TOPICS IN ADVANCED QUANTUM MECHANICS

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Addison-Wesley Publishing Company
The Advanced Book Program
Redwood City, California • Menlo Park, California • Reading, Massachusetts
New York • Don Mills, Ontario • Wokingham, United Kingdom • Amsterdam
Bonn • Sydney • Singapore • Tokyo • Madrid • San Juan

Physics Editor: *Barbara Holland*
Production Manager: *Pam Suwinsky*
Cover Design: *Michael Fender*
Electronic Formatting: *Superscript Typography*

Library of Congress Cataloging-in-Publication Data

Holstein, Barry R., 1943–
Topics in advanced quantum mechanics/Barry R. Holstein
p. cm.
"Advanced book program."
Includes index.
1. Quantum theory 1 Title
QC174.12.H64 1991 530 1'2–dc20 91-3832
ISBN 0-201-50820-6

The Advanced Book Program
350 Bridge Parkway
Redwood City, CA 94065

1 2 3 4 5 6 7 8 9 10-MA-95 94 93 92

Acknowledgments

It is a pleasure to acknowledge the contributions of those who have helped to make a this book a reality—the late Julius Ashkin who first instructed me in these matters and on whose unpublished lecture notes a good deal of chapters I, II, IV and VII is based, Mrs. Claire Ashkin for granting permission to use this material, a generation of students who participated in the development of this course, Roger "TeX" Gilson who transformed my handwritten notes into a typeset manuscript, and finally Margaret "M_AC"Donald, who skillfully generated the many figures.

Prologue

About fifteen years ago I was asked by the chairman to teach the graduate course in advanced quantum mechanics. This was the third semester of the quantum sequence and was to include relativistic quantum mechanics as well as an introduction to quantum field theory. I had intended to begin my discussion with the Dirac equation and to develop covariant perturbation theory via Green's function methods. However, I soon found that the students needed a better grounding in the related nonrelativistic techniques before lauching into a full relativistic treatment. Thus I developed material on nonrelativistic Feynman diagrams and their application to electromagnetic processes. I searched at that time for a suitable text, but was unable to find any that really fit my needs—Bjorken and Drell's *Relativistic Quantum Mechanics* was excellent for the relativistic aspects but contained no corresponding nonrelativistic material; Sakurai's *Advanced Quantum Mechanics* was much better in this regard but used the "wrong" metric; *etc.* Since I did not wish to see the students buried into their notebooks trying to keep up with my blackboard work, I developed a set of lecture notes and problems covering this material, some being substantially based upon a similar course which I had taken from Julius Ashkin at Carnegie-Mellon University a decade earlier. My notes were duplicated and handed out to the students.

Since that time, I have taught the course on four or five additional occasions. Each time I included some new material with additional problems and related notes. It is these notes and problems which are collected below into what I hope is a coherent volume. Problems are inserted not at the end of each chapter but rather in the flow of the discussion. Since this was a graduate course, I have taken the liberty of assuming a thorough grounding in basic quantum mechanics at the level of say Merzbacher's *Quantum Mechanics* and have used $\hbar = 1$ and $c = 1$ throughout, except where they are required for clarity. The charge on the electron is denoted by e and is taken to be negative — $e = -|e|$. Contraction of four vectors is accomplished via the metric tensor

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

i.e. $A \cdot B = A^\mu B_\mu = A^\mu \eta_{\mu\nu} B^\nu = A^0 B^0 - \vec{A} \cdot \vec{B}$ and the conventions of Bjorken and Drell with respect to Dirac matrices are followed, except for γ_5 for which the negative of their definition is chosen. Unit vectors are denoted by a hat symbol, as are abstract operators, but it should be clear from the context which is which.

There is clearly too much discussed here to be included in a single semester course, and there is a good deal of material, usually included at the end of each chapter and designated by an asterisk, which is supplementary to the major content—intended rather to whet the appetite of adventuresome readers. It is my hope, nevertheless, that other instructors will find useful material herein to supplement their own presentations of advanced quantum mechanics and that in turn a new generation of students will be exposed to the excitement which I first felt twenty five years ago.

