FOUNDATIONS OF ALGEBRA AND ANALYSIS

An Elementary Approach

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AND
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PREFACE

For the most part the topics treated in this book may be found in any of the multitude of books that have appeared in recent years treating such subjects as college algebra and trigonometry, introductory college mathematics, pre-calculus mathematics, etc. For more than ten years we have been involved in teaching and developing those freshman mathematics courses for which these books were written. During that time we have used and perused a number of these books. We realize that no book will please any teacher in all respects—every book has its weaknesses as well as its strengths. However, we have felt for some time that the vast majority of these books lack a basic ingredient—*mathematical integrity*.

When we say a book lacks integrity, we mean that it is a collection of more or less unrelated topics, treated in a cursory fashion, with no unifying principles, and in which the "rules of the game" have not been clearly stated. An excessive number of undefined terms are introduced, frequently without being so labeled. Definitions, if given at all, are often inadequate. Proofs of statements about real numbers are frequently made to rely on concepts and axioms from geometry which have nothing to do with real numbers. The result is that the student does not know what he can and cannot assume in carrying out proofs and solving problems.

Our purpose then in writing this book is to develop a body of mathematical ideas in a logical, coherent fashion with a minimum of undefined terms; to stress the nature of proof—its relationship to undefined terms, axioms, and definitions; to present material which is essential, not only to a proper treatment of calculus, but also to a sound development of advanced algebra and analysis.

Occasionally when developing a particular topic in this manner, a basic theorem is required whose proof is somewhat lengthy and difficult. However, subsequent theorems follow with considerable ease. In some instances it may be advisable to state the basic theorem without proof and to proceed with the rest of the topic. The proof can then be discussed at a later time.

In many instances proofs of theorems have been relegated to the problem sets, and occasionally ideas essential to subsequent developments have been introduced in the exercises. As an aid to the instructor, problems which may be subsequently referred to, or which are especially important, are marked with a †. Difficult problems are marked with an *

Frequently we have found it necessary to introduce examples and problems involving concepts not yet developed in the text (for example, real numbers and mathematical induction in Chapter 1). This is necessary in order to provide motivation and to illustrate the concepts under discussion. However, such examples are not part of the logical development, which could well proceed without them. These illustrative examples are quite simple and are taken from mathematics which most students have studied in high school.

Definitions of several fundamental concepts (such as *function*) appear in more than one chapter. This permits a greater degree of flexibility in the use of the book in that certain chapters may then be omitted with a minimum of "back-tracking." In this regard we list below several suggestions for selecting chapters to suit various one-semester courses. The entire book may be used for a year course of 8–10 units.

1. College Algebra and Trigonometry (4–5 units)
   Chapters 2, 4, 5, 6, 8, 9, 11 (and the first part of Chapter 1).
2. Number Systems (for teachers' institutes) (3–4 units)
   Chapters 1, 2, 3, 4, 5.
3. Cultural Mathematics (Liberal Arts students) (3–4 units)
   Chapters 1, 2, 6, 10.
4. Foundations of Mathematics (elementary) (3 units)
   Chapters 1, 2, 10.

The Instructor's Manual contains suggestions for abridging the various chapters and also provides supplementary solution sets.

A.R.L.
G.C.P.

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J. V. Uspensky, *Theory of Equations* (New York: McGraw-Hill Book Company, 1948). Problems 2(b,c,d,e), 3, 7(g,h), page 173; problem 9(b,c), page 181; problem 8(b,c), page 184; problem 8(b,c,d), page 417; problem 11(a), page 436; problems 1(e), 2(a,b,c,d), page 441; problems 2(e), 3(a,b,c,d,e), 4(a,b,c), 5(a), page 442; problem 5(b), page 443; problem 4(a,b,c,d), page 462.

Ross A. Beaumont and Richard W. Ball, *Introduction to Modern Algebra* (New York: Holt, Rinehart and Winston, Inc., 1954). Problems 8(c), 9(b), page 436; Exercise 2, page 440.

