SILARG VI

PROCEEDINGS OF THE 6TH
SIMPOSIO LATINO AMERICANO DE RELATIVIDAD Y GRAVITATION

Rio de Janeiro, Brasil
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PREFACE

The Simposio Latino-Americano de Relatividad y Gravitacion — SILARG — is a periodical scientific meeting organized and attended by Latin-American relativists, in which physicists from all countries of South and Central America join together to acknowledge and discuss general aspects of Relativity, Gravitation, Cosmology and related areas, and to communicate the results of recent research work undertaken on these subjects in our region. On occasion, several lectures from other countries have been invited to deliver short courses on selected topics of interest; this year, we have had the pleasure of receiving Professors Y. Choquet-Bruhat (Université de Paris) and M.A.H. MacCallum (Queen Mary College, London).

In this sixth version of the SILARG Proceedings, we present original research articles which have been communicated in the VI SILARG held in Rio de Janeiro, on July 13–18, 1987, and which are not intended to appear in this form elsewhere.

On behalf of the organizing committee of the VI SILARG we warmly thank the participation of our Latin-American friends and colleagues whose presence in Rio made possible the realization of such a successful reunion and especially Drs. J. Tiomno and M.D. Maia for their invaluable help in preparing this edition.

M. Novello

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We would like to thank the international (UNESCO and CLAF) and Brazilian (FINEP, CNPq, CAPES and CBPF) sponsors for their financial support which made this Meeting possible.

We thank the local organizing committee, Drs. Ligia M.C.S. Rodrigues, Isaias Costa and Marcelo Rebouças, for their dedication and enthusiasm in all phases of the Reunion.

M. Novello

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INVITED LECTURES

Global Problems in General Relativity.

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Université Paris VI

INTRODUCTION

The equations which govern at a classical level the fundamental physical interactions - except gravitation - are all semi linear partial differential equations on a given manifold, usually the Minkowski space-time M_4, eventually another manifold with a given globally hyperbolic metric, of dimension 4 or of higher dimensions in unified theories. The first fundamental problem, to insure the causality of the corresponding dynamics - a reasonable requirement at the classical level - is the well posedness of the Cauchy problem for these field equations at least for smooth initial data and some intervall of time.

The role of constraints, and of gauge freedom are now well understood and this local Cauchy problem is well known to have a satisfactory solution. Quite remarkably it has also been possible to prove the global existence on Minkowski space time of solutions of many of the fundamental field equations, either for essentially arbitrary data (Yang-Mills equations with Higgs source) or for small data and zero rest mass (Yang-Mills with Higgs and spinor sources). These results, and the asymptotic behaviour of the solutions which has been obtained in a number of cases ([5],[7]), are specially important in view of quantization.

The proof of these global existence theorems rests essentially on the geometric properties of the fields and of the equations they satisfy.

For the gravitational interaction - through its geometric interpretation is the most anciently discovered - the problem is much more difficult, because the gravitational field does not obey semi linear differential equations on a given manifold, but the Einstein equations which are quasi linear partial differential equations on a manifold which is not a priori given. In fact an einsteinian space time is a pair (V,g) with V a differentiable manifold and g a pseudo-riemannian metric of hyperbolic signature which satisfies Einstein's equations on V; by its very definition a solution is always global on its carrier manifold.

The "regularity" conditions on the metric are not a priori imposed, we just require that this metric satisfies Einstein's equations in some strong or weak sense. This means, if we consider for instance the vacuum case, that its Ricci tensor vanishes, as a function for smooth metrics, or at least as a distribution in other cases.

Examples 1) The gravitational shock waves ([17],[2]) are admissible Einsteinian space times.

2) The exterior Schwarzschild metric

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta \, d\varphi^2)$$

is a space time solution of Einstein equations on the manifold $\mathbb{R}^4$

minus $B \times \mathbb{R}$, B ball $r \leqslant 2m$ of $\mathbb{R}^3$.

A fundamental principle of general relativity is that its carrier space is a differentiable manifold, that is an equivalence class of atlases; the fact that it is possible to define V by one chart diffeomorphic to $\mathbb{R}^4$, and that g is defined there is not of fundamental physical relevance since coordinates have no physical meaning.