Amherst, Massachusetts
December, 1991

Contents

CHAPTER I

PROPAGATOR METHODS

I.1 BASIC QUANTUM MECHANICS

The fundamental problem of quantum mechanics is to determine the time development of quantum states. That is, given a state vector $|\Psi(0)\rangle$ at time $t = 0$, what is the state at a later time $t - |\psi(t)\rangle$? The answer is provided by the Schrödinger equation

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle \tag{1.1}$$

where $\hat{H}$ is the Hamiltonian operator. Usually one sees this equation expressed in terms of the coordinate space projection of the state vector—*i.e.* the wavefunction $\psi(x,t)$ where†

$$\psi(x,t) \equiv \langle x|\psi(t)\rangle \quad . \tag{1.2}$$

The time-evolution of the wavefunction is then given by

$$i\frac{\partial}{\partial t}\langle x|\psi(t)\rangle = \left\langle x\left|\hat{H}\right|\psi(t)\right\rangle \quad . \tag{1.3}$$

In order to evaluate the matrix element on the right we can insert a complete set of co-ordinate states

$$1 = \int_{-\infty}^{\infty} dx' |x'\rangle\langle x'| \tag{1.4}$$

yielding

$$i\frac{\partial}{\partial t}\langle x|\psi(t)\rangle = \int_{-\infty}^{\infty} dx' \left\langle x\left|\hat{H}\right|x'\right\rangle \langle x'|\psi(t)\rangle \quad . \tag{1.5}$$

Finally we need to interpret the operator matrix element $\left\langle x|\hat{H}|x'\right\rangle$. In general, the Hamiltonian $\hat{H}$ can be written in terms of kinetic and potential energy components as

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) \quad . \tag{1.6}$$

Here $\hat{x}|x> = x|x>$ with $<x|x'> = \delta(x-x')$ so

$$\langle x|V(\hat{x})|x'\rangle = V(x)\langle x|x'\rangle = V(x)\delta(x-x') \quad . \tag{1.7}$$

In order to represent the kinetic energy piece we can insert a complete set of momentum states such that $\hat{p}|p> = p|p>$ with $<p|p'> = 2\pi\delta(p-p')$. Then

$$1 = \int_{-\infty}^{\infty} \frac{dp}{2\pi}|p\rangle\langle p| \tag{1.8}$$

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† For simplicity of notation, we shall work here in one dimension. However, generalization to three dimensions is obvious.

yielding

$$\left\langle x \left| \frac{\hat{p}^2}{2m} \right| x' \right\rangle = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \langle x|p\rangle \frac{p^2}{2m} \langle p|x'\rangle \quad . \tag{1.9}$$

Since $\langle x|p\rangle$ is simply a plane wave

$$\langle x|p\rangle = e^{ipx} \tag{1.10}$$

we have

$$\begin{aligned}\left\langle x \left| \frac{\hat{p}^2}{2m} \right| x' \right\rangle &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{p^2}{2m} e^{ip(x-x')} \\ &= -\frac{1}{2m} \frac{\partial^2}{\partial x^2} \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ip(x-x')} \\ &= -\frac{1}{2m} \frac{\partial^2}{\partial x^2} \delta(x-x') \quad .\end{aligned} \tag{1.11}$$

Substitution back into Eq. 1.3 yields

$$\begin{aligned}& i\frac{\partial}{\partial t} \langle x|\psi(t)\rangle = i\frac{\partial}{\partial t}\psi(x,t) \\ &= \left(-\frac{1}{2m} \frac{\partial^2}{\partial x^2} + V(x)\right) \int_{-\infty}^{\infty} dx' \, \delta(x-x') \langle x'|\psi(t)\rangle \\ &= H(x)\psi(x,t)\end{aligned} \tag{1.12}$$

which is the usual version of the Schrödinger equation, where

$$H(x) = -\frac{1}{2m} \frac{\partial^2}{\partial x^2} + V(x) \quad . \tag{1.13}$$

provides the representation of the operator $\hat{H}$ in coordinate space. For a free particle this reduces to the simple form

