Lecture Notes in Mathematics

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Holomorphic Dynamical Systems

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Preface

The theory of holomorphic dynamical systems is a subject of increasing interest in mathematics, both for its challenging problems and for its connections with other branches of pure and applied mathematics.

A holomorphic dynamical system is the datum of a complex variety and a holomorphic object (such as a self-map or a vector field) acting on it. The study of a holomorphic dynamical system consists in describing the asymptotic behavior of the system, associating it with some invariant objects (easy to compute) which describe the dynamics and classify the possible holomorphic dynamical systems supported by a given manifold. The behavior of a holomorphic dynamical system is pretty much related to the geometry of the ambient manifold (for instance, hyperbolic manifolds do no admit chaotic behavior, while projective manifolds have a variety of different chaotic pictures). The techniques used to tackle such problems are of various kinds: complex analysis, methods of real analysis, pluripotential theory, algebraic geometry, differential geometry, topology.

To cover all the possible points of view of the subject in a unique occasion has become almost impossible, and the CIME session in Cetraro on Holomorphic Dynamical Systems was not an exception. On the other hand the selection of the topics and of the speakers made it possible to focus on a number of important topics in the discrete and in the continuous setting, both for the local and for the global aspects, providing a fascinating introduction to many key problems of the current research. The CIME Course aimed to give an ample description of the phenomena occurring in central themes of holomorphic dynamics such as automorphisms and meromorphic self-maps of projective spaces, of entire maps on complex spaces and holomorphic foliations in surfaces and higher dimensional manifolds, enlightening the different techniques used and bringing the audience to the borderline of current research topics. This program, with its interdisciplinary characterization, drew the attention and the participation of young researchers and experienced mathematicians coming from different backgrounds: complex analysis and geometry, topology, ordinary differential equations and number theory. We are sure that the present volume will serve the same purpose. We briefly describe here the papers that stemmed from the courses and constitute the Chapters of this volume.

In his lectures, Marco Abate outlines the local theory of iteration in one and several variables. He studies the structure of the stable set K_f of a selfmap f of a neighborhood U of a fixed point, describing both the topological structure of K_f and the dynamical nature of the (global) dynamical system $(K_f, f_{|K_f})$. One important way to study a local holomorphic dynamical system consists in replacing it by an equivalent but simpler system. Following a traditional approach, Abate considers three equivalence relations - topological, holomorphic and formal conjugacy - and discusses normal forms and invariants in all these cases. He starts surveying the one-dimensional theory, which is fairly complete, even though there are still some open problems, and then he presents what is known in the multidimensional case, that is an exciting mixture of deep results and still unanswered very natural questions.

The lectures of Eric Bedford provide an introduction to the dynamics of the automorphisms of rational surfaces. The first part is devoted to polynomial automorphisms of $\mathbf{C}^2$ and in particular to the complex Hénon maps, the most heavily studied family of invertible holomorphic maps. The investigations of the Hénon maps can be guided by the study of the dynamics of polynomial maps of one variable, a very rich and classical topic. Although the Hénon family is only partially understood, its methods and results provide motivation and guidance for the understanding of other types of automorphisms. In the second part of the notes, Bedford considers the geometry of compact rational surfaces with the illustration of some examples of their automorphisms. In contrast with the case of the polynomial automorphisms of $\mathbf{C}^2$, not much is known about neither the set of all rational surface automorphisms, nor about a dynamical classification of them.

The theory of foliations by Riemann surfaces is central in the study of continuous aspects of holomorphic dynamics. In his lecture notes, Marco Brunella describes the state of the art of the topic and reports on his results on the uniformisation theory of foliations by curves on compact Kähler manifolds. Each leaf is uniformized either by the unit disk or by $\mathbf{C}$ or by the projective line. Brunella explains how the universal covers may be patched together to form a complex manifold with good properties and he studies the analytic properties of this manifold, in particular regarding holomorphic convexity. In turn this leads to results on the distribution of parabolic leaves inside the foliation and to positivity statements concerning the canonical bundle of the foliation, generalizing results of Arakelov on fibrations by algebraic curves.

Sibony's course in the CIME session was based on the lecture notes by Tien-Cuong Dinh and Nessim Sibony that are included in this volume. This contribution, which could be a stand alone monograph for depth and extension, gives a broad presentation to the most recent developments of pluripotential methods, and to the theory of positive closed currents, in dynamics in Several Complex Variables. The notes concentrate on the dynamics of endomorphisms of projective spaces and the polynomial-like mappings. Green currents and equilibrium measure are constructed to study quantitative properties and speed of convergence for endomorphisms of projective spaces; equidistribution problems and ergodic properties are also treated. For polynomial-like mappings, the equilibrium measure of maximal entropy is constructed and equidistribution properties of points are proved, under suitable dynamical degree assumptions. The tools introduced here are of independent interest and can be applied in other dynamical problems. The presentation includes all the necessary prerequisites about plurisubharmonic functions and currents, making the text self-contained and quite accessible.

In his lectures, Schleicher studies iteration theory of entire holomorphic functions in one complex variable, a field of research that has been quite active in recent years. A review of dynamics of entire maps, which includes the classical and well developed theory of polynomial dynamics, serves to introduce the main topic: the consideration of transcendental maps. The notes study key dynamical properties of large classes of transcendental functions and of special prototypical families of entire maps such as the exponential family $z \mapsto \lambda e^z$ or the cosine family $z \mapsto ae^z + be^{-z}$. It turns out that some aspects of the dynamics of transcendental entire maps are inspired by the polynomial theory, others are very different and exploit all the power of deep results from complex analysis. Transcendental dynamics turns out to be a largely yet unexplored and fascinating area of research where surprising mathematical results - that sometimes had been constructed artificially in other branches of mathematics - arise in a natural way.

It is a real pleasure to thank the speakers for their great lectures and all the authors for the beautiful contributions to this volume. We also like to thank all the participants for their interest and enthusiasm that created a very warm and stimulating scientific atmosphere. We like to express our gratitude to CIME for sponsoring the summer school, and to the LAMI (Laboratorio di Applicazioni della Matematica all'Ingegneria, Università della Calabria) for its financial contribution. Our particular gratitude goes to Pietro Zecca and Elvira Mascolo for their continuous support, and to their collaborators Carla Dionisi and Maria Giulia Bartaloni for their invaluable help.

Graziano Gentili, Jacques Guenot and Giorgio Patrizio

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Contents

Discrete Holomorphic Local Dynamical Systems

Marco Abate

Abstract This chapter is a survey on local dynamics of holomorphic maps in one and several complex variables, discussing in particular normal forms and the structure of local stable sets in the non-hyperbolic case, and including several proofs and a large bibliography.

1 Introduction

Let us begin by defining the main object of study in this survey.

Definition 1.1. Let M be a complex manifold, and $p \in M$. A *(discrete) holomorphic local dynamical system* at p is a holomorphic map $f \colon U \to M$ such that $f(p) = p$, where $U \subseteq M$ is an open neighbourhood of p; we shall also assume that $f \not\equiv \mathrm{id}_U$. We shall denote by $\mathrm{End}(M,p)$ the set of holomorphic local dynamical systems at p.

Remark 1.2. Since we are mainly concerned with the behavior of f nearby p, we shall sometimes replace f by its restriction to some suitable open neighbourhood of p. It is possible to formalize this fact by using germs of maps and germs of sets at p, but for our purposes it will be enough to use a somewhat less formal approach.

Remark 1.3. In this survey we shall never have the occasion of discussing continuous holomorphic dynamical systems (i.e., holomorphic foliations). So from now on all dynamical systems in this paper will be discrete, except where explicitly noted otherwise.

To talk about the dynamics of an $f \in \mathrm{End}(M,p)$ we need to define the iterates of f. If f is defined on the set U, then the second iterate $f^2 = f \circ f$ is defined

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on $U \cap f^{-1}(U)$ only, which still is an open neighbourhood of p. More generally, the k-th iterate $f^k = f \circ f^{k-1}$ is defined on $U \cap f^{-1}(U) \cap \cdots \cap f^{-(k-1)}(U)$. This suggests the next definition:

Definition 1.4. Let $f \in \mathrm{End}(M, p)$ be a holomorphic local dynamical system defined on an open set $U \subseteq M$. Then the *stable set* K_f of f is

$$K_f = \bigcap_{k=0}^{\infty} f^{-k}(U).$$

In other words, the stable set of f is the set of all points $z \in U$ such that the *orbit* $\{f^k(z) \mid k \in \mathbb{N}\}$ is well-defined. If $z \in U \setminus K_f$, we shall say that z (or its orbit) *escapes* from U.

Clearly, $p \in K_f$, and so the stable set is never empty (but it can happen that $K_f = \{p\}$; see the next section for an example). Thus the first natural question in local holomorphic dynamics is:

(Q1) *What is the topological structure of* K_f?

For instance, when does K_f have non-empty interior? As we shall see in Proposition 4.1, holomorphic local dynamical systems such that p belongs to the interior of the stable set enjoy special properties.

Remark 1.5. Both the definition of stable set and Question 1 (as well as several other definitions and questions we shall see later on) are topological in character; we might state them for local dynamical systems which are continuous only. As we shall see, however, the *answers* will strongly depend on the holomorphicity of the dynamical system.

Definition 1.6. Given $f \in \mathrm{End}(M, p)$, a set $K \subseteq M$ is *completely f-invariant* if $f^{-1}(K) = K$ (this implies, in particular, that K is *f-invariant*, that is $f(K) \subseteq K$).

Clearly, the stable set K_f is completely f-invariant. Therefore the pair (K_f, f) is a discrete dynamical system in the usual sense, and so the second natural question in local holomorphic dynamics is

(Q2) *What is the dynamical structure of* (K_f, f)?

For instance, what is the asymptotic behavior of the orbits? Do they converge to p, or have they a chaotic behavior? Is there a dense orbit? Do there exist proper f-invariant subsets, that is sets $L \subset K_f$ such that $f(L) \subseteq L$? If they do exist, what is the dynamics on them?

To answer all these questions, the most efficient way is to replace f by a "dynamically equivalent" but simpler (e.g., linear) map g. In our context, "dynamically equivalent" means "locally conjugated"; and we have at least three kinds of conjugacy to consider.

Definition 1.7. Let $f_1 \colon U_1 \to M_1$ and $f_2 \colon U_2 \to M_2$ be two holomorphic local dynamical systems at $p_1 \in M_1$ and $p_2 \in M_2$ respectively. We shall say that f_1 and f_2 are *holomorphically* (respectively, *topologically*) *locally conjugated* if there are open neighbourhoods $W_1 \subseteq U_1$ of p_1, $W_2 \subseteq U_2$ of p_2, and a biholomorphism (respectively, a homeomorphism) $\varphi \colon W_1 \to W_2$ with $\varphi(p_1) = p_2$ such that

$$f_1 = \varphi^{-1} \circ f_2 \circ \varphi \qquad \text{on} \qquad \varphi^{-1}\big(W_2 \cap f_2^{-1}(W_2)\big) = W_1 \cap f_1^{-1}(W_1).$$

If $f_1 \colon U_1 \to M_1$ and $f_2 \colon U_2 \to M_2$ are locally conjugated, in particular we have

$$f_1^k = \varphi^{-1} \circ f_2^k \circ \varphi \quad \text{on} \quad \varphi^{-1}\big(W_2 \cap \cdots \cap f_2^{-(k-1)}(W_2)\big) = W_1 \cap \cdots \cap f_1^{-(k-1)}(W_1)$$

for all $k \in \mathbb{N}$ and thus

$$K_{f_2|_{W_2}} = \varphi(K_{f_1|_{W_1}}).$$

So the local dynamics of f_1 about p_1 is to all purposes equivalent to the local dynamics of f_2 about p_2.

Remark 1.8. Using local coordinates centered at $p \in M$ it is easy to show that any holomorphic local dynamical system at p is holomorphically locally conjugated to a holomorphic local dynamical system at $O \in \mathbb{C}^n$, where $n = \dim M$.

Whenever we have an equivalence relation in a class of objects, there are classification problems. So the third natural question in local holomorphic dynamics is

(Q3) *Find a (possibly small) class $\mathscr{F}$ of holomorphic local dynamical systems at $O \in \mathbb{C}^n$ such that every holomorphic local dynamical system f at a point in an n-dimensional complex manifold is holomorphically (respectively, topologically) locally conjugated to a (possibly) unique element of $\mathscr{F}$, called the* holomorphic *(respectively,* topological*)* normal form *of f.*

Unfortunately, the holomorphic classification is often too complicated to be practical; the family $\mathscr{F}$ of normal forms might be uncountable. A possible replacement is looking for invariants instead of normal forms:

(Q4) *Find a way to associate a (possibly small) class of (possibly computable) objects, called* invariants, *to any holomorphic local dynamical system f at $O \in \mathbb{C}^n$ so that two holomorphic local dynamical systems at O can be holomorphically conjugated only if they have the same invariants. The class of invariants is furthermore said* complete *if two holomorphic local dynamical systems at O are holomorphically conjugated if and only if they have the same invariants.*

As remarked before, up to now all the questions we asked made sense for topological local dynamical systems; the next one instead makes sense only for holomorphic local dynamical systems.

A holomorphic local dynamical system at $O \in \mathbb{C}^n$ is clearly given by an element of $\mathbb{C}_0\{z_1, \ldots, z_n\}^n$, the space of n-uples of converging power series in $z_1, \ldots, z_n$

without constant terms. The space $\mathbb{C}_0\{z_1,\dots,z_n\}^n$ is a subspace of the space $\mathbb{C}_0[[z_1,\dots,z_n]]^n$ of n-uples of formal power series without constant terms. An element $\Phi \in \mathbb{C}_0[[z_1,\dots,z_n]]^n$ has an inverse (with respect to composition) still belonging to $\mathbb{C}_0[[z_1,\dots,z_n]]^n$ if and only if its linear part is a linear automorphism of $\mathbb{C}^n$.

Definition 1.9. We say that two holomorphic local dynamical systems f_1, $f_2 \in \mathbb{C}_0\{z_1,\dots,z_n\}^n$ are *formally conjugated* if there is an invertible $\Phi \in \mathbb{C}_0[[z_1,\dots,z_n]]^n$ such that $f_1 = \Phi^{-1} \circ f_2 \circ \Phi$ in $\mathbb{C}_0[[z_1,\dots,z_n]]^n$.

It is clear that two holomorphically locally conjugated holomorphic local dynamical systems are both formally and topologically locally conjugated too. On the other hand, we shall see examples of holomorphic local dynamical systems that are topologically locally conjugated without being neither formally nor holomorphically locally conjugated, and examples of holomorphic local dynamical systems that are formally conjugated without being neither holomorphically nor topologically locally conjugated. So the last natural question in local holomorphic dynamics we shall deal with is

(Q5) *Find normal forms and invariants with respect to the relation of formal conjugacy for holomorphic local dynamical systems at $O \in \mathbb{C}^n$.*

In this survey we shall present some of the main results known on these questions, starting from the one-dimensional situation. But before entering the main core of the paper I would like to heartily thank François Berteloot, Kingshook Biswas, Filippo Bracci, Santiago Diaz-Madrigal, Graziano Gentili, Giorgio Patrizio, Mohamad Pouryayevali, Jasmin Raissy and Francesca Tovena, without whom none of this would have been written.

