Mathematics for Computer Graphics

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MATHEMATICS FOR COMPUTER GRAPHICS

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Preface

The purpose of this book is to further the applying of mathematics to computer graphics. It is written from the conviction that not only is mathematics beautiful in itself, but that applying mathematics is a beautiful thing to do.

We include sections giving the reader what can serve as a reminder of, or as a compact introduction to vectors, matrices, groups, and complex numbers, to the level required for the main exposition. A high priority is given to visual illustration and examples. There are exercises which the reader can use if desired, at strategic points in the text, gathered together and sometimes extended at the end of each chapter. When an exercise number is marked thus √, there is an answer or hint in *selected answers* just before the index. In the introduction we suggest some easy ways in, where to find purely pictures, light reading, or material requiring diligent commitment, depending on what rewards one seeks at the time.

These days, the mathematics of computer graphics could surely not be compassed in a single book. We do not for example build up from scratch the apparatus of elementary calculus and coordinate geometry. However we do offer a study of transformations and symmetry in the plane, with applications to producing patterns by computer (Chapters 1-6). We classify length-preserving transformations in 3-space, with techniques from vectors, matrices and geometry, and show how to actually *do* a considerable variety of things, culminating in rotation by quaternions and in-betweening solid motions in 3-space (Chapters 7-9). The topology part of the book (Chapters 10-13) contains a variety of applications, interesting and important in their own right, such as Newton's method of solution, extrema of continuous maps, and the Principle of Linear Programming. At the same time it gives the base for properties and constructions of fractal images via iterated function systems and Mandelbrot and Julia sets (Chapters 14-16).

Much of the material of this book began as a graduate course in the Summer of 1988, for PhD students in computer graphics at the Ohio State University. My thanks are due to Rick Parent for encouraging the idea of such a course, to Phil Huneke for enabling me to give it as a visiting faculty member in the mathematics department, and to Charles Csuri for encouraging the idea of the book. I thank Eiichi Bannai for his original invitation to come to Ohio for our joint work (on t-designs). I am indebted to Robin McLeod and Tektronix for an Academic Scholarship award and for an instructive week of computer graphics at Tektronix Beaverton, Oregon.

A further part of the book was developed from a course for Final Year mathematics students at the University of Glasgow. I thank my department for three months leave to begin the book. I thank the Glasgow Mathematical Association, Maclaurin Society, and St. Andrews University mathematics department, for opportunities to air some ideas to an audience.

I am much indebted to John Patterson for checking the draft text with care and detail beyond the call of duty, and for many helpful comments. My thanks too for comments on parts of the text, to Ian Anderson, John Jeacocke, Alistair Kilgour, James Logie, Adam McBride, Finlay Mc Naughtan, Ian Murphy, and Edmund Robertson. In a different vein, I thank Keiran Clenaghan for running Computing Science student projects based on Chapters 1-6, and the many enthusiastic school students who spur my faith in the seventeen plane patterns each Open Day. Also Jim McNally for finance and publicity via our Enterprise in Higher Education department. On the last two counts thanks are due to Scotsys Computers for their kind loans of equipment. Also on the British scene, I thank Rae Earnshaw for

organising excellent international conferences that helped acquaint me with the people and ideas in computer graphics.

The text was prepared as camera-ready copy by the author in Microsoft Word on an Apple Macintosh IIci and printed on an Apple Laserwriter. Typesetting of mathematical formula was done mainly with Formulator, and diagrams drawn with Superpaint. Further software used, in addition to Pascal programs written by the author, was: The Game of fractal images (H.-O.Peitgen, H.Jürgens, D.Saupe, M.Parmet), The Desktop fractal design system (M.F. Barnsley), and Explorer (J.Dirksen, β-test copy). In this connection, my thanks to Adrian Bowman, who lent me his IIcx when the latter two programs failed on the IIci, and as a source of images for the colour plates (care of the Glasgow University photographic unit).

Final thanks to Keith van Rijsbergen, head of computing science at Glasgow, who created the series in which this book appears, and to Cambridge University Press for their resolute patience until the book was finished.

Stuart G. Hoggar
Glasgow, January 1992

Introduction

> The person in the being mode will come to the lecture with an idea, a question, in mind. He will not attempt to write down everything he hears, but will afterwards emerge knowing more than the person who did. (Freely quoted from Erich Fromm *To have or to be*.)

It is expected that, rather than work through the whole book, readers will wish to browse or to look up particular topics. To this end we give a fairly extended introduction, list of symbols, and index. Each chapter begins with a table of its contents. The book is in four interconnected parts (the connections are outlined at the end of the Introduction):

I	*The plane*	Chapters 1-6.
II	*3-space*	Chapters 7-9.
III	*Topology*	Chapters 10-13.
IV	*Fractals*	Chapters 14-16.

In each case the easiest chapter is the first cited, but it would be a pity to stop there. Indeed the results of Chapters 1-2 are foundational for all four parts, sometimes leading to very pleasant shortcuts of an argument or calculation. On the other hand Chapter 10, whilst essentially avoiding topology, may be used as a pointer to the rest of the book. One aid to taking in information is first to go through following a substructure and let the rest take care of itself (a surprising amount of the rest gets tacked on). To facilitate this, each description of a part is followed by a quick trip through that part, which the reader may care to follow.

An easy way in. If it is true that one picture is worth a thousand words then an easy but fruitful way into this book is to browse through selected pictures. Here is one suggestion. Start with the plane pattern examples of Chapter 5: Section 5.7 and Exercises 5. For some more, see Examples 4.20 and Exercises 4. An early illustration of a Julia set is Figure 11.14. More illustrations of fractal sets in black and white (not to be despised) are found in (most of) Figures 10.6 to 10.18 (curves), Figures 14.10 to 14.25, Figure 15.9, and Figures 16.18 to 16.26. Finally, move to the colour plates.

Chapters 1-6 (Part I.) The mathematics is geared towards producing patterns automatically by computer, allocating some design decisions to a user. We begin with isometries - those transformations of the plane which preserve distance and hence shape, but which may switch left handed objects into right handed ones (such isometries are called indirect). In this part of the book we work geometrically, without recourse to matrices. In Chapter 1 we show that isometries fall into two classes: the direct ones are rotations or translation, and the indirect ones reflections or glides. In Chapter 2 we derive the rules for combining isometries and introduce groups and the Dihedral group in particular. In a short Chapter 3 we apply the theory so far to classifying all 1-dimensional or 'braid' patterns into seven types (Table 3.1).

From Chapter 4 especially we consider symmetries or 'symmetry operations' on a plane pattern. That is, those isometries which send a pattern onto itself, each part going to another with the same size and shape (see Figure 1.3 and ff.). A plane pattern is one having translation symmetries in two non-parallel directions. Thus examples are wallpaper patterns, floor tilings, carpets, patterned textiles, and the Escher interlocking pattern of the frontispiece. We prove the Crystallographic restriction, that rotational symmetries of a plane pattern must be multiples of a 1/2, 1/3, 1/4, or 1/6 turn (1/5 is not allowed). We show that plane patterns, are made up of parallelogram shaped cells, falling into five types (Figure 4.9). The chapter concludes with examples.

In Chapter 5, guided by the conclusions of Chapter 4, we deduce the existence of seventeen pattern types, each with its own set of interacting symmetry operations. In Section 5.8 we include a flow chart for deciding into which type any given pattern fits, plus a fund of test examples. In Chapter 6 we draw some threads together by proving that the seventeen proposed categories really are distinct according to a rigorous definition of 'equivalent' patterns (Section 6.1), and that every pattern must fall into one of the categories provided it is 'discrete' (there is a *lower* limit on how far any of its symmetries can move the pattern). By this stage we increasingly use the idea that, because the composition of two symmetries is a third, the set of all symmetries of a pattern form a group (the definition is recalled in Section 2.5). In Section 6.3 we consider various kinds of regularity upon which a pattern may be based, via techniques of Coxeter graphs and Wythoff's construction (they apply in higher dimensions to give polyhedra). Finally in Section 6.4 we concentrate the theory towards building an algorithm to construct (e.g. by computer) a pattern of any type from a modest user input, based on a smallest replicating unit called a fundamental region (we also offer software on a disc, see the end of the chapter).

Chapters 1-6: a quick trip. Read the introduction to Chapter 1 then note Theorem 1.18 on what isometries of the plane turn out to be. Note from Theorem 2.1 how they can all be expressed in terms of reflections, and the application of this in Example 2.6 to composing rotations about distinct points. Look through Table 2.2 for anything that surprises you. Theorem 2.12 is vital information and this will become apparent later. Do the exercise above Figure 2.19. Omit Chapter 3 for now. Read the first four pages of Chapter 4 then pause for the Crystallographic restriction (Theorem 4.15). Proceed to Figure 4.9, Genesis of the five net types, note Examples 4.20, and try Exercise 6 at the end of the chapter yourself. Get the main message of Chapter 5 by using the scheme of Section 5.8 to identify pattern types in Exercises 5 at the end of the chapter (examples with answers are given in Section 5.7). Finish in Chapter 6 by looking through Section 6.4 on 'Creating plane patterns' and recreate the one in question 13 of Exercises 6 (end of the chapter) by finding one fundamental region.

Chapters 7-9 (Part II.) These chapters build from two to three-dimensional geometry and, by contrast with earlier chapters, are matrix oriented. In Chapter 7, after recapitulating the basics of 3-d vectors and coordinates we explain left handed versus right handed triples of vectors and their use in coordinate systems. The scalar product of vectors is introduced, with its relation to geometry. Now we consider matrices, determinants, and some applications such as: calculating areas and volumes, determining whether a triple is left or right handed, calculating various types of vector products. Finally we show how to determine the matrix of any plane isometry, using a 3 by 3 matrix if translation is involved.

Chapter 8 is about isometries in 3-space (now translation is included in a 4 by 4 matrix). We begin by proving that isometries can necessarily be represented in matrix form, determine the effect of a change of coordinate axes, and characterise isometries by their determinant as always preserving or always reversing right handedness. We determine the matrices for general reflection, rotation, translation, glides, rotary reflections and screw isometries, showing that these exhaust the possibilities for 3-d isometries. We introduce a number of techniques for going back and forth between the geometry and matrix of an isometry. For example determining from a rotation matrix the direction and position of the axis, the angle and its sense. Thus we easily compute the composition of any two of the six types of isometry. Example 8.29 is an application in Molecular graphics. The usefulness of eigenvalues and eigenvectors for some of these calculations appears. Along the way we derive some formulae and methods of practical value which deserve to be better known, such as (8.11), Example 8.40, Theorems 8.42, 8.49 and Corollary 8.52. We conclude with a list of where twenty such 'how to' solutions are found in the chapter (a possible quick way in).

Chapter 9 aims to show some benefits of using quaternions as a technique for calculating with rotations, which comes into its own especially when, in setting up animation of solid objects, we wish to move smoothly between a whole sequence of translated and rotated images (key frames) of the object. We cover first the basics of complex numbers a+b**i**, where a,b are real and **i** is a symbol whose square is -1. Indeed the complex numbers are the smallest set of 'numbers' to include both the real numbers (which they do in the form a+0**i**) and a square root of -1. A remarkable consequence of this inclusion is that not only does every quadratic equation now have a solution, but so does every polynomial equation, of whatever degree. This is the famous 'Fundamental Theorem of Algebra' (proved in Chapter 13 but first used, for eigenvalues, in Chapter 8). However the fact that generalises into 3-d rotations by quaternions is this: a complex number may be viewed as a point in the plane with polar coordinates r,θ. Then, to multiply two numbers, we multiply the values of r, but *add* the angles (see Figure 9.5). We introduce the special arithmetic rules (9.15) for quaternions a+b**i**+c**j**+d**k**, where **j**,**k** are further square roots of -1. On reaching Example 9.36 we are ready to calculate compositions of 3-d rotations by using quaternions. Some nice test cases are provided by composing symmetry operations of Platonic solids (tetrahedron, square, icosahedron..). Finally in Section 9.4 we show how in-betweening over a series of key frames may be done by constructing a Bézier-type curve through corresponding points (unit quaternions) on the three-dimensional sphere in real 4-space.

Chapters 7-9: a quick trip. Do Exercise 7.11 at the end of Chapter 7, checking out the background of vectors, matrices and determinants in (7.4), Definitions 7.3 and 7.29, Theorem 7.31 (c),(d), Theorem 7.32, Rules 7.20 and preceding definition. For Chapter 8, pick out three things that look new or interesting from the how to do list on page 177. If inclined, follow an example and do an exercise on each. In Chapter 9, if new to complex numbers, first read pages 180 to 186 and note Figure 9.7. For quaternions and rotations look through Section 9.2.1. Use Theorem 9.30 to do Exercise 25 on page 216, using the method of Example 9.38.

Chapters 10-13 (Part III.) Chapter 10 is a fairly easy intoduction to ideas which reappear throughout the rest of the book, with plenty of diagrams and pictures. We show first how some phenomena of coastlines and land frontiers can now be 'explained' in fractal terms. It proves very instructive to introduce Mandelbrot's initiator-generator construction for curves, from snowflake, to the Sierpinski gasket as curve, to the plane-filling type. Its reformulation in terms of plane transformations (going beyond isometries but using them) points to powerful techniques for fractals, which appear first in Chapter 13.

With Chapter 11 we come for the first time to topology. The basics of metric spaces are covered, where 'metric' signifies that we allow a variety of concepts of distance. This is emphatically not the esoteric for its own sake, for from Chapter 13 on we reap the benefit of laws of distance applying to a concept of distance between visual images. The idea of distance leads to open sets, closed sets, and thence to rigorous and usable definitions of interior and boundary. Then to a definition of continuous function which dispenses with ε's and δ's and makes some otherwise hard-to-prove results much easier.

Chapter 12 plays the role of topology, part 2. We define compact sets and prove their important equivalence in n-space to closed and bounded sets. Some famous results follow on the benefits of compactness. Much flows from: the image of a compact set under a continuous function is compact. For example (i) every linear function from n- to m-space has a bound on the factor $|f(\mathbf{x})|/|\mathbf{x}|$ by which it scales the length of a vector $\mathbf{x}$ (important for iterated function systems in Chapter 14), and (ii) a real linear function on a compact subset of n-space attains its maximum and minimum on the boundary, from which follows the Principle of Linear Programming, (12.9). The other key idea in this chapter is connectedness, agreeing with an intuitive idea in obvious cases. Similarly to the compact case, the image of a connected set under a continuous function is connected. This has easy applications to a proof of the Intermediate Value Theorem and working with 'sides' of a (hyper)plane. Many results here (as in Chapter 11) are the basis of later study of Mandelbrot, Julia, and other fractal sets.

Chapter 13 introduces the Hausdorff distance between two pictures interpreted as subsets of the plane. We prove that the distance laws of Chapter 11 for a metric space are satisfied. Now we define the 'collage map', sending a plane set E to a new set ψE determined by N plane transformations. We prove the far-reaching Contraction Mapping Theorem, 13.14 which shows via Theorem 13.28 when the successive image pictures $\psi^k E$ approach a unique one (this is followed up in Chapter 14). Hausdorff distance also plays an important role in Chapter 15. A second and interesting application of the Contraction Mapping Theorem is to the iterative solution of polynomial equations, showing why Newton's method is better than most. We also use it to show the existence and uniqueness of a class of differential equations. In the final Section, 13.4, we tie some loose ends such as the approximation of a continuous function by a sequence of step functions (see the 'measures' part of Chapter 15) and a proof of the Fundamental Theorem of Algebra by topology and winding numbers.

Chapters 10-13 : A quick trip. Use (10.7) to calculate the dimension of the Sierpinski gasket in Figure 10.7. Examine Figure 10.14 and read page 232. In Chapter 11 note Definitions 11.1, 11.2 and 11.34 then follow Examples 11.41, using the index as necessary. In Chapter 12, take the meaning of *compact* from Theorem 12.26, note Theorem 12.28, and follow the proof of Application 12.29. Do Exercise 12.8 on page 301 similarly to Example 12.30 (but don't forget the hint). For Chapter 13, start by reading the first two pages. Note the (fact of) applications of Theorem 13.14 to solving equations in Section 13.2.2. Note the existence and uniqueness of fractals (Theorem 13.28) following from the topology Theorem 13.27 (again, use the index as necessary), and its illustration in Figures 13.8. You may like to see how winding numbers are applied in Section 13.4.2. For the idea of distance between two pictures, try Exercise 13.7 on page 327.

Chapters 14-16 (Part IV.) This last part of the book is mostly about fractals. Whilst it contains material of a technical nature (after all, we are dealing with foundations), there is much that is relatively easy to pick up. Chapter 14 covers the iterated function systems (IFS) highlighted and developed especially by Barnsley (1988) and co-workers. We define an IFS, based on N contractive maps w_i of the plane that are affine (linear plus translation), and its attractor $\mathcal{A}$, a subset of the plane to be interpreted and viewed as an 'image', or picture. In a framework from Chapter 13, $\mathcal{A}$ is the limit of a sequence of sets A_n, where A_{n+1} is the collage $w_1(A_n) \cup w_2(A_n) \cup ... \cup w_N(A_n)$. After some classic attractors, such as Barnsley's fern, we study affine maps and how to work with them in the IFS and computer screen context. Now we take the reader through some typical attractors with small N (tree, anvil, 'Moscow by night') and give practice in recognising the transformations that produce such results. We consider the possibility of finding the component transformations of an IFS to produce more general images and show the value of the Contraction Mapping Theorem and its accompanying bound (often translated as the 'Collage Theorem'). We demonstrate the process for a face requiring twelve transformations.

Chapter 15 is about the Random Iteration Algorithm (RIA) of Barnsley and Demko (1985). It produces the attractor of an IFS not by a series of approximations $\psi^k E$ but as a sequence of points which build up to it. Each successive point is obtained by applying a random choice of transformation w_i from those of the IFS (Definition 15.23). We analyse the algorithm's effectiveness using an addressing scheme for attractors which is a generalisation of the binary, ternary or decimal expansions of numbers on the real line. We investigate whether the RIA might work more efficiently with a less 'random' number generator for choosing the w_i. Hutchinson's idea (1981), that measure theory ought to help in working with attractors, comes to fruition very naturally via the RIA (Barnsley and Demko). Not only shades of grey but of colour are 'measured out', even down to the pixel level. By Section 15.4, having covered some of the technicalities, we are well placed to introduce Hausdorff dimension, based as it is on measures. (We take a fractal set as one whose Hausdorff dimension differs from its topological dimension.) We include the result that a self-similar set has Hausdorff dimension equal to its easily calculated similarity dimension.

In Chapter 16 we come to some of the hardest and yet some of the most spectacular things covered in this book, arising from the dynamics of iterated complex functions f(z). Thus we start with a function $f: \mathbf{C} \to \mathbf{C}$ (where $\mathbf{C}$ is the plane regarded as complex numbers), an initial point z_0, and observe the behaviour of the sequence $z_0, z_1, z_2, ..$, (called the *orbit* of z_0) where $z_{n+1} = f(z_n)$. Importantly, this process could be the iterative solution of an equation (see later). Very influential on the overall outcome is the orbit's behaviour near a fixed point α of f(z), i.e. one for which $f(\alpha) = \alpha$. If the derivative $f'(\alpha)$ has modulus less than 1 then points z sufficiently near α are drawn towards it ($f^n(z) \to \alpha$ as $n \to \infty$). Such points constitute the *basin of attraction* $A(\alpha)$, whose boundary is the *Julia set* of f. We prove results leading to the computer generation of Julia sets (Constