for E, and let $\mathcal{EI}_E$ be the set of ergodic, invariant probability measures for E. Then we have:

Theorem 3.3. (Ergodic Decomposition – Farrell [F], Varadarajan [V]) *Let E be a countable Borel equivalence relation on X. Then $\mathcal{I}_E, \mathcal{EI}_E$ are Borel sets in the standard Borel space $P(X)$ of probability measures on X.*

Now suppose $\mathcal{I}_E \neq \emptyset$. Then $\mathcal{EI}_E \neq \emptyset$, and there is a Borel surjection $\pi : X \to \mathcal{EI}_E$ such that

1. *π is E-invariant,*
2. *if $X_e = \{x : \pi(x) = e\}$, then $e(X_e) = 1$ and $E|X_e$ has a unique invariant measure, namely e, and*
3. *if $\mu \in \mathcal{I}_E$, then $\mu = \int \pi(x) \, d\mu(x)$.*

Moreover, π is uniquely determined in the sense that, if π' is another such map, then $\{x : \pi(x) \neq \pi'(x)\}$ is null with respect to all measures in $\mathcal{I}_E$.

4 Isomorphism and Orbit Equivalence

We say that (X, E), (Y, F) are (Borel) *isomorphic* ($E \cong_B F$) if there is a Borel bijection $\pi : X \to Y$ such that

$$\forall x, y \in X\, (xEy \Leftrightarrow \pi(x)F\pi(y))\,.$$

We say that (X, E, μ), (Y, F, ν) are (Borel almost everywhere) *isomorphic* if there are conull Borel invariant sets $A \subseteq X$, $B \subseteq Y$ and a Borel isomorphism π of $(A, E|A)$ with $(B, F|B)$ such that $\pi_*\mu = \nu$. Note that ergodicity is preserved under isomorphism. If Γ acts in a Borel fashion on X, Y, then the actions are (Borel) *isomorphic* if there is a Borel bijection $\pi : X \to Y$ such that

$$\forall x \in X \forall \gamma \in \Gamma\, (\pi(\gamma \cdot x) = \gamma \cdot \pi(x))\,.$$

Finally, if Γ acts in a measure preserving fashion on (X, μ), (Y, ν), then the actions are (Borel almost everywhere) *isomorphic* or *conjugate* if there are conull Borel invariant sets $A \subseteq X$, $B \subseteq Y$ and a Borel isomorphism π of the actions of Γ on A, B such that $\pi_*\mu = \nu$. It is a classical problem of ergodic theory to classify (e.g., when $\Gamma = \mathbb{Z}$) the probability measure preserving, ergodic actions up to conjugacy. (Recall here the results of Ornstein and others.) Actions of Γ, Δ on X, Y, respectively, are *orbit equivalent* if $E^X_\Gamma \cong_B E^Y_\Delta$. Similarly measure preserving actions of Γ, Δ on $(X, \mu), (Y, \nu)$, respectively, are *orbit equivalent* if there are conull Borel invariant sets $A \subseteq X$, $B \subseteq Y$ and a Borel isomorphism π of E^A_Γ, E^B_Δ with $\pi_*\mu = \nu$.

II

Amenability and Hyperfiniteness

5 Amenable Groups

Suppose X is a set. A *finitely additive probability measure* (f.a.p.m.) on X is a map

$$\mu : \text{Power}(X) \to [0,1],$$

where $\text{Power}(X) = \{A : A \subseteq X\}$, such that $\mu(X) = 1$ and $\mu(A \cup B) = \mu(A) + \mu(B)$, if $A \cap B = \emptyset$. A f.a.p.m. μ on a countable group Γ is *left-invariant* if

$$\forall \gamma \in \Gamma \forall A \subseteq \Gamma \left(\mu(\gamma A) = \mu(A)\right),$$

and a countable group Γ is *amenable* if it admits a left-invariant f.a.p.m.

Example 5.1. Suppose Γ is finite. Then

$$\mu(A) = |A|/|\Gamma|$$

defines a left-invariant f.a.p.m. on Γ, thus Γ is amenable.

Example 5.2. The group $\Gamma = \mathbb{Z}$ is amenable. Let $\mathcal{U}$ be a non-atomic ultrafilter on $\mathbb{N}$. Then

$$\mu(A) = \lim_{n \to \mathcal{U}} \frac{|A \cap \{-n, \dots, n\}|}{2n+1}$$

defines a left-invariant f.a.p.m. on Γ. Here $\lim_{n \to \mathcal{U}} a_n$, where $\{a_n\}$ is a bounded sequence of reals, denotes the unique real a such that for each neighborhood N of a the set $\{n : a_n \in N\}$ is in $\mathcal{U}$.

A *mean* on Γ is a linear functional $m : l^\infty(\Gamma) \to \mathbb{C}$ such that

(i) m is positive, i.e., $f \geq 0 \Rightarrow m(f) \geq 0$, and
(ii) $m(1) = 1$, where 1 denotes the constant function with value 1.

