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The Representation Theory of the Symmetric Group

Gordon James
Sidney Sussex College
Cambridge, Great Britain

Adalbert Kerber
University of Bayreuth
Bayreuth, Federal Republic of Germany

Foreword by
P. M. Cohn
University of London, Bedford College

Introduction by
G. de B. Robinson
University of Toronto

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Editor's Statement

A large body of mathematics consists of facts that can be presented and described much like any other natural phenomenon. These facts, at times explicitly brought out as theorems, at other times concealed within a proof, make up most of the applications of mathematics, and are the most likely to survive change of style and of interest.

This ENCYCLOPEDIA will attempt to present the factual body of all mathematics. Clarity of exposition, accessibility to the non-specialist, and a thorough bibliography are required of each author. Volumes will appear in no particular order, but will be organized into sections, each one comprising a recognizable branch of present-day mathematics. Numbers of volumes and sections will be reconsidered as times and needs change.

It is hoped that this enterprise will make mathematics more widely used where it is needed, and more accessible in fields in which it can be applied but where it has not yet penetrated because of insufficient information.

GIAN-CARLO ROTA

Foreword

The theory of group representation has its roots in the character theory of abelian groups, which was formulated first for cyclic groups in the context of number theory (Gauss, Dirichlet, but already implicit in the work of Euler), and later generalized by Frobenius and Stickelberger to any finite abelian groups. For an abelian group all irreducible representations (over $\mathbb{C}$) are of course 1-dimensional and hence are completely described by their characters. The representation theory of finite groups emerged around the turn of the century as the work of Frobenius, Schur, and Burnside. While it applied in principle to any finite group, the symmetric group S_n was a simple but important special case; —simple because its characters and irreducible representations could already be found in the rational field, important because every finite group could be embedded in some symmetric group.

Moreover, the theory can be applied whenever we have a symmetric group action on a linear space. Perhaps the simplest example is the case of a bilinear form $f(x, y)$. No theory is required to decompose f into a symmetric part: $s(x, y)=f(x, y)+f(y, x)$ and an antisymmetric part: $a(x, y)=f(x, y)-f(y, x)$. These are of course just (bases for the 1-dimensional modules affording) the irreducible representations of S_2, 1-dimensional because S_2 is abelian. Taking next a trilinear form $f(x, y, z)$, we have again the symmetric and antisymmetric parts:

$$s(x, y, z)=\sum_{\sigma} f(\sigma x, \sigma y, \sigma z),$$

$$a(x, y, z)=\sum_{\sigma} \operatorname{sgn}\sigma f(\sigma x, \sigma y, \sigma z),$$

where σ ranges over all permutations of x, y, z. No other linear combination of f's is only multiplied by a scalar factor by the S_3-action (such a factor would have to be 1 or $\operatorname{sgn}\sigma$, because every permutation is a product of transpositions), but we can find pairs of linear combinations spanning a 2-dimensional S_3-module, e.g.

$$p=f(x, y, z)+f(y, x, z)-f(z, y, x)-f(z, x, y)$$
$$q=f(z, y, x)+f(y, z, x)-f(x, y, z)-f(x, z, y)$$

Here p is obtained by 'symmetrizing' x, y and 'antisymmetrizing' x, z, and q

is obtained by interchanging x, z in p. If (x, y) denotes the transposition of x, y etc., then we have

$$(x, y)=p, \quad (x, z)p=q, \quad (y, z)p=-p-q, \quad (x, y)q=-p-q,$$
$$(x, z)q=p, \quad (y, z)q=q.$$

Thus we obtain the representation

$$(x, y)\to\begin{pmatrix}1 & -1\\ 0 & -1\end{pmatrix}, \quad (x, z)\to\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad (y, z)\to\begin{pmatrix}-1 & 0\\ -1 & 1\end{pmatrix},$$
$$(x, y, z)\to\begin{pmatrix}0 & -1\\ 1 & -1\end{pmatrix}, \quad (x, z, y)\to\begin{pmatrix}-1 & 1\\ -1 & 0\end{pmatrix},$$

which is an irreducible 2-dimensional representation of S_3.