$$H_0(x) = -\frac{1}{2m} \frac{\partial^2}{\partial x^2} \quad . \tag{1.14}$$

Time Development Operator

An alternative formulation of this problem is in terms of the time development operator $\hat{U}(t, t')$ defined via

$$\hat{U}(t,t')|\psi(t')\rangle = \begin{cases} |\psi(t)\rangle & t \geq t' \\ 0 & t < 0 \end{cases} \tag{1.15}$$

with the boundary condition

$$\lim_{t \to t'+} \hat{U}(t,t') = 1 \quad . \tag{1.16}$$

For the case of a free particle, obeying

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}_0|\psi(t)\rangle \tag{1.17}$$

the solution for $\hat{U}^{(0)}(t,0)$ is

$$\hat{U}^{(0)}(t,0) = \theta(t)\exp\left(-i\hat{H}_0 t\right) \quad , \tag{1.18}$$

where

$$\theta(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases} \tag{1.19}$$

is the usual theta function. For example, if

$$\psi(x,0) = \frac{1}{(2\pi\sigma^2)^{1/4}}\exp\left(-\frac{(x-a)^2}{4\sigma^2}\right) \tag{1.20}$$

we find

$$\begin{aligned} \psi(x,t) &= < x|\hat{U}^{(0)}(t,0)|\psi(0) > = \int_{-\infty}^{\infty} dx' < x|\hat{U}^{(0)}(t,0)|x' >< x'|\psi(0) > \\ &= e^{-iH_0(x)t}\psi(x,0) = e^{-iH_0(x)t}\frac{1}{(2\pi\sigma^2)^{1/4}}\exp\left(-\frac{(x-a)^2}{4\sigma^2}\right) \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{it}{2m}\right)^n \frac{\partial^{2n}}{\partial x^{2n}}\exp\left(-\frac{(x-a)^2}{4\sigma^2}\right)\frac{1}{(2\pi\sigma^2)^{1/4}} \quad . \end{aligned} \tag{1.21}$$

Although one could straightforwardly evaluate this power series, it is easier to note the identity [Bl 68]

$$\frac{\partial^2}{\partial x^2}\frac{1}{\sqrt{\rho}}\exp\left(-\frac{(x-a)^2}{4\rho}\right) = \frac{\partial}{\partial\rho}\frac{1}{\sqrt{\rho}}\exp\left(-\frac{(x-a)^2}{4\rho}\right) \quad . \tag{1.22}$$

Then using

$$\exp\left(\alpha\frac{\partial}{\partial z}\right)f(z) = \sum_{n=0}^{\infty}\frac{\alpha^n}{n!}\frac{\partial^n}{\partial z^n}f(z) = f(z+\alpha) \tag{1.23}$$

we find

$$\psi(x,t) = \left(\frac{\sigma^2}{2\pi}\right)^{1/4}\frac{1}{\left(\sigma^2 + i\frac{t}{2m}\right)^{1/2}}\exp\left(-\frac{(x-a)^2}{4\left(\sigma^2 + i\frac{t}{2m}\right)}\right) \quad . \tag{1.24}$$

We note that

$$|\psi(x,t)|^2 = \frac{1}{\sqrt{2\pi}\,\sigma(t)}\exp\left(-\frac{(x-a)^2}{2\sigma^2(t)}\right) \quad \text{with} \quad \sigma(t) = \left(\sigma^2 + \frac{t^2}{4m^2\sigma^2}\right)^{1/2} \tag{1.25}$$

which obviously exhibits the canonical spreading experienced by such a wavepacket.