Example : The smooth metric g given on $\mathbb{R}^4$

$$dt^2 = \frac{dt^2}{(1+t^2)^2} - \sum_{i=1}^{3} (dx^i)^2$$

is a smooth Einsteinian space time on $\mathbb{R}^3 \times \mathbb{R}$, but it exists only for the finite amount of proper time

$$\int_{-\infty}^{\infty} \frac{dt}{1+t^2} = \pi \quad .$$

It is isometric to the slice of Minkowski space time given by the manifold $\mathbb{R}^3 \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with $dT^2 - \sum_{i=1}^{3} (dX^i)^2$, by the diffeomorphism

$$f : T = \mathrm{tg}\, t \quad , \quad x^i = X^i \quad .$$

More generally a space time $(\hat{V}, \hat{g})$ is called an extension of (V,g) if there exists an embedding F of V into $\hat{V}$ such that the pull back of $\hat{g}$ on V by F is equal to g :

$$g = F^* \hat{g} \quad .$$

We could say that a space time (V,g) solution of Einstein equations is a global solution if it is inextendible - but we would like to have information about the proper time of its existence.

Proper time, in relativity, is relative to an observer, i.e. to a time like world line : there are therefore several possibilities for the definition of "existence for an infinite proper time". A strong version is that all time like [respectively null] C^1 curves have infinite length [respectively affine parameter]. A weaker version is that these properties are satisfied by all time like and null geodesics : such space times are called time like and null geodesically complete.

The singularity theorems proved by Penrose and Hawking in the late sixties give a number of situations where einsteinian space times, with sources satisfying some positivity conditions, are incomplete.

These theorems rely essentially on Raychanduri equation - a consequence of the geodesic derivation formula - which imply that time like or null geodesics have a tendency to converge in such space times.

However the study of the global structure of an Einsteinian space time evolving from generic Cauchy data is still an essentially open problem. In particular it is not yet known if for initial data on $\mathbb{R}^3$ satisfying the constraints, and near from the Minkowskian ones (i.e. metric γ and symmetric 2-tensor k with e - γ and k small, e the euclidean metric), there is a complete einsteinian vacuum space time. This problem - the stability of Minkowski space time - is under active study by D. Christodoulou and S. Klainerman, with collaboration of S.T. Yau.

It seems that the answer will be positive. We recall in § 1 some recent results on global existence and non existence for some non linear equations, and a theorem which applies to the second variation of Einstein equations in harmonic coordinates.

For large - but physically reasonable - data one does not expect the space time to be complete, but one may hope that for the classical general relativity that we are studying, which governs phenomena at our scale, the causality will be preserved : we shall see in §4 recent results which have been obtained on the "cosmic censorship conjecture".

1. SOME RECENT RESULTS ON NON LINEAR WAVE EQUATIONS

For an appropriate choice of gauge the usual equations of classical field theories, Yang-Mills and Einstein equations take the form in local coordinates of a second order, quasi-diagonal quasi-linear system for a set u of N unknown functions u_I on $\mathbb{R}^d$

$$g^{\alpha\beta}(u,\partial u)\ \partial^2_{\alpha\beta}\ u_I + f_I(u,\partial u) = 0 \tag{1-1}$$

where ∂u denotes the set of partial derivatives of u, and these equations are such that they admit the solution u = 0 and their linearization at u = 0 reduces to the d'Alembert equation

$$\Box v = 0 \quad , \quad \Box = \eta^{\alpha\beta}\ \partial^2_{\alpha\beta} \quad , \ \eta_{\alpha\beta} \text{ the Minkowski metric ,} \tag{1-2}$$

that is they are such that

$$g^{\alpha\beta}(0,0) = \eta^{\alpha\beta} \tag{1-3}$$

$$f'_I(0,0) = 0 \tag{1-4}$$

where $f'_I(u,\partial u)$ denotes the derivative at the point u, ∂u of the function, on $\mathbb{R}^N \times \mathbb{R}^{dN}$ given $(u, \partial u) \to f_I(u, \partial u)$.

The Yang Mills equations in Lorentz gauge have these properties, as well as the Einstein equations in harmonic coordinates if, in this case, one sets $u_I = g_{\lambda\mu} - \eta_{\lambda\mu}$.

It has been proved by S. Klainerman [12], through the use of a Nash-Moser type regularization, and by D. Christodoulou by the conformal method used previously for Yang Mills fields ([8],[5]) that under the hypothesis (1-3) and (1-4) the Cauchy problem with data on $x^0 = 0$ (i.e. $\mathbb{R}^{d-1}$) in appropriate functional spaces and small in appropriate functional spaces and small in the corresponding norm, has a global solution - i.e. a solution on $\mathbb{R}^d$, if $d \geqslant 5$.

In the most common physical case d = 4 the same conclusion holds if one adds to the previous conditions the additional one

$$f''_I(0,0).(v,\ell w) = 0 \ , \ \forall\, I, \ \forall\, v, \ w \text{ and } \ell \text{ such that } \eta^{\alpha\beta}\ell_\alpha\ell_\beta = 0 \qquad (1\text{-}5)$$

The notation $f''_I(0,0)$ stands for the second derivative of f_I taken at u = 0 and $\partial u = 0$: it is a quadratic form in the set of scalars v and the set of covectors ∂v.

