Matrix Differential Calculus

with Applications in Statistics and Econometrics

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Matrix Differential Calculus with Applications in Statistics and Econometrics

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Matrix Differential Calculus
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Preface

There has been a long-felt need for a book that gives a self-contained and unified treatment of matrix differential calculus, specifically written for econometricians and statisticians. The present book is meant to satisfy this need. It can serve as a textbook for advanced undergraduates and postgraduates in econometrics and as a reference book for practising econometricians. Also mathematical statisticians and psychometricians may find something to their liking in the book.

When used as a textbook it can provide a full semester course. No previous knowledge of matrix algebra or calculus is required, but some knowledge in these fields does help. The basics of matrix algebra as deemed necessary for a proper understanding of the main subject of the book are supplied in the first of the book's six parts. The book also contains the essentials of multivariable calculus but geared to and often phrased in terms of differentials.

The sequence in which the chapters are read is not of great consequence. It is fully conceivable that practitioners start with Part Three (Differentials: the practice) and dependent on their predilections carry on to any of the applied Parts Five or Six. Those who want a full understanding of the underlying theory should read the whole book, although even then they could go through the necessary matrix algebra when the specific need arises.

Matrix differential calculus as presented in this book is based on differentials, and this sets the book apart from other books in this area. The approach via differentials is in our opinion superior to any other existing approach. Our leading idea is that differentials are more congenial to multivariable functions as they crop up in econometrics, mathematical statistics or psychometrics than derivatives, although from a theoretical point of view the two concepts are equivalent. When there is a specific need for derivatives they will be obtained from differentials.

The book falls into six parts. Part One deals with matrix algebra. It lists—and often also proves—items like the Schur, Jordan and singular-value decompositions, concepts like the Hadamard and Kronecker products, the vec operator, the commutation and duplication matrices, the Moore–Penrose inverse, and results on bordered matrices (and their determinants) and (linearly restricted) quadratic forms.

Part Two, which forms the theoretical heart of the book, is entirely devoted to a thorough development of the theory of differentials. It presents the essentials of calculus but geared to and phrased in terms of differentials. First and second differentials are defined, 'identification' rules for Jacobian and Hessian matrices are given, and chain rules derived. A separate chapter on the theory of (constrained) optimization in terms of differentials concludes this part.

Part Three is the practical core of the book. It contains the rules for working with differentials, lists the differentials of important scalar, vector and matrix functions (*inter alia* eigenvalues, eigenvectors and the Moore–Penrose inverse) and supplies 'identification' tables for Jacobian and Hessian matrices.

Part Four (one chapter on inequalities) owes its existence to our feeling that econometricians should be conversant with inequalities like the Cauchy–Schwarz and Minkowski (and extensions thereof), but should also master a powerful result like Poincaré's separation theorem. This part is to some extent also the case history of a disappointment. When we started writing this book we had the ambition to derive all inequalities by means of matrix differential calculus. After all, every inequality can be rephrased as the solution of an optimization problem. This proved to be an illusion, due to the fact that the Hessian matrix in most cases is singular at the optimum point.

Part Five is entirely devoted to applications of matrix differential calculus to the linear regression model. There is an exhaustive treatment of estimation problems related to the fixed part of the model under various assumptions concerning ranks and (other) constraints. It further has topics relating to the stochastic part of the model, viz. estimation of the error variance and prediction of the error term. There is also a small section on sensitivity analysis. An introductory chapter deals with the necessary statistical preliminaries.

Part Six deals with maximum likelihood estimation, which is of course an ideal source for demonstrating the power of the propagated techniques. In the first of three chapters, several models are being analysed, *inter alia* the multivariate normal distribution, the errors-in-variables model and the nonlinear regression model. There is a discussion of how to deal with symmetry and positive definiteness, and special attention is given to the information matrix. The second chapter in this part deals with simultaneous equations under normality conditions. It investigates both identification and estimation problems, subject to various (non)linear constraints on the parameters. There is a separate treatment of full-information maximum likelihood (FIML) and limited-information maximum likelihood (LIML) with special attention to the derivation of asymptotic variance matrices. The final chapter addresses itself to various psychometric problems, *inter alia* principal components, multimode component analysis, factor analysis and canonical correlation.

All chapters contain many exercises. These are frequently meant to be complementary to the main text. References to equations, theorems and sections are given as follows: equation (1) refers to an equation within the same section; (2.1) refers to an equation in section 2 within the same chapter; and (3.2.1) refers to an equation in section 2 of Chapter 3. Similarly, we refer to theorems and sections within the same chapter by a single serial number, and to theorems and sections in other chapters by double numbers. The notation is mostly standard, except that matrices and vectors are printed in italic, not in bold face. Special symbols are used to denote derivative D and Hessian H. The differential operator is denoted by (roman) d.

A large number of books and papers have been published on the theory and applications of matrix differential calculus. Without attempting to describe their relative virtues and particularities, the interested reader may wish to consult Dwyer and MacPhail (1948), Bodewig (1959), Wilkinson (1965), Dwyer (1967), Neudecker (1967, 1969), Tracy and Dwyer (1969), Tracy and Singh (1972), McDonald and Swaminathan (1973), MacRae (1974), Balestra (1976), Bentler and Lee (1978), Henderson and Searle (1979), Wong and Wong (1979, 1980), Nel (1980), Rogers (1980), Wong (1980, 1985), Graham (1981), McCulloch (1982), Schönemann (1985), Magnus and Neudecker (1985), Pollock (1985) and Don (1986). The papers by Henderson and Searle (1979) and Nel (1980) and Rogers' (1980) book contain extensive bibliographies.

The two authors share the responsibility for Parts One, Three, Five and Six, although any new results in Part One are due to Magnus. Parts Two and Four are due to Magnus, although Neudecker contributed some results to Part Four. Magnus is also responsible for the writing and organization of the final text.

We wish to thank our colleagues F. J. H. Don, R. D. H. Heijmans, D. S. G. Pollock and R. Ramer for their critical remarks and contributions. The greatest obligation is owed to Sue Kirkbride at the London School of Economics who patiently and cheerfully typed and retyped the various versions of the book. Partial financial support was provided by the Netherlands Organization for the Advancement of Pure Research (Z. W. O.) and the Suntory Toyota International Centre for Economics and Related Disciplines at the London School of Economics.

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