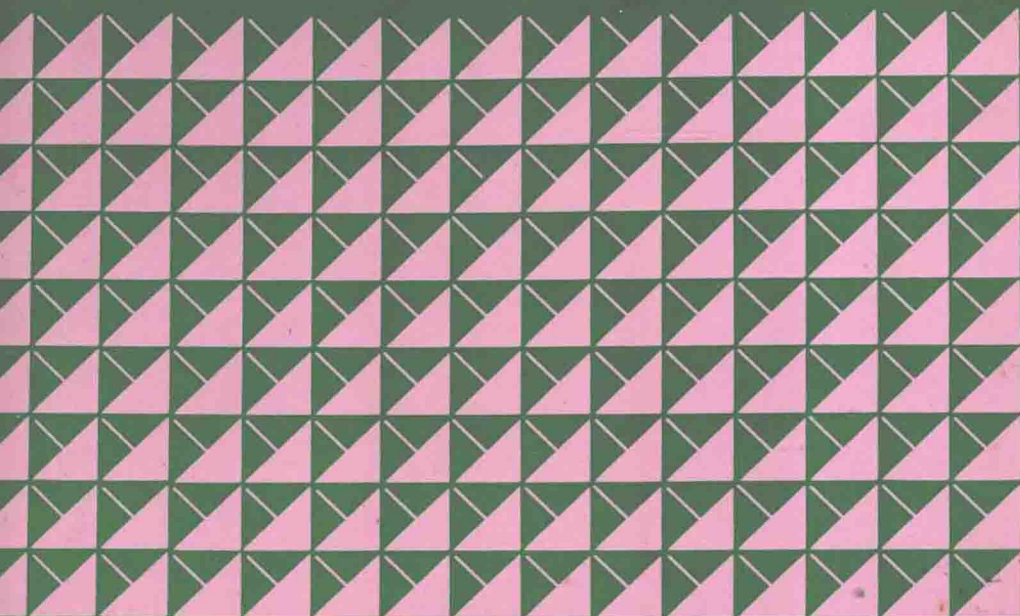


Matrix Differential Calculus

**with Applications in
Statistics and Econometrics**



**Jan R. Magnus
Heinz Neudecker**

*Matrix Differential Calculus
with Applications in
Statistics and Econometrics*

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Matrix Differential Calculus
with Applications in
Statistics and Econometrics

Preface

There has been a long-felt need for a book that gives a self-contained and unified treatment of matrix differential calculus, specifically written for econometricians and statisticians. The present book is meant to satisfy this need. It can serve as a textbook for advanced undergraduates and postgraduates in econometrics and as a reference book for practising econometricians. Also mathematical statisticians and psychometricians may find something to their liking in the book.

When used as a textbook it can provide a full semester course. No previous knowledge of matrix algebra or calculus is required, but some knowledge in these fields does help. The basics of matrix algebra as deemed necessary for a proper understanding of the main subject of the book are supplied in the first of the book's six parts. The book also contains the essentials of multivariable calculus but geared to and often phrased in terms of differentials.

The sequence in which the chapters are read is not of great consequence. It is fully conceivable that practitioners start with Part Three (Differentials: the practice) and dependent on their predilections carry on to any of the applied Parts Five or Six. Those who want a full understanding of the underlying theory should read the whole book, although even then they could go through the necessary matrix algebra when the specific need arises.

Matrix differential calculus as presented in this book is based on differentials, and this sets the book apart from other books in this area. The approach via differentials is in our opinion superior to any other existing approach. Our leading idea is that differentials are more congenial to multivariable functions as they crop up in econometrics, mathematical statistics or psychometrics than derivatives, although from a theoretical point of view the two concepts are equivalent. When there is a specific need for derivatives they will be obtained from differentials.

The book falls into six parts. Part One deals with matrix algebra. It lists—and often also proves—items like the Schur, Jordan and singular-value decompositions, concepts like the Hadamard and Kronecker products, the vec operator, the commutation and duplication matrices, the Moore–Penrose inverse, and results on bordered matrices (and their determinants) and (linearly restricted) quadratic forms.

Part Two, which forms the theoretical heart of the book, is entirely devoted to a thorough development of the theory of differentials. It presents the essentials of

calculus but geared to and phrased in terms of differentials. First and second differentials are defined, 'identification' rules for Jacobian and Hessian matrices are given, and chain rules derived. A separate chapter on the theory of (constrained) optimization in terms of differentials concludes this part.