2 One Complex Variable: The Hyperbolic Case

Let us then start by discussing holomorphic local dynamical systems at $0 \in \mathbb{C}$. As remarked in the previous section, such a system is given by a converging power series f without constant term:

$$f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots \in \mathbb{C}_0\{z\}.$$

Definition 2.1. The number $a_1 = f'(0)$ is the *multiplier* of f.

Since $a_1 z$ is the best linear approximation of f, it is sensible to expect that the local dynamics of f will be strongly influenced by the value of a_1. For this reason we introduce the following definitions:

Definition 2.2. Let $a_1 \in \mathbb{C}$ be the multiplier of $f \in \mathrm{End}(\mathbb{C},0)$. Then

- if $|a_1| < 1$ we say that the fixed point 0 is *attracting;*
- if $a_1 = 0$ we say that the fixed point 0 is *superattracting;*

- if $|a_1| > 1$ we say that the fixed point 0 is *repelling;*
- if $|a_1| \neq 0, 1$ we say that the fixed point 0 is *hyperbolic;*
- if $a_1 \in S^1$ is a root of unity, we say that the fixed point 0 is *parabolic* (or *rationally indifferent*);
- if $a_1 \in S^1$ is not a root of unity, we say that the fixed point 0 is *elliptic* (or *irrationally indifferent*).

As we shall see in a minute, the dynamics of one-dimensional holomorphic local dynamical systems with a hyperbolic fixed point is pretty elementary; so we start with this case.

Remark 2.3. Notice that if 0 is an attracting fixed point for $f \in \mathrm{End}(\mathbb{C},0)$ with non-zero multiplier, then it is a repelling fixed point for the inverse map $f^{-1} \in \mathrm{End}(\mathbb{C},0)$.

Assume first that 0 is attracting for the holomorphic local dynamical system $f \in \mathrm{End}(\mathbb{C},0)$. Then we can write $f(z) = a_1 z + O(z^2)$, with $0 < |a_1| < 1$; hence we can find a large constant $M > 0$, a small constant $\varepsilon > 0$ and $0 < \delta < 1$ such that if $|z| < \varepsilon$ then

$$|f(z)| \leq (|a_1| + M\varepsilon)|z| \leq \delta|z|. \tag{1}$$

In particular, if Δ_ε denotes the disk of center 0 and radius ε, we have $f(\Delta_\varepsilon) \subset \Delta_\varepsilon$ for $\varepsilon > 0$ small enough, and the stable set of $f|_{\Delta_\varepsilon}$ is Δ_ε itself (in particular, a one-dimensional attracting fixed point is always stable). Furthermore,

$$|f^k(z)| \leq \delta^k |z| \to 0$$

as $k \to +\infty$, and thus every orbit starting in Δ_ε is attracted by the origin, which is the reason of the name "attracting" for such a fixed point.

If instead 0 is a repelling fixed point, a similar argument (or the observation that 0 is attracting for f^{-1}) shows that for $\varepsilon > 0$ small enough the stable set of $f|_{\Delta_\varepsilon}$ reduces to the origin only: all (non-trivial) orbits escape.

It is also not difficult to find holomorphic and topological normal forms for one-dimensional holomorphic local dynamical systems with a hyperbolic fixed point, as shown in the following result, which can be considered as the beginning of the theory of holomorphic dynamical systems:

Theorem 2.4 (Kœnigs, 1884 [Kœ]). *Let $f \in \mathrm{End}(\mathbb{C},0)$ be a one-dimensional holomorphic local dynamical system with a hyperbolic fixed point at the origin, and let $a_1 \in \mathbb{C}^* \setminus S^1$ be its multiplier. Then:*

(i) *f is holomorphically (and hence formally) locally conjugated to its linear part $g(z) = a_1 z$. The conjugation φ is uniquely determined by the condition $\varphi'(0) = 1$.*
(ii) *Two such holomorphic local dynamical systems are holomorphically conjugated if and only if they have the same multiplier.*
(iii) *f is topologically locally conjugated to the map $g_<(z) = z/2$ if $|a_1| < 1$, and to the map $g_>(z) = 2z$ if $|a_1| > 1$.*

Proof. Let us assume $0 < |a_1| < 1$; if $|a_1| > 1$ it will suffice to apply the same argument to f^{-1}.