Norman Miller and R. E. K. Rourke, *Plane Trigonometry and Statics* (Toronto: The Macmillan Company of Canada, Ltd., 1943). Problems 19, 22, 24, 25, page 332; problems 8, 12, page 345; problem 25, page 346; problems 11, 12, page 359.

CONTENTS

I

LOGIC AND SETS

1 INTRODUCTION

In the study of logic we are concerned with *propositions* and *propositional functions*. The latter will be developed after the concept of "function" has been introduced.

A proposition is a statement to which can be assigned one and only one of the values, "true," or "false," which are considered as *undefined* or *primitive* terms. In fact the word "statement" shall remain undefined. We assume that the terms "true" and "false" have meaning to us, and we require of a proposition only that it have one value or the other, *but not both.* A statement not having this property is not a proposition and will not be considered in the development of our logical system. The following are examples of propositions:

(1) $2 + 3 = 7$.
(2) There are a trillion stars in the universe.
(3) There are 52 states in the United States.

Note that a statement may be a proposition even though there may exist no immediate or practical method for determining whether it is true or false (in our system).

On the other hand, consider the following "statement."

The statement in this box is false.

If the statement is "true," then it is "false" and vice versa. Hence this statement is *not* a proposition. We shall regard such statements (collections of words) as meaningless.

N.B. In the development of the theory of logic and sets we shall make generous use of what is usually called "High-School Mathematics." This will be done primarily in the examples and exercises for the purposes of illustrating and clarifying the concepts. In this regard we shall use the terms "true," "false" in their usual sense. For example, $2 + 3 = 5$ is a true proposition, while $5 > 11$ (5 is greater than 11) is a false proposition. In subsequent chapters certain of the topics comprising High-School Mathematics will be developed from basic principles.

2 COMPOUND PROPOSITIONS: CONNECTIVES

Propositions may be combined in various ways to form other propositions. A mechanism for doing this is called a *connective*. When connectives are used to combine propositions, the result is called a *compound proposition*. The five basic connectives used in logic—conjunction, disjunction, negation, implication, and equivalence—are treated in §§3–7. We shall denote arbitrary propositions by $p, q, r, \ldots$.

3 CONJUNCTION: AND

We may join two propositions p, q with the word "and" and write "p and q," or in symbols "$p \wedge q$." The statement "$p \wedge q$" then becomes a proposition provided we specify how the *truth values*, "true," "false," are to be assigned. Letting "T" stand for "true" and "F" for "false," we can accomplish this by constructing a *truth table* for "$p \wedge q$." The symbols $p, q, p \wedge q$ are placed at the head of a column of the table (Figure 1–1),

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Figure 1–1

with "$p \wedge q$" heading the last column. We next fill in the columns headed by p, q with all possible combinations of T and F. (For two symbols there are 2^2 combinations.) We then *assign values* to the boxes in the column headed by "$p \wedge q$." The manner in which we do this is arbitrary, but our choice is guided by the kind of mathematical system we wish to construct. The table in Figure 1–1 is the one usually given for "$p \wedge q$."

Observe that "$p \wedge q$" is false in every case except that in which both propositions have value T. "$p \wedge q$" is called the *conjunction of p with q*

(in that order). The connectives "or," "not," "implies," "is equivalent to," will now be taken up in order.

4 DISJUNCTION: OR

"p or q" in symbols "$p \vee q$" is defined by the table in Figure 1–2. Note that a disjunction is true whenever *either* or *both* of the propositions are true. It is false only when both propositions are false.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Figure 1–2

5 NEGATION: NOT

"Not p," in symbols, "$\sim p$" is defined by the table in Figure 1–3. Note that negation merely reverses the truth value of the proposition.

p	$\sim p$
T	F
F	T

Figure 1–3

6 IMPLICATION: IMPLIES

"p implies q," in symbols, "$p \Rightarrow q$," is defined by the table in Figure 1–4. The proposition preceding the arrow is called the *antecedent* or the *hypothesis* of the implication, while the proposition following the arrow is called the *consequent* or the *conclusion*. *An implication is true in every case*

p	q	$p \Longrightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Figure 1–4

except the one with true antecedent and false consequent. It is important to note that a knowledge of the truth value of an implication gives no information about the truth value of the antecedent and the consequent separately. Thus, by asserting the truth of an implication, "$p \Rightarrow q$," one merely asserts that it is *not the case that p is true and q is false.*

Examples.