Part Three is the practical core of the book. It contains the rules for working with differentials, lists the differentials of important scalar, vector and matrix functions (*inter alia* eigenvalues, eigenvectors and the Moore–Penrose inverse) and supplies 'identification' tables for Jacobian and Hessian matrices.

Part Four (one chapter on inequalities) owes its existence to our feeling that econometricians should be conversant with inequalities like the Cauchy–Schwarz and Minkowski (and extensions thereof), but should also master a powerful result like Poincaré's separation theorem. This part is to some extent also the case history of a disappointment. When we started writing this book we had the ambition to derive all inequalities by means of matrix differential calculus. After all, every inequality can be rephrased as the solution of an optimization problem. This proved to be an illusion, due to the fact that the Hessian matrix in most cases is singular at the optimum point.

Part Five is entirely devoted to applications of matrix differential calculus to the linear regression model. There is an exhaustive treatment of estimation problems related to the fixed part of the model under various assumptions concerning ranks and (other) constraints. It further has topics relating to the stochastic part of the model, viz. estimation of the error variance and prediction of the error term. There is also a small section on sensitivity analysis. An introductory chapter deals with the necessary statistical preliminaries.

Part Six deals with maximum likelihood estimation, which is of course an ideal source for demonstrating the power of the propagated techniques. In the first of three chapters, several models are being analysed, *inter alia* the multivariate normal distribution, the errors-in-variables model and the nonlinear regression model. There is a discussion of how to deal with symmetry and positive definiteness, and special attention is given to the information matrix. The second chapter in this part deals with simultaneous equations under normality conditions. It investigates both identification and estimation problems, subject to various (non)linear constraints on the parameters. There is a separate treatment of full-information maximum likelihood (FIML) and limited-information maximum likelihood (LIML) with special attention to the derivation of asymptotic variance matrices. The final chapter addresses itself to various psychometric problems, *inter alia* principal components, multimode component analysis, factor analysis and canonical correlation.

All chapters contain many exercises. These are frequently meant to be complementary to the main text. References to equations, theorems and sections are given as follows: equation (1) refers to an equation within the same section; (2.1) refers to an equation in section 2 within the same chapter; and (3.2.1) refers to an equation in section 2 of Chapter 3. Similarly, we refer to theorems and sections within the same chapter by a single serial number, and to theorems and sections

in other chapters by double numbers. The notation is mostly standard, except that matrices and vectors are printed in *italic*, not in bold face. Special symbols are used to denote derivative D and Hessian H . The differential operator is denoted by (roman) d .

A large number of books and papers have been published on the theory and applications of matrix differential calculus. Without attempting to describe their relative virtues and particularities, the interested reader may wish to consult Dwyer and MacPhail (1948), Bodewig (1959), Wilkinson (1965), Dwyer (1967), Neudecker (1967, 1969), Tracy and Dwyer (1969), Tracy and Singh (1972), McDonald and Swaminathan (1973), MacRae (1974), Balestra (1976), Bentler and Lee (1978), Henderson and Searle (1979), Wong and Wong (1979, 1980), Nel (1980), Rogers (1980), Wong (1980, 1985), Graham (1981), McCulloch (1982), Schönemann (1985), Magnus and Neudecker (1985), Pollock (1985) and Don (1986). The papers by Henderson and Searle (1979) and Nel (1980) and Rogers' (1980) book contain extensive bibliographies.

The two authors share the responsibility for Parts One, Three, Five and Six, although any new results in Part One are due to Magnus. Parts Two and Four are due to Magnus, although Neudecker contributed some results to Part Four. Magnus is also responsible for the writing and organization of the final text.

We wish to thank our colleagues F. J. H. Don, R. D. H. Heijmans, D. S. G. Pollock and R. Ramer for their critical remarks and contributions. The greatest obligation is owed to Sue Kirkbride at the London School of Economics who patiently and cheerfully typed and retyped the various versions of the book. Partial financial support was provided by the Netherlands Organization for the Advancement of Pure Research (Z. W. O.) and the Suntory Toyota International Centre for Economics and Related Disciplines at the London School of Economics.