It was Alfred Young's achievement to find a natural classification of all the irreducible representations of S_n in terms of 'Young tableaux', which are essentially the different ways of fully symmetrizing and antisymmetrizing. The n-symbols permuted are arranged in a diagram so that rows are symmetrized and columns antisymmetrized. In the above example we symmetrized x, y and antisymmetrized x, z; this is indicated by the tableau

x	y
z	

Young's derivation via tableaux was even more direct than Frobenius' and Schur's earlier method, using bialternants or S-functions, although these functions are useful in formulating combinations of representations such as plethysm.

There have been many accounts of the theory, from various points of view, and often the original sources have been hard to follow. It is good to have a general treatment, —by two authors who have both made substantial original contributions, —which combines the best of previous accounts, and systematizes and adds much that is new. After a clear exposition of Young's approach (in modern terms) they present an improved version of Specht modules giving a characteristic-free treatment and leading to a practical algorithm for estimating dimensions. The applications to combinatorics include Polya's enumeration theory, and also the less well known work of Redfield, and there is a separate chapter on the connection with representations of the general linear groups.

The comprehensive treatment, with helpful suggestions for further reading, very full references, various tables of characters, as well as the interesting historical introduction by G. de B. Robinson, will all help to make 'James-Kerber' the standard work on the subject.

P. M. Cohn

Introduction

In this introduction to the work of James and Kerber I should like to survey briefly the story of developments in the representation theory of the symmetric group. Detailed references will not be possible, but it seems worthwhile to glance at the background which has aroused so much interest in recent years.

The idea of a group goes back a long way and is inherent in the study of the regular polyhedra by the Greeks. It was Galois who systematically developed the connection with algebraic equations, early in the nineteenth century. Not long after, the geometrical relationship between the lines on a general cubic surface and the bitangents of a plane quartic curve aroused the interest of Hesse and Cayley, with a significant contribution by Schläfli in 1858 [1, Chapter IX].* Jordan in his *Traité des Substitutions*, 1870 [2], and Klein in his *Vorlesungen über das Ikosaeder*, 1884 [3], added new dimensions to Galois's work. The first edition of Burnside's *Theory of Groups of Finite Order* appeared in 1897, just at the time when Frobenius's papers in the *Berliner Sitzungsberichte* were changing the whole algebraic approach. With the appearance of *Schur's Thesis* [4] in 1901, the need for a revision of Burnside's work became apparent.

Burnside began his preface to the second edition, which appeared in 1911 [5], with the comment: "Very considerable advance in the theory of groups of finite order has been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers...." His preface concludes with the remark: "I owe my best thanks to the Rev. Alfred Young, M.A., Rector of Birdbrook, Essex, and former Fellow of Clare College, Cambridge, who read the whole of the book as it passed through the Press. His careful criticism has saved me from many errors and his suggestions have been of great help to me."

Alfred Young was born in 1873 and graduated from Cambridge in 1895. His first paper, "The irreducible concomitants of any number of binary quartics," appeared in the *Proceedings of the London Mathematical Society* in 1899. It had been refereed by Burnside, who told him to read the works of Frobenius and Schur; unfortunately Young knew no German, so it was not till after the war that he was able to incorporate their ideas in his important *QSA* series.

*References will be found at the end of this Introduction.

My own contact with Alfred Young began in 1929. I graduated from the University of Toronto in 1927 and was much interested in geometry, owing largely to the presence on our staff of Jacques Chapelon from Paris. I was fortunate in obtaining a small scholarship at St. John's College, Cambridge, where my first supervisor was M. H. A. Newman. Under his guidance I began to read topology. No group theory was taught in Toronto or Cambridge in those days, but its significance in topology fascinated me. Soon this became apparent to Newman, and he arranged for me to be transferred to Alfred Young as a graduate student. Young came in to Cambridge once a week to lecture. He and his wife stayed at the Blue Boar Hotel, just across the street from St. John's, where I would go to visit him. The geometrical aspects of group theory continued to interest me, and I attended Baker's tea party every week. This was where I met Donald Coxeter and several other geometers to whom I refer in the Introduction to *Young's Collected Papers*, published in Toronto [6], 1977.

After earning my Ph.D. in Cambridge in 1931, I returned to the University of Toronto. My work on the symmetric group continued, and with the cooperation of J. S. Frame and Philip Hall, yielded the dimension formulae for the irreducible representations of S_n and GL_d over the real field.