(i) Choose $0 < \delta < 1$ such that $\delta^2 < |a_1| < \delta$. Writing $f(z) = a_1 z + z^2 r(z)$ for a suitable holomorphic germ r, we can clearly find $\varepsilon > 0$ such that $|a_1| + M\varepsilon < \delta$, where $M = \max_{z \in \overline{\Delta_\varepsilon}} |r(z)|$. So we have

$$|f(z) - a_1 z| \le M|z|^2 \qquad \text{and} \qquad |f^k(z)| \le \delta^k |z|$$

for all $z \in \overline{\Delta_\varepsilon}$ and $k \in \mathbb{N}$.

Put $\varphi_k = f^k / a_1^k$; we claim that the sequence $\{\varphi_k\}$ converges to a holomorphic map $\varphi \colon \Delta_\varepsilon \to \mathbb{C}$. Indeed we have

$$\begin{aligned} |\varphi_{k+1}(z) - \varphi_k(z)| &= \frac{1}{|a_1|^{k+1}} \left| f\big(f^k(z)\big) - a_1 f^k(z) \right| \\ &\le \frac{M}{|a_1|^{k+1}} |f^k(z)|^2 \le \frac{M}{|a_1|} \left(\frac{\delta^2}{|a_1|} \right)^k |z|^2 \end{aligned}$$

for all $z \in \overline{\Delta_\varepsilon}$, and so the telescopic series $\sum_k (\varphi_{k+1} - \varphi_k)$ is uniformly convergent in Δ_ε to $\varphi - \varphi_0$.

Since $\varphi_k'(0) = 1$ for all $k \in \mathbb{N}$, we have $\varphi'(0) = 1$ and so, up to possibly shrink ε, we can assume that φ is a biholomorphism with its image. Moreover, we have

$$\varphi\big(f(z)\big) = \lim_{k \to +\infty} \frac{f^k\big(f(z)\big)}{a_1^k} = a_1 \lim_{k \to +\infty} \frac{f^{k+1}(z)}{a_1^{k+1}} = a_1 \varphi(z),$$

that is $f = \varphi^{-1} \circ g \circ \varphi$, as claimed.

If ψ is another local holomorphic function such that $\psi'(0) = 1$ and $\psi^{-1} \circ g \circ \psi = f$, it follows that $\psi \circ \varphi^{-1}(\lambda z) = \lambda \psi \circ \varphi^{-1}(z)$; comparing the expansion in power series of both sides we find $\psi \circ \varphi^{-1} \equiv \mathrm{id}$, that is $\psi \equiv \varphi$, as claimed.

(ii) Since $f_1 = \varphi^{-1} \circ f_2 \circ \varphi$ implies $f_1'(0) = f_2'(0)$, the multiplier is invariant under holomorphic local conjugation, and so two one-dimensional holomorphic local dynamical systems with a hyperbolic fixed point are holomorphically locally conjugated if and only if they have the same multiplier.