Proposition 5.3. *If m is a mean, then $||m|| = 1$.*

Proof. For $f \in l^\infty(\Gamma)$, find $\alpha \in \mathbb{T}$ such that $\alpha m(f) = |m(f)|$. Then

$$\begin{aligned} |m(f)| &= \alpha m(f) \\ &= m(\alpha f) \\ &= \text{Re } m(\alpha f) \\ &= m(\text{Re } \alpha f) \\ &\leq ||\text{Re } \alpha f||_\infty \\ &\leq ||\alpha f||_\infty \\ &\leq ||f||_\infty, \end{aligned}$$

and the claim follows. ⊣

There is a canonical correspondence between means and f.a.p.m.'s, given by associating to each mean m the f.a.p.m. defined by

$$\mu(A) = m(1_A),$$

where 1_A = the *characteristic function* of A, and by associating to each f.a.p.m. μ the mean defined by

$$m(f) = \int f \, d\mu.$$

A mean m is *left-invariant* if

$$\forall \gamma \in \Gamma (m(\gamma \cdot f) = m(f)),$$

where $\gamma \cdot f(\delta) = f(\gamma^{-1}\delta)$. Thus

Proposition 5.4. *Γ is amenable $\Leftrightarrow$ Γ admits a left-invariant mean.*

By considering $\gamma \mapsto \gamma^{-1}$, it is easy to see that left-invariance can be replaced with right-invariance in the definition of amenability. In fact,

Proposition 5.5. *Γ is amenable $\Leftrightarrow$ Γ admits a 2-sided invariant f.a.p.m.*

Proof. Let μ_l be a left-invariant f.a.p.m. on Γ, define $\mu_r(A) = \mu_l(A^{-1})$, and observe that

$$\mu(A) = \int \mu_l(A\gamma^{-1}) \, d\mu_r(\gamma)$$

gives the desired 2-sided invariant f.a.p.m. ⊣

Next we turn to closure properties of amenability.

Proposition 5.6. *Suppose Γ is a countable group.*

(i) If Γ is amenable and $\Delta \leq \Gamma$, then Δ is amenable.
(ii) If $N \trianglelefteq \Gamma$, then Γ is amenable $\Leftrightarrow$ $N, \Gamma/N$ are amenable. In particular, it follows that the amenable groups are closed under epimorphic images and finite products.
(iii) Γ is amenable $\Leftrightarrow$ every finitely generated subgroup of Γ is amenable.

Proof. To see (i), let μ be a left-invariant f.a.p.m. on Γ, let $T \subseteq \Gamma$ consist of one point from every right coset $\Delta\gamma$, $\gamma \in \Gamma$, and note that

$$\nu(A) = \mu(AT)$$

is a left-invariant f.a.p.m. on Δ.

To see ($\Rightarrow$) of (ii), it remains to check that if Γ is amenable, then Γ/N is amenable. But if μ is a left-invariant f.a.p.m. for Γ, then

$$\nu(A) = \mu\left(\bigcup A\right)$$

defines a left-invariant f.a.p.m. on Γ/N.

To see ($\Leftarrow$) of (ii), let μ be a left-invariant f.a.p.m. on N. Then μ induces a f.a.p.m. μ_C on each $C \in \Gamma/N$, given by

$$\mu_C(A) = \mu(\gamma^{-1}A),$$

where $\gamma \in C$. As μ is left-invariant, μ_C is independent of the choice of γ. Now let ν be a left-invariant f.a.p.m. on Γ/N, and observe that

$$\lambda(A) = \int_{\Gamma/N} \mu_C(A \cap C)\ d\nu(C)$$

defines a left-invariant f.a.p.m. on Γ.

To see (iii), suppose that $\{\Gamma_n\}_{n\in\mathbb{N}}$ is an increasing, *exhaustive* (i.e., whose union is Γ) sequence of amenable subgroups of Γ, let μ_n be a left-invariant f.a.p.m. on Γ_n, fix a non-atomic ultrafilter $\mathcal{U}$ on $\mathbb{N}$, and observe that

$$\mu(A) = \lim_{n\to\mathcal{U}} \mu_n(A \cap \Gamma_n)$$

defines a left-invariant f.a.p.m. on Γ. ⊣

Corollary 5.7. *Solvable groups are amenable.*

Proof. First, we note that abelian groups are amenable. By Proposition 5.6, it suffices to show that finitely generated abelian groups are amenable. But all such groups are direct products of finite groups with copies of $\mathbb{Z}$, and are therefore amenable by Proposition 5.6 and Examples 5.1 and 5.2.

Now suppose G is solvable. Then we can find

$$\{1\} = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n = G$$

such that H_{i+1}/H_i is abelian, for all $i < n$, and the corollary now follows from repeated applications of Proposition 5.6. ⊣

Given two sets $X, Y \subseteq \Gamma$, we write $X \sim Y$ if there are partitions $X_1, \dots, X_n$ of X and $Y_1, \dots, Y_n$ of Y and group elements $\gamma_1, \dots, \gamma_n \in \Gamma$ such that $\gamma_i X_i = Y_i$, for all $1 \le i \le n$. A group Γ is *paradoxical* if there are disjoint sets $A, B \subseteq \Gamma$ such that $A \sim B \sim \Gamma$.