We can equivalently perform the above calculation in momentum space, where the time development operator has the simple form

$$\left\langle p\left|\hat{U}^{(0)}(t,t')\right|p'\right\rangle = < p|\exp(-i\hat{p}^2(t-t')/2m)|p' > \theta(t-t')$$

$$= \exp -i\frac{p^2}{2m}(t-t') < p|p' > \theta(t-t') = \exp -i\frac{p^2}{2m}(t-t')2\pi\delta(p-p')\theta(t-t') \quad . \tag{1.26}$$

If

$$\langle x|\psi(0)\rangle = \frac{1}{(2\pi\sigma^2)^{1/4}} \exp\left(-\frac{(x-a)^2}{4\sigma^2}\right) \tag{1.27}$$

we have

$$\begin{aligned} \langle p|\psi(0)\rangle &= \int_{-\infty}^{\infty} dx \, \langle p|x\rangle \langle x|\psi(0)\rangle \\ &= \int_{-\infty}^{\infty} dx \, e^{-ipx} \frac{1}{(2\pi\sigma^2)^{1/4}} \exp\left(-\frac{(x-a)^2}{4\sigma^2}\right) \\ &= (8\pi\sigma^2)^{1/4} \exp\left(-\sigma^2 p^2\right) \exp\left(-ipa\right) \quad . \end{aligned} \tag{1.28}$$

Then

$$\begin{aligned} \langle p|\psi(t)\rangle &= \int_{-\infty}^{\infty} \frac{dp'}{2\pi} < p|\hat{U}^{(0)}(t,0)|p' >< p'|\psi(0) > \\ &= (8\pi\sigma^2)^{1/4} \exp\left(-\sigma^2 p^2 - ipa - i\frac{p^2}{2m}t\right)\theta(t) \quad . \end{aligned} \tag{1.29}$$

We can return to coordinate space via

$$\begin{aligned} \langle x|\psi(t)\rangle &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} \langle x|p\rangle \langle p|\psi(t)\rangle \\ &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ipx} \left(8\pi\sigma^2\right)^{1/4} \exp\left(-\sigma^2 p^2 - ipa - i\frac{p^2}{2m}t\right)\theta(t) \\ &= \left(\frac{\sigma^2}{2\pi}\right)^{1/4} \frac{1}{\left(\sigma^2 + i\frac{t}{2m}\right)^{1/2}} \exp - \frac{(x-a)^2}{4\left(\sigma^2 + i\frac{t}{2m}\right)}\theta(t) \end{aligned} \tag{1.30}$$

which agrees precisely with Eq. 1.24 found via coordinate space methods.

PROBLEM I.1.1

Wave Packet Spreading: A Paradox

It was demonstrated above using the identity

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial z}\right) z^{-1/2} \exp\left(-\frac{x^2}{4z}\right) = 0$$

that a Gaussian wavepacket

$$\psi(x, t=0) = \left(2\pi\sigma^2\right)^{-1/4} \exp\left(-\frac{x^2}{4\sigma^2}\right)$$

evolves in time via

$$\psi(x,t) = \xi^{-1/2} \left(2\pi\sigma^2\right)^{-1/4} \exp\left(-\frac{x^2}{4\sigma^2\xi}\right)$$

where

$$\xi = 1 + i\frac{t}{2m\sigma^2} \quad .$$

Then

$$|\psi(x,t)|^2 = \left(2\pi\sigma^2(t)\right)^{-1/2} \exp\left(-\frac{x^2}{2\sigma^2(t)}\right)$$

where

$$\sigma^2(t) = \sigma^2 + \frac{t^2}{4m^2\sigma^2} \quad .$$

i) Show that

$$\int_{-\infty}^{\infty} dx\, |\psi(x,t)|^2 = 1$$

$$\int_{-\infty}^{\infty} dx\, x^2\, |\psi(x,t)|^2 = \sigma^2(t)$$

so that the wavepacket remains normalized to unity but has a width

$$\sigma(t) = \sqrt{\sigma^2 + \frac{t^2}{4m^2\sigma^2}}$$

which evolves with time. This is simply the usual "spreading" of a quantum mechanical wave packet.