The condition (1-5) says that the quadratic form vanishes whenever v is arbitrary and ∂v is replaced by a set ℓw with ℓ a null covector and w arbitrary.

Example 1 : $f(u) = u^p$, p a positive integer, then

$$f'(u).(v) = p\, u^{p-1} v \qquad , \qquad f''(u).(v) = p(p-1)\, u^{p-2}\, v^2$$

$$f''(0) = 0 \quad \text{if} \quad p \geqslant 3 \quad .$$

Example 2 : (harmonic map equation)

$$f^I(u,\partial u) = \eta^{\alpha\beta}\, \Gamma_J{}^I{}_K(u)\ \partial_\alpha\, u^J\ \partial_\beta\, u^K$$

$$f''^I(0,0).(v,\partial v) \equiv \eta^{\alpha\beta}\, \Gamma_J{}^I{}_K(0)\ \partial_\alpha\, v^J\ \partial_\beta\, v^K$$

$$f''^I(0,0).(v,\ell w) \equiv \eta^{\alpha\beta}\, \Gamma_J{}^I{}_K(0)\ \ell_\alpha\, w^J\ \ell_\beta\, w^K = 0 \quad \text{if} \quad \eta^{\alpha\beta}\, \ell_\alpha\, \ell_\beta = 0$$

The condition (1-5) is called the null condition. It is not satisfied by Einstein equations in harmonic coordinates - nor by Yang-Mills equations in lorentz gauge.

The theorems of Klainerman-Christodoulou shows the stability of Minkowski space times of dimensions $d > 4$.

Note that these theorems hold for initial data in functional spaces which imply fall off at space infinity : they cannot be used to prove through a method of descent global existence for Kaluza-Klein theories in dimension 4.

In the case of equations with a mass term,

$$g^{\alpha\beta}(u,\partial u)\ \partial^2_{\alpha\beta}u + mu + f(u,\partial u) = 0 \quad , \quad m > 0$$

a global existence result in dimension 4, under the conditions (1-3), (1-4) has been given by S. Klainerman [13] : he uses estimates derived also from the other invariances of the minkowskian wave operator $\Box$, not only the usual energy estimates linked with its time invariance.

On the other hand it is known that the equation (1-1) with $m = 0$ in dimension 4 does not always have global solutions if the null condition is not satisfied. It has been proved by F. John [11] that the equation

$$\Box\, u = u^2$$

has no global solution for any C^∞ initial data with compact support. This result indicates that a global existence theorem for Einstein equations in dimension 4 will not be obtained just by looking at these equations in harmonic coordinates.

2. GLOBAL INSTABILITY FOR EINSTEIN EQUATIONS IN HARMONIC COORDINATES

In harmonic coordinates Einstein equations in vacuum are of the form

$$R_I^{(h)}(u) \equiv \frac{1}{2} g^{\alpha\beta}(u)\, \partial^2_{\alpha\beta}\, u_I + f_I(u,\partial u) = 0 \quad ,$$
$$(u_I) = (g_{\lambda\mu} - \eta_{\lambda\mu}) \;,\; I = 1,\ldots, 10 \quad . \tag{2-1}$$

$$f_I(u,\partial u) = - g^{\alpha\beta}\, g^{\rho\sigma}\, \partial_\beta\, g_{\mu\sigma}\, \partial_\alpha\, g_{\nu\sigma} + g^{\alpha\rho}\, g^{\beta\sigma}\, [\mu,\rho\sigma][\nu,\alpha\beta] \quad .$$

We consider these equations on $\mathbb{R}^3 \times \mathbb{R}$. One shows that there cannot exist a neighborhood Ω of 0 in the space $\left(x\ C_o^\infty\ (\mathbb{R}^3)\right)^{20}$ such that for each initial data set $u_I(x^i,0)$, $\partial_o\, u_I(x^i,0)$ in Ω the equations (2-1) have a global solution which decays uniformly like $1/r$ at space infinity. Indeed if there was such a neighborhood then the linearized equations at $u = 0$ would have a solution on $\mathbb{R}^4$ with this same property, and also the second variation of these equations. The first variation equations at u = 0 reduce to the Minkowski wave operator

$$\frac{1}{2} \square\, h_I = 0 \qquad (\text{we have } f'_I(0,0) = 0) \tag{2-2}$$

where h can be thought of as the tangent $u'(0)$ to a 1 parameter family of solutions $u(\lambda)$ such that $u(0) = 0$. The second derivative $u''(0)$, denoted k, satisfies the second variation equations which reduce to