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Jan R. Magnus
Heinz Neudecker

Contents

Preface

xv

Part One—Matrices

1	Basic properties of vectors and matrices	3
1	Introduction,	3
2	Sets,	3
3	Matrices: addition and multiplication,	4
4	The transpose of a matrix,	5
5	Square matrices,	6
6	Linear forms and quadratic forms,	7
7	The rank of a matrix,	8
8	The inverse,	9
9	The determinant,	9
10	The trace,	10
11	Partitioned matrices,	11
12	Eigenvalues and eigenvectors,	12
13	Complex matrices,	15
14	Schur's decomposition theorem,	16
15	The Jordan decomposition,	17
16	The singular-value decomposition,	18
17	Further results concerning eigenvalues,	18
18	Positive (semi)definite matrices,	21
19	Three further results for positive definite matrices,	23
20	A useful result,	24
	Miscellaneous exercises,	25
	Bibliographical notes,	26
2	Kronecker products, the vec operator and the Moore–Penrose inverse	27
1	Introduction,	27
2	The Kronecker product,	27

- 3 *Eigenvalues of a Kronecker product*, 28
- 4 *The vec operator*, 30
- 5 *The Moore–Penrose (MP) inverse*, 32
- 6 *Existence and uniqueness of the MP inverse*, 32
- 7 *Some properties of the MP inverse*, 33
- 8 *Further properties*, 34
- 9 *The solution of linear equation systems*, 36
- Miscellaneous exercises*, 38
- Bibliographical notes*, 38

3 Miscellaneous matrix results

40

- 1 *Introduction*, 40
- 2 *The adjoint matrix*, 40
- 3 *Proof of Theorem 1*, 41
- 4 *Two results concerning bordered determinants*, 43
- 5 *The matrix equation $AX = 0$* , 44
- 6 *The Hadamard product*, 45
- 7 *The commutation matrix K_{mn}* , 46
- 8 *The duplication matrix D_n* , 48
- 9 *Relationship between D_{n+1} and D_n , I*, 50
- 10 *Relationship between D_{n+1} and D_n , II*, 52
- 11 *Conditions for a quadratic form to be positive (negative) subject to linear constraints*, 53
- 12 *Necessary and sufficient conditions for $r(A:B) = r(A) + r(B)$* , 56
- 13 *The bordered Gramian matrix*, 57
- 14 *The equations $X_1A + X_2B' = G_1$, $X_1B = G_2$* , 60
- Bibliographical notes*, 62

Part Two—Differentials: the theory

4 Mathematical preliminaries

65

- 1 *Introduction*, 65
- 2 *Interior points and accumulation points*, 65
- 3 *Open and closed sets*, 66
- 4 *The Bolzano–Weierstrass theorem*, 69
- 5 *Functions*, 70
- 6 *The limit of a function*, 70
- 7 *Continuous functions and compactness*, 71
- 8 *Convex sets*, 72
- 9 *Convex and concave functions*, 75
- Bibliographical notes*, 77

5	Differentials and differentiability	78
1	<i>Introduction</i> , 78	
2	<i>Continuity</i> , 78	
3	<i>Differentiability and linear approximation</i> , 80	
4	<i>The differential of a vector function</i> , 82	
5	<i>Uniqueness of the differential</i> , 84	
6	<i>Continuity of differentiable functions</i> , 84	
7	<i>Partial derivatives</i> , 85	
8	<i>The first identification theorem</i> , 87	
9	<i>Existence of the differential, I</i> , 88	
10	<i>Existence of the differential, II</i> , 89	
11	<i>Continuous differentiability</i> , 91	
12	<i>The chain rule</i> , 91	
13	<i>Cauchy invariance</i> , 93	
14	<i>The mean-value theorem for real-valued functions</i> , 93	
15	<i>Matrix functions</i> , 94	
16	<i>Some remarks on notation</i> , 96	
	<i>Miscellaneous exercises</i> , 97	
	<i>Bibliographical notes</i> , 98	
6	The second differential	99
1	<i>Introduction</i> , 99	
2	<i>Second-order partial derivatives</i> , 99	
3	<i>The Hessian matrix</i> , 100	
4	<i>Twice differentiability and second-order approximation, I</i> , 101	
5	<i>Definition of twice differentiability</i> , 102	
6	<i>The second differential</i> , 103	
7	<i>(Column) symmetry of the Hessian matrix</i> , 105	
8	<i>The second identification theorem</i> , 107	
9	<i>Twice differentiability and second-order approximation, II</i> , 108	
10	<i>Chain rule for Hessian matrices</i> , 110	
11	<i>The analogue for second differentials</i> , 111	
12	<i>Taylor's theorem for real-valued functions</i> , 112	
13	<i>Higher-order differentials</i> , 113	
14	<i>Matrix functions</i> , 114	
	<i>Bibliographical notes</i> , 115	
7	Static optimization	116
1	<i>Introduction</i> , 116	
2	<i>Unconstrained optimization</i> , 116	
3	<i>The existence of absolute extrema</i> , 118	
4	<i>Necessary conditions for a local minimum</i> , 119	