ii) Derive the time evolution of the Gaussian wavepacket without exploiting the identity by using a power series expansion

$$\begin{aligned}\psi(x,t) &= e^{-iH_0(x)t}\psi(x,0) \\ &= (2\pi\sigma^2)^{-1/4}\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\left(i\frac{t}{2m}\right)^{\ell}\frac{\partial^{2\ell}}{\partial x^{2\ell}}\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{x^2}{4\sigma^2}\right)^n \quad .\end{aligned}$$

iii) Now suppose that

$$\psi(x,0) = N\begin{cases}\exp\left(-\frac{a^2}{a^2-x^2}\right) & |x| < a \\ 0 & |x| > a\end{cases}$$

where N is a normalization constant. Although this functional form may look a bit strange, a little thought should convince one that the wavefunction and all its derivatives are continuous at any point on the real line. However, it is easy to see that

$$e^{-iH_0(x)t}\psi(x,0) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(i\frac{t}{2m}\right)^n\frac{\partial^{2n}}{\partial x^{2n}}\psi(x,0)$$

vanishes for all time if $|x| \geq a$ since

$$\frac{\partial^{2n}}{\partial x^{2n}}0 = 0 \quad .$$

Hence, this type of wavepacket apparently does not undergo spreading. Is this assertion correct? If not, where have we made an error in our analysis and what does the actual time evolved wavefunction look like [HoS 72]?

I.2 THE PROPAGATOR

One can evaluate the co-ordinate space matrix element of the time development operator by transforming to momentum space and back again.

$$\begin{aligned}D_F^{(0)}(x',t;x,0) &\equiv < x'|\hat{U}^{(0)}(t,0)|x> = \left\langle x'\left|e^{-i\hat{H}_0 t}\right|x\right\rangle \theta(t)\\ &= \theta(t)\int_{-\infty}^{\infty}\frac{dp}{2\pi}\langle x'|p\rangle\, e^{-i\frac{p^2}{2m}t}\langle p|x\rangle = \theta(t)\int_{-\infty}^{\infty}\frac{dp}{2\pi}e^{ip(x'-x)-i\frac{p^2}{2m}t}\\ &= \theta(t)\sqrt{\frac{m}{2\pi it}}\exp i\frac{m(x'-x)^2}{2t}\ .\end{aligned} \tag{2.1}$$

D_F is usually called the "propagator," as it gives the amplitude for a particle produced at position x at time 0 to "propagate" to position x' at time t.

Just as a check we can verify that this form of the propagator does indeed generate the time development of the freely moving Gaussian wavefunction

$$\begin{aligned}\psi(x',t) &= \int_{-\infty}^{\infty} dx\, D_F^{(0)}(x',t;x,0)\psi(x,0)\\ &= \int_{-\infty}^{\infty} dx\sqrt{\frac{m}{2\pi it}}\exp\frac{im(x'-x)^2}{2t}\,\frac{1}{(2\pi\sigma^2)^{1/4}}\exp\left(-\frac{(x-a)^2}{4\sigma^2}\right)\\ &= \left(\frac{\sigma^2}{2\pi}\right)^{1/4}\exp\left(-\frac{(x'-a)^2}{4\left(\sigma^2+i\frac{t}{2m}\right)}\right)\frac{1}{\left(\sigma^2+i\frac{t}{2m}\right)^{1/2}}\end{aligned} \tag{2.2}$$

in complete agreement with expression derived in sect. I.1.

Path Integrals and the Propagator

Before going further, it is useful to note an alternative way by which the propagator can be calculated—the Feynman path integral [FeH 65]

$$D_F(x',t;x,0) = \int \mathcal{D}\left[x(t)\right]\exp\frac{iS\left[x(t)\right]}{\hbar} \tag{2.3}$$

where the notation is that the integral represents a sum over *all* paths $x(t)$ connecting the initial and final spacetime points—$x, 0$ and x', t respectively. For each path there is a weighting factor given by $\exp\frac{iS}{\hbar}$ where $S = \int dt L[x(t)]$ is the classical action associated with that path. The path integration can be carried out by dividing the time interval $0 - t$ into n slices of width ϵ. This provides a set of times t_i spaced a distance ϵ apart between the values 0 and t. At each time t_i we select a point x_i. A path is constructed by connecting all possible x_i points so selected by straight lines as shown in Figure I.1 and the path integral is written (setting $\hbar = 1$) as

$$D_F(x',t;x,0) = \lim_{n\to\infty}\frac{1}{A^n}\prod_{i=1}^{n-1}\left(\int_{-\infty}^{\infty}dx_i\right)\exp iS^{(0)} \tag{2.4}$$