Richard Brauer was on staff in Toronto 1935–1948, and his interest in representation theory was responsible for much of the development which took place in those years and later, while he was at Ann Arbor 1949–1952, and Harvard 1952–1978. In 1958 I was invited to lecture at the Australian universities, and my *Representation Theory of the Symmetric Group* appeared in 1961. In 1968 my wife and I went to Christchurch, New Zealand, for three months, and it was during this period that I became interested in the application of group theory to physics. W. T. Sharp in Toronto had obtained his Ph.D. with Wigner in Princeton, and I made contact with Wybourne in New Zealand and with Biedenharn in the U.S., and attended a seminar in Bochum in West Germany in 1969. It was there that I met Adalbert Kerber and many other interesting people. Not long after, I was in touch with Gordon James, who got his Ph.D. in Cambridge with J. G. Thompson. Then when the representation theory gathering was held in Oberwolfach in 1975 I had a chance to talk with many group theorists whose writings I had read, but had never met. Afterwards my wife and I paid a brief visit to the Kerbers in historic Aachen.

It was in the autumn of 1975 that Gordon James came to spend a year at the University of Toronto. He and Kerber had begun to work on this book and we had many conversations; Kerber was largely interested in wreath products, while James had begun writing his considerable number of papers on modular theory. A number of errors had appeared in the decomposition matrices at the end of my book [7], and James has done much to improve their construction in this volume.

In April of 1976 Foata organized another gathering of group theorists in Strasbourg. He did a beautiful job, exploiting the charm of the city and its university to bring together a large number of speakers [8] on various aspects and applications of the symmetric group. Having been invited by Professor McConnell, I gave a repeat performance of my Strasbourg talk in Dublin. This was my first visit to Ireland, and it gave me much pleasure to see J. L. Synge, who had been on our staff in Toronto for many years.

It was in June 1978 that T. V. Narayana of the University of Alberta in Edmonton arranged a gathering at the University of Waterloo. He had become involved in Young's work, and his volume *Lattice Path Combinatorics with Statistical Applications* was published in our Exposition Series in 1979. The proceedings of Young Day has just appeared [9] with an introduction by J. S. Frame and a paper generalizing the hook-formulae for $O_n(2)$.

The last gathering in Oberwolfach which I attended was in January 1979. This book was well on its way but we all regretted that publication would be so long delayed. Its content contributes much to complete the picture, from the point of view of the representation theory of S_n, but there remains the question raised by Frame's work:—Could there be a degree formula for the irreducible representations of S_n or GL_d over a finite field? Future research may provide the answer.

In conclusion, let me refer to *The Theory of Partitions* [10] by Andrews, which has appeared in this series. Professor Rota's comment is worth quoting: "Professor Andrews has written the first thorough survey of this many-sided field. The specialist will consult it for the more recondite results, the student will be challenged by many deceptively simple facts, and the applied scientist may locate in it a missing identity to organize his data." When Young's *Collected Papers* came out, Andrews wrote a most interesting review of his work [11], listing 121 papers based on the original ideas of this remarkable man. The accompanying portrait of Alfred Young was sent to me by Professor Garnir.

It has given me much pleasure to work with the authors of this book, and I wish its readers every satisfaction, as well as the best of luck in further developing these ideas.

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G. DE B. ROBINSON

Preface

The purpose of this book is to provide an account of both the ordinary and the modular representation theory of the symmetric groups. The range of applications of this theory is vast, varying from theoretical physics through combinatorics to the study of polynomial identity algebras, and new uses are still being found. So diverse are the questions which arise that we feel justified in hoping the reader might find that some part of our text inspires him to undertake research of his own into one of the many unsolved problems in this elegant branch of mathematics.

There are several different ways of approaching symmetric group representations, and while we have tried to illuminate parts of the theory by giving more than one description of it, we have made no effort to cover every view of the subject.

The ordinary representation theory of the symmetric groups was first developed by Frobenius, but the greatest contribution to the early material came from Alfred Young. Since Young's main interest lay in quantitative substitutional analysis, it is difficult for a modern mathematican to understand his papers. The reader is referred to the book *Substitutional Analysis* by D. E. Rutherford for a pleasant account of a great part of Young's work. Both Frobenius's and Young's collected works are now available. We include an account of the group algebra and its idempotents, along the lines pursued by Young, since the symmetric group is one of the very rare cases where many aspects of general representation theory can be described explicitly. This also helps us to motivate the introduction of many combinatorial structures which turn out to be useful. The combinatorics are themselves a fascinating and fruitful basis for further study, and they continue to inspire many research papers.