- 5 Sufficient conditions for a local minimum: first-derivative test, 121
- 6 Sufficient conditions for a local minimum: second-derivative test, 122
- 7 Characterization of differentiable convex functions, 124
- 8 Characterization of twice differentiable convex functions, 127
- 9 Sufficient conditions for an absolute minimum, 128
- 10 Monotonic transformations, 129
- 11 Optimization subject to constraints, 130
- 12 Necessary conditions for a local minimum under constraints, 131
- 13 Sufficient conditions for a local minimum under constraints, 135
- 14 Sufficient conditions for an absolute minimum under constraints, 139
- 15 A note on constraints in matrix form, 140
- 16 Economic interpretation of Lagrange multipliers, 141
- Appendix: the implicit function theorem, 142
- Bibliographical notes, 144

Part Three—Differentials: the practice

- 8 Some important differentials 147
 - 1 Introduction, 147
 - 2 Fundamental rules of differential calculus, 147
 - 3 The differential of a determinant, 149
 - 4 The differential of an inverse, 151
 - 5 The differential of the Moore–Penrose inverse, 152
 - 6 The differential of the adjoint matrix, 155
 - 7 On differentiating eigenvalues and eigenvectors, 157
 - 8 The differential of eigenvalues and eigenvectors:
the real symmetric case, 158
 - 9 The differential of eigenvalues and eigenvectors:
the general complex case, 161
 - 10 Two alternative expressions for $d\lambda$, 163
 - 11 The second differential of the eigenvalue function, 166
 - 12 Multiple eigenvalues, 167
 - Miscellaneous exercises, 167
 - Bibliographical notes, 169
- 9 First-order differentials and Jacobian matrices 170
 - 1 Introduction, 170
 - 2 Classification, 170
 - 3 Bad notation, 171
 - 4 Good notation, 173
 - 5 Identification of Jacobian matrices, 174
 - 6 The first identification table, 175

7	<i>Partitioning of the derivative</i> , 175	
8	<i>Scalar functions of a vector</i> , 176	
9	<i>Scalar functions of a matrix, I: trace</i> , 177	
10	<i>Scalar functions of a matrix, II: determinant</i> , 178	
11	<i>Scalar functions of a matrix, III: eigenvalue</i> , 180	
12	<i>Two examples of vector functions</i> , 181	
13	<i>Matrix functions</i> , 182	
14	<i>Kronecker products</i> , 184	
15	<i>Some other problems</i> , 185	
	<i>Bibliographical notes</i> , 187	
10	Second-order differentials and Hessian matrices	188
1	<i>Introduction</i> , 188	
2	<i>The Hessian matrix of a matrix function</i> , 188	
3	<i>Identification of Hessian matrices</i> , 189	
4	<i>The second identification table</i> , 190	
5	<i>An explicit formula for the Hessian matrix</i> , 191	
6	<i>Scalar functions</i> , 192	
7	<i>Vector functions</i> , 194	
8	<i>Matrix functions, I</i> , 194	
9	<i>Matrix functions, II</i> , 195	
Part Four—Inequalities		
11	Inequalities	199
1	<i>Introduction</i> , 199	
2	<i>The Cauchy–Schwarz inequality</i> , 199	
3	<i>Matrix analogues of the Cauchy–Schwarz inequality</i> , 201	
4	<i>The theorem of the arithmetic and geometric means</i> , 202	
5	<i>The Rayleigh quotient</i> , 203	
6	<i>Concavity of λ_1, convexity of λ_n</i> , 204	
7	<i>Variational description of eigenvalues</i> , 205	
8	<i>Fischer’s min–max theorem</i> , 206	
9	<i>Monotonicity of the eigenvalues</i> , 208	
10	<i>The Poincaré separation theorem</i> , 209	
11	<i>Two corollaries of Poincaré’s theorem</i> , 210	
12	<i>Further consequences of the Poincaré theorem</i> , 211	
13	<i>Multiplicative version</i> , 212	
14	<i>The maximum of a bilinear form</i> , 213	
15	<i>Hadamard’s inequality</i> , 214	
16	<i>An interlude: Karamata’s inequality</i> , 215	
17	<i>Karamata’s inequality applied to eigenvalues</i> , 217	