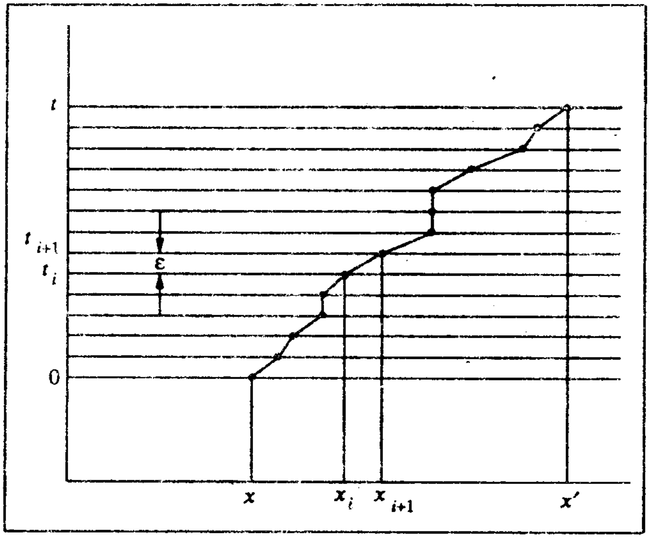

Fig. I.1: A particular time slice used in calculation of the propagator.

where A is a normalization constant which defines the measure—note that there is one factor of A for each straight line segment. In the limit as $\epsilon \to 0$ we can evaluate the action for each line segment in the infinitesimal approximation. For the free particle we have

$$\begin{aligned} S^{(0)} &= \int_0^t dt' L\left(x(t'), \dot{x}(t'), t'\right) = \int_0^t dt' \frac{1}{2} m \dot{x}^2(t') \\ &= \sum_{i=1}^{n} \frac{1}{2} m \frac{(x_i - x_{i-1})^2}{\epsilon} \quad \text{with} \quad x_0 = x \quad \text{and} \quad x_n = x' \end{aligned}$$

The integrations may be performed sequentially

$$\begin{aligned} &\int_{-\infty}^{\infty} dx_1 \exp i\frac{m}{2\epsilon}\left((x_1 - x_0)^2 + (x_2 - x_1)^2\right) = \sqrt{\frac{2\pi i\epsilon}{2m}} \exp i\frac{m}{2\cdot 2\epsilon}(x_2 - x_0)^2 \\ &\int_{-\infty}^{\infty} dx_2 \exp i\frac{m}{2\epsilon}\left(\frac{1}{2}(x_2 - x_0)^2 + (x_3 - x_2)^2\right) = \sqrt{\frac{2\pi i\epsilon\cdot 2}{3m}} \exp i\frac{m}{3\cdot 2\epsilon}(x_3 - x_0)^2 \\ &\quad \vdots \\ &\int_{-\infty}^{\infty} dx_{n-1} \exp i\frac{m}{2\epsilon}\left(\frac{1}{n-1}(x_{n-1} - x_0)^2 + (x_n - x_{n-1})^2\right) \\ &= \sqrt{\frac{2\pi i\epsilon(n-1)}{nm}} \exp i\frac{m}{n\cdot 2\epsilon}(x_n - x_0)^2 \end{aligned} \tag{2.5}$$

yielding

$$D_F^{(0)}(x',t;x,0)=\left(\frac{2\pi i\epsilon}{m}\right)^{(n-1)/2}\frac{1}{A^n}\frac{1}{\sqrt{n}}\exp i\frac{m}{2n\epsilon}(x'-x)^2 \tag{2.6}$$

The constant A may be determined by use of the completeness condition

$$\begin{aligned}\psi(x',t)&=\int_{-\infty}^{\infty}dx<x'|\hat{U}^{(0)}(t,0)|x><x|\psi(0)>\\&=\int_{-\infty}^{\infty}dxD_F^{(0)}(x',t;x,0)\psi(x,0)\ .\end{aligned} \tag{2.7}$$