(1) $2 > 5 \Rightarrow 4 = 2$ (F $\Rightarrow$ F, true)
(2) $2 > 5 \Rightarrow 2 + 3 = 5$ (F $\Rightarrow$ T, true)
(3) $6 > 3 \Rightarrow 2 + 3 = 5$ (T $\Rightarrow$ T, true)
(4) $6 > 3 \Rightarrow 4 = 2$ (T $\Rightarrow$ F, false)
(5) $[(2 > 3) \wedge (3 > 4)] \Rightarrow 2 > 4$ (F $\Rightarrow$ F, true)

Other terminologies for "$p \Rightarrow q$" are:

"If p, then q;"
"p is sufficient for q;"
"q is necessary for p;"
"q, if p;"
"p only if q."

7 EQUIVALENCE: IS EQUIVALENT TO

"p is equivalent to q," in symbols "$p \Leftrightarrow q$" is defined by the table in Figure 1–5. Thus we see that an equivalence is true when and only when the two propositions have the *same* truth value. Whenever $p \Leftrightarrow q$ is true, we say that p, q are (logically) equivalent propositions. *Thus any two false propositions, or any two true propositions are equivalent.*

Other terminologies for $p \Leftrightarrow q$ are:

"p if and only if q" (abbreviated "p iff q");
"p is necessary and sufficient for q" (abbreviated "p is n.a.s. for q").

p	q	$p \Longleftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Figure 1–5

N.B. In the sequel, whenever a proposition such as an implication "$p \Rightarrow q$" or an equivalence "$p \Leftrightarrow q$" is asserted without qualifying remarks, or the student is asked in an exercise to prove that "$p \Rightarrow q$" or that "$p \Leftrightarrow q$," the student should always interpret this to mean that the implication or the equivalence is *true*, or should be proved true (see problems 5–10). Thus the statement "x is a square implies x is a rectangle" should be interpreted "It is true that x is a square implies x is a rectangle."

Example 1. Let p, q, r have truth values T, F, T, respectively. Determine the truth values of the following:

(1) $p \wedge \sim q$
(2) $(p \vee q) \Rightarrow r$
(3) $(p \wedge \sim r) \Rightarrow \sim q$
(4) $[\sim(p \wedge q)] \Leftrightarrow \sim(p \vee \sim r)$

Solution.

(1) $p \wedge \sim q$
$\mathrm{T} \wedge \sim\mathrm{F}$
$\mathrm{T} \wedge \mathrm{T}$, true
$\therefore$ $p \wedge \sim q$ is true

(2) $(p \vee q) \Rightarrow r$
$(\mathrm{T} \vee \mathrm{F}) \Rightarrow \mathrm{T}$
$\mathrm{T} \Rightarrow \mathrm{T}$, true
$\therefore$ $(p \vee q) \Rightarrow r$ is true

(3) $(p \wedge \sim r) \Rightarrow \sim q$
$(\mathrm{T} \wedge \sim\mathrm{T}) \Rightarrow \sim\mathrm{F}$
$(\mathrm{T} \wedge \mathrm{F}) \Rightarrow \mathrm{T}$
$\mathrm{F} \Rightarrow \mathrm{T}$, true
$\therefore$ $(p \wedge \sim r) \Rightarrow \sim q$ is true

(4) $[\sim(p \wedge q)] \Leftrightarrow \sim(p \vee \sim r)$

$[\sim(T \wedge F)] \Leftrightarrow \sim(T \vee \sim T)$

$\sim F \Leftrightarrow \sim(T \vee F)$

$T \Leftrightarrow \sim T$

$T \Leftrightarrow F$, false

$\therefore$ $[\sim(p \wedge q)] \Leftrightarrow \sim(p \vee \sim r)$ is false

Example 2. Show that for all propositions p, q the following proposition is true:

$$(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$$

Solution. We construct a truth table (Figure 1–6) whose last column is headed by the given proposition.

p	q	$p \Rightarrow q$	$\sim q$	$\sim p$	$\sim q \Rightarrow \sim p$	$(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Figure 1–6

Since the last column consists entirely of "true" values, the given proposition is true for all propositions, p, q.