- 18 *An inequality concerning positive semidefinite matrices*, 217
- 19 *A representation theorem for $(\Sigma a_i^p)^{1/p}$* , 218
- 20 *A representation theorem for $(\text{tr } A^p)^{1/p}$* , 219
- 21 *Hölder's inequality*, 220
- 22 *Concavity of $\log |A|$* , 222
- 23 *Minkowski's inequality*, 223
- 24 *Quasilinear representation of $|A|^{1/n}$* , 224
- 25 *Minkowski's determinant theorem*, 227
- 26 *Weighted means of order p* , 227
- 27 *Schlömilch's inequality*, 229
- 28 *Curvature properties of $M_p(x, a)$* , 230
- 29 *Least squares*, 232
- 30 *Generalized least squares*, 233
- 31 *Restricted least squares*, 233
- 32 *Restricted least squares: matrix version*, 235
- Miscellaneous exercises*, 237
- Bibliographical notes*, 239

Part Five—The linear model

12 Statistical preliminaries

243

- 1 *Introduction*, 243
- 2 *The cumulative distribution function*, 243
- 3 *The joint density function*, 244
- 4 *Expectations*, 244
- 5 *Variance and covariance*, 245
- 6 *Independence of two random variables (vectors)*, 247
- 7 *Independence of n random variables (vectors)*, 249
- 8 *Sampling*, 249
- 9 *The one-dimensional normal distribution*, 249
- 10 *The multivariate normal distribution*, 250
- 11 *Estimation*, 252
- Miscellaneous exercises*, 253
- Bibliographical notes*, 253

13 The linear regression model

254

- 1 *Introduction*, 254
- 2 *Affine minimum-trace unbiased estimation*, 255
- 3 *The Gauss–Markov theorem*, 256
- 4 *The method of least squares*, 258
- 5 *Aitken's theorem*, 259
- 6 *Multicollinearity*, 261
- 7 *Estimable functions*, 263

- 8 Linear constraints: the case $\mathcal{M}(R') \subset \mathcal{M}(X')$, 264
- 9 Linear constraints: the general case, 267
- 10 Linear constraints: the case $\mathcal{M}(R') \cap \mathcal{M}(X') = \{0\}$, 270
- 11 A singular variance matrix: the case $\mathcal{M}(X) \subset \mathcal{M}(V)$, 271
- 12 A singular variance matrix: the case $r(X'V^+X) = r(X)$, 273
- 13 A singular variance matrix: the general case, I, 274
- 14 Explicit and implicit linear constraints, 275
- 15 The general linear model, I, 277
- 16 A singular variance matrix: the general case, II, 278
- 17 The general linear model, II, 281
- 18 Generalized least squares, 282
- 19 Restricted least squares, 283
- Miscellaneous exercises, 285
- Bibliographical notes, 286

14 Further topics in the linear model

287

- 1 Introduction, 287
- 2 Best quadratic unbiased estimation of σ^2 , 287
- 3 The best quadratic and positive unbiased estimator of σ^2 , 288
- 4 The best quadratic unbiased estimator of σ^2 , 290
- 5 Best quadratic invariant estimation of σ^2 , 292
- 6 The best quadratic and positive invariant estimator of σ^2 , 293
- 7 The best quadratic invariant estimator of σ^2 , 294
- 8 Best quadratic unbiased estimation in the multivariate normal case, 295
- 9 Bounds for the bias of the least-squares estimator of σ^2 , I, 297
- 10 Bounds for the bias of the least-squares estimator of σ^2 , II, 299
- 11 The prediction of disturbances, 300
- 12 Predictors that are best linear unbiased with scalar variance matrix (BLUS), 301
- 13 Predictors that are best linear unbiased with fixed variance matrix (BLUF), I, 303
- 14 Predictors that are best linear unbiased with fixed variance matrix (BLUF), II, 305
- 15 Local sensitivity of the posterior mean, 306
- 16 Local sensitivity of the posterior precision, 308
- Bibliographical notes, 309

Part Six—Applications to maximum likelihood estimation

15 Maximum likelihood estimation

312

- 1 Introduction, 312
- 2 The method of maximum likelihood (ML), 312
- 3 ML estimation of the multivariate normal distribution, 314