The development of the modular representation theory of symmetric groups was started by T. Nakayama, who derived some p-modular properties of symmetric groups S_n, for $n<2p$, and stated a conjecture about the p-block structure of symmetric groups S_n of arbitrary degree n. This conjecture was proved jointly by R. Brauer and G. de B. Robinson. Robinson and his coworkers developed these methods rapidly to study the decomposition numbers of the symmetric groups. The situation as it was in 1961 is described in Robinson's book.

Later, these methods were combined by the present authors with modular results derived from W. Specht's alternative approach to ordinary representation theory. Specht showed how to derive representations by considering submodules of a polynomial ring $F[x_1,\ldots,x_n]$, and this method yielded interesting results without referring to the characteristic of the field F. Using modules isomorphic to those of Specht, we explain another approach to the ordinary representations, at the same time extracting information about the modular theory.

There is a more recent approach to a characteristic-free representation theory of S_n starting with a basis of a certain intertwining space which was originally derived for invariant-theoretical purposes by G.-C. Rota. We shall not present this method, as we anticipate that it will be described in another book.

The main application which we cover involves the representations of an arbitrary group, using symmetry operators on tensor space, but we also discuss combinatorics, wreath products, and permutation groups. The standard reference for symmetric functions is the book of D. E. Littlewood, but I. G. Macdonald's recent volume gives an up-to-date account of many of the results.

Both authors are greatly indebted to Professor Robinson and the University of Toronto for their generous hospitality during several visits, and they wish to record their gratitude to Professor Robinson, without whose continued enthusiasm, encouragement, and advice the book would not have been written.

In conclusion, we would like to express our thanks for the important help and criticism we received from so many colleagues and from our students, whom we have taken pleasure in working with and who have given us so much useful advice. We mention in particular H. Boerner, R. W. Carter, M. Clausen, N. Esper, H. K. Farahat, A. Golembiowski, M. Klemm, W. Lehmann, A. O. Morris, J. Neubüser, H. Pahlings, M. H. Peel, F. Sänger, D. Stockhofe, J. Tappe, K.-J. Thürlings and B. Wagner.

G. D. JAMES
A. KERBER

Contents

List of Symbols

Symbol	Meaning	Definition on p.
$a_i(\pi)$	The number of i-cycles occurring in the cycle notation of π	7
$a(\pi)$	The cycle type $(a_1(\pi), \dots, a_n(\pi))$ of π	10
$a(f; \pi)$	The type of $(f; \pi)$	139
$a_i^\beta(m)$	The p-residue of the node which m replaces in t_i^β	307
A_n	The alternating group on $\{1, \dots, n\}$	7
AS_λ	The alternating representation of S_λ	17
B_k	The kth Bell number	228
$c(\pi)$	The number of pairwise disjoint cyclic factors of π	5
$\mathbb{C}$	The field of complex numbers	—
C^α	The conjugacy class of S_n consisting of permutations with cycle partition α	12
$C^{\alpha\pm}$	The two conjugacy classes of A_n arising from splitting class C^α of S_n	12
$C^A(\pi)$	The conjugacy class in A_n of the permutation π	11
$C_A(\pi)$	The centralizer in A_n of the permutation π	11
$C^S(\pi)$	The conjugacy class in S_n of the permutation π	11
$C_S(\pi)$	The centralizer in S_n of the permutation π	11
char S_n	The ring of generalized characters of S_n	39
Cyc(H)	The cycle index of H	170
Cyc(H, D)	The generalized cycle index of H with respect to χ^D	171
Cyc($H \| p$)	Pólya insertion of p in H	170
d_α	The depth of α	99
$d^1_{\alpha,\beta}$	A p-modular decomposition number of α	243
$d_i^\alpha(r, s)$	The axial distance between r and s in t_i^α	123
diag G^*	The diagonal subgroup of G^*	134
det β	The determinant of a Gram matrix for S^β	313
D^β	$S^\beta/(S^\beta \cap S^{\beta\perp})$ for β p-regular	299
$D^1_{n,p}$	The p-modular decomposition matrix of S_n	244