(iii) Since $|a_1| < 1$ it is easy to build a topological conjugacy between g and $g_<$ on Δ_ε. First choose a homeomorphism χ between the annulus $\{|a_1|\varepsilon \le |z| \le \varepsilon\}$ and the annulus $\{\varepsilon/2 \le |z| \le \varepsilon\}$ which is the identity on the outer circle and given by $\chi(z) = z/(2a_1)$ on the inner circle. Now extend χ by induction to a homeomorphism between the annuli $\{|a_1|^k \varepsilon \le |z| \le |a_1|^{k-1}\varepsilon\}$ and $\{\varepsilon/2^k \le |z| \le \varepsilon/2^{k-1}\}$ by prescribing

$$\chi(a_1 z) = \frac{1}{2}\chi(z).$$

Putting finally $\chi(0) = 0$ we then get a homeomorphism χ of Δ_ε with itself such that $g = \chi^{-1} \circ g_< \circ \chi$, as required. □

Remark 2.5. Notice that $g_<(z) = \frac{1}{2}z$ and $g_>(z) = 2z$ cannot be topologically conjugated, because (for instance) $K_{g_<}$ is open whereas $K_{g_>} = \{0\}$ is not.

Remark 2.6. The proof of this theorem is based on two techniques often used in dynamics to build conjugations. The first one is used in part (i). Suppose that we would like to prove that two invertible local dynamical systems $f, g \in \mathrm{End}(M,p)$ are conjugated. Set $\varphi_k = g^{-k} \circ f^k$, so that

$$\varphi_k \circ f = g^{-k} \circ f^{k+1} = g \circ \varphi_{k+1}.$$

Therefore if we can prove that $\{\varphi_k\}$ converges to an invertible map φ as $k \to +\infty$ we get $\varphi \circ f = g \circ \varphi$, and thus f and g are conjugated, as desired. This is exactly the way we proved Theorem 2.4.(i); and we shall see variations of this technique later on.

To describe the second technique we need a definition.

Definition 2.7. Let $f\colon X \to X$ be an open continuous self-map of a topological space X. A *fundamental domain* for f is an open subset $D \subset X$ such that

(i) $f^h(D) \cap f^k(D) = \emptyset$ for every $h \neq k \in \mathbb{N}$;
(ii) $\bigcup\limits_{k \in \mathbb{N}} f^k(\overline{D}) = X$;
(iii) if $z_1, z_2 \in \overline{D}$ are so that $f^h(z_1) = f^k(z_2)$ for some $h > k \in \mathbb{N}$ then $h = k+1$ and $z_2 = f(z_1) \in \partial D$.

There are other possible definitions of a fundamental domain, but this will work for our aims.

Suppose that we would like to prove that two open continuous maps $f_1 : X_1 \to X_1$ and $f_2\colon X_2 \to X_2$ are topologically conjugated. Assume we have fundamental domains $D_j \subset X_j$ for f_j (with $j = 1, 2$) and a homeomorphism $\chi : \overline{D_1} \to \overline{D_2}$ such that

$$\chi \circ f_1 = f_2 \circ \chi \tag{2}$$

on $\overline{D_1} \cap f_1^{-1}(\overline{D_1})$. Then we can extend χ to a homeomorphism $\tilde{\chi}\colon X_1 \to X_2$ conjugating f_1 and f_2 by setting

$$\tilde{\chi}(z) = f_2^k\big(\chi(w)\big), \tag{3}$$

for all $z \in X_1$, where $k = k(z) \in \mathbb{N}$ and $w = w(z) \in \overline{D}$ are chosen so that $f_1^k(w) = z$. The definition of fundamental domain and (2) imply that $\tilde{\chi}$ is well-defined. Clearly $\tilde{\chi} \circ f_1 = f_2 \circ \tilde{\chi}$; and using the openness of f_1 and f_2 it is easy to check that $\tilde{\chi}$ is a homeomorphism. This is the technique we used in the proof of Theorem 2.4.(iii); and we shall use it again later.

Thus the dynamics in the one-dimensional hyperbolic case is completely clear. The superattracting case can be treated similarly. If 0 is a superattracting point for an $f \in \mathrm{End}(\mathbb{C}, 0)$, we can write

$$f(z) = a_r z^r + a_{r+1} z^{r+1} + \cdots$$

with $a_r \neq 0$.