If we pick $t=\epsilon<<<1$ then

$$\psi(x',\epsilon)\cong\int_{-\infty}^{\infty}dx\frac{1}{A}\exp i\frac{m(x'-x)^2}{2\epsilon}\psi(x,0)+\dots\ . \tag{2.8}$$

Since ϵ is small, the exponential will rapidly oscillate and thereby wash out the integral unless $x\cong x'$. Thus, we can write

$$\begin{aligned}\psi(x',\epsilon)&\cong\psi(x',0)\frac{1}{A}\int_{-\infty}^{\infty}dx\exp i\frac{m(x'-x)^2}{2\epsilon}+\dots\\&=\psi(x',0)\frac{1}{A}\sqrt{\frac{2\pi i\epsilon}{m}}+\dots\ .\end{aligned} \tag{2.9}$$

Hence in order to have the correct behavior as $\epsilon\to 0$ we must pick

$$A=\sqrt{\frac{2\pi i\epsilon}{m}} \tag{2.10}$$

so that the free propagator becomes, using $t=n\epsilon$

$$D_F^{(0)}(x',t;x,0)=\sqrt{\frac{m}{2\pi in\epsilon}}\exp\frac{im}{2n\epsilon}(x'-x)^2=\sqrt{\frac{m}{2\pi it}}\exp\frac{im(x'-x)^2}{2t} \tag{2.11}$$

in complete agreement with the expression derived via more conventional means (*cf.* Eq. 2.1).

The reason that the propagator can be written as a path integral can be understood by using the completeness relation

$$1=\int_{-\infty}^{\infty}dx_i\ |x_i\rangle\langle x_i|\ . \tag{2.12}$$

For later use, we shall give the derivation here for the general case involving interaction with a potential $V(\hat{x})$. Starting with the definition

$$D_F(x_f,t_f;x_i,t_i)=\left\langle x_f\left|\exp -i\hat{H}(t_f-t_i)\right|x_i\right\rangle\theta(t_f-t_i) \tag{2.13}$$

and breaking the time interval $t_f - t_i$ (assumed to be positive) into n discrete steps of size

$$\epsilon = \frac{t_f - t_i}{n} \tag{2.14}$$

we can write

$$\begin{aligned} D_F(x_f, t_f; x_i, t_i) = \int_{-\infty}^{\infty} dx_1 \ldots dx_{n-1} \left\langle x_n \left| e^{-i\epsilon\hat{H}} \right| x_{n-1} \right\rangle \\ \cdot \left\langle x_{n-1} \left| e^{-i\epsilon\hat{H}} \right| x_{n-2} \right\rangle \ldots \left\langle x_1 \left| e^{-i\epsilon\hat{H}} \right| x_0 \right\rangle \quad . \end{aligned} \tag{2.15}$$

In the limit of large n the time slices become infinitesimal and

$$\begin{aligned} \left\langle x_\ell \left| e^{-i\epsilon\hat{H}} \right| x_{\ell-1} \right\rangle &= \left\langle x_\ell \left| \exp -i\epsilon \left(\frac{\hat{p}^2}{2m} + V(\hat{x}) \right) \right| x_{\ell-1} \right\rangle \\ &\approx \exp -i\epsilon V(x_\ell) \left\langle x_\ell \left| e^{-i\epsilon\frac{\hat{p}^2}{2m}} \right| x_{\ell-1} \right\rangle + \mathcal{O}(\epsilon^2) \quad . \end{aligned} \tag{2.16}$$

Introducing a complete set of momentum states, we have

$$\begin{aligned} \left\langle x_\ell \left| e^{-i\epsilon\frac{\hat{p}^2}{2m}} \right| x_{\ell-1} \right\rangle &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ip(x_\ell - x_{\ell-1})} e^{-i\epsilon\frac{p^2}{2m}} \\ &= \sqrt{\frac{m}{2\pi i\epsilon}} \exp i\frac{m}{2\epsilon}(x_\ell - x_{\ell-1})^2 \end{aligned} \tag{2.17}$$