▶ EXERCISE 1

1. Let p, q, r have truth values F, F, T, respectively. Determine the truth value for each of the following:

   (a) $(\sim\sim p) \Leftrightarrow q$
   (b) $(p \wedge \sim r) \Rightarrow \sim q$
   (c) $p \vee (q \wedge r)$
   (d) $(p \vee q) \wedge (p \vee r)$
   (e) $p \Rightarrow (q \vee r)$
   (f) $(p \wedge q) \Rightarrow \sim(q \vee r)$
   (g) $[\sim(p \vee q)] \Leftrightarrow [(\sim p) \wedge (\sim q)]$
   (h) $[p \Rightarrow (q \vee r)] \Leftrightarrow [(p \wedge \sim q) \Rightarrow r]$

2. Given p: Grass is green (true)
   q: The sky is blue (true)
   r: $2 + 3 = 7$ (false)

   Express each of the following in symbolic form using p, q, r and connectives, and determine the truth value of each:

(a) Grass is green, and $2 + 3 \neq 7$.
(b) If the sky is not blue, then grass is not green.
(c) Grass is green is necessary for $2 + 3 = 7$.
(d) Grass is green or the sky is blue, is necessary and sufficient for $2 + 3 \neq 7$.
(e) $2 + 3 = 7$ is a sufficient condition for the grass to be green.

3. Given p: The earth is round (true)
q: $7 > 10$ (false)
r: Men are beasts (false)

Express the following compound propositions in good English and determine the truth value of each.

(a) $p \Rightarrow (q \vee r)$
(b) $(p \wedge \sim r) \Leftrightarrow (\sim q)$
(c) $[\sim(p \wedge q)] \Rightarrow (r \vee \sim p)$
(d) $\sim(p \vee \sim r)$
(e) $[\sim(p \wedge r)] \Leftrightarrow [(\sim p) \vee (\sim r)]$

4. Let p, q, r be unspecified propositions: Show that whatever the truth values of p, q, r may be, the following compound propositions are true.

(a) $(\sim\sim p) \Leftrightarrow p$
(b) $[p \vee (q \vee r)] \Leftrightarrow [(p \vee q) \vee r]$
(c) $(p \Rightarrow q) \Leftrightarrow (q \vee \sim p)$
(d) $(p \Rightarrow q) \Leftrightarrow \sim(p \wedge \sim q)$
(e) $[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$
(f) $[\sim(p \wedge q)] \Leftrightarrow [(\sim p) \vee (\sim q)]$

5. Let p be a true proposition. Let q be an arbitrary proposition. Prove $q \Rightarrow p$.

6. Given: $(\sim p) \Rightarrow q$ for every q.
Prove: p is true.

7. Let p, q be arbitrary propositions.
Prove: (a) $(p \wedge q) \Rightarrow p$
(b) $p \Rightarrow (p \vee q)$

8. Let p be false. Prove $p \Rightarrow q$ for every q.

9. Given: $p \Rightarrow q$ and $r \Rightarrow s$
Prove: (a) $(p \wedge r) \Rightarrow (q \wedge s)$
(b) $(p \vee r) \Rightarrow (q \vee s)$
(*Hint:* Consider the possible combinations of truth values for p, q, r, s.)

10. Given: $p \Rightarrow q$
Prove: (a) $(p \wedge q) \Leftrightarrow p$
(b) $(p \vee q) \Leftrightarrow q$