and, taking the continuum limit, we find the path integral prescription

$$\begin{aligned} D_F(x_f, t_f; x_i, t_i) &= \lim_{n\to\infty} \left(\frac{m}{2\pi i\epsilon} \right)^{\frac{n}{2}} \int_{-\infty}^{\infty} dx_{n-1} \ldots dx_1 \\ &\quad \times \exp i \sum_{\ell=1}^{n} \left(m\frac{(x_\ell - x_{\ell-1})^2}{2\epsilon} - \epsilon V(x_\ell) \right) \\ &= \int \mathcal{D}[x(t)] \exp iS[x(t)] \end{aligned} \tag{2.18}$$

where

$$S[(t)] = \int_{t_1}^{t_2} dt \left(\frac{1}{2} m\dot{x}^2(t) - V(x(t)) \right) \tag{2.19}$$

is the classical action.

Classical Connection

Perhaps the most peculiar and fascinating aspect of this prescription is that *all* paths connecting the spacetime endpoints must be included in the summation. This appears to be in total contradiction with the classical mechanics result that a particle traverses a well-defined trajectory. The resolution of this apparent paradox may be found by explicitly restoring the dependence on $\hbar$ and noting that the path integral prescription is given by

$$\sum_{x(t)} \exp iS[x(t)]/\hbar \tag{2.20}$$

Classical physics results as $\hbar \to 0$, and in this limit a slight change in the path $x(t)$ produces a huge change in phase and hence little or no contribution to the path summation except for trajectories $\bar{x}(t)$ for which the action is stationary—*i.e.*, Hamilton's principle

$$\left.\frac{\delta S[x(t)]}{\delta x}\right|_{x(t)=\bar{x}(t)} = 0 \ . \tag{2.21}$$

In order to find such a path we take

$$\begin{aligned} 0 &= S\left[\bar{x}(t) + \delta x(t)\right] - S\left[\bar{x}(t)\right] \\ &= \int_0^t dt' \left[\frac{m}{2}\left(\dot{\bar{x}}(t') + \delta\dot{x}(t')\right)^2 - \frac{m}{2}(\dot{\bar{x}}(t'))^2 - V(\bar{x}(t') + \delta x(t')) + V(\bar{x}(t'))\right] \\ &= \int_0^t dt' \left(m\dot{\bar{x}}(t')\delta\dot{x}(t') - V'(\bar{x}(t'))\delta x(t')\right) + \mathcal{O}((\delta x)^2) \ . \end{aligned} \tag{2.22}$$

integrate by parts and use the feature that the endpoints of the path are fixed, *i.e.*, $\delta x(0) = \delta x(t) = 0$. Then

$$0 = -\int_0^t dt' (m\ddot{\bar{x}}(t') + V'(\bar{x}(t')))\delta x(t') \ . \tag{2.23}$$

so that the trajectory which satisfies the stationary phase condition for arbitrary $\delta x(t')$ must obey

$$m\ddot{\bar{x}} + V'(\bar{x}) = 0 \tag{2.24}$$

which is just the classical mechanics prescription for the motion of a freely moving particle, *i.e.*, $\bar{x}(t) = x_{\mathrm{cl}}(t)$. In the limit $\hbar \to 0$ the classical trajectory represents the *only* path contributing to the path integral and the paradox is resolved.

One can also get a feel for the meaning of the propagator by noting that since

$$\begin{aligned} \langle x|\psi(t)\rangle &= \left\langle x \left| e^{-i\hat{H}_0 t} \right| \psi(0)\right\rangle = \int_{-\infty}^{\infty} dx' \left\langle x \left| e^{-i\hat{H}_0 t} \right| x'\right\rangle \langle x'|\psi(0)\rangle \\ &= \int_{-\infty}^{\infty} dx' \, D_F^{(0)}(x,t;x',0) \, \langle x'|\psi(0)\rangle \end{aligned} \tag{2.25}$$

if we take

$$\langle x'|\psi(0)\rangle = \delta(x') \tag{2.26}$$

so that at $t = 0$ the particle is located precisely at the origin, then

$$D_F^{(0)}(x,t;0,0) = \sqrt{\frac{m}{2\pi i t}} \exp \frac{imx^2}{2t} = \langle x|\psi(t)\rangle \ . \tag{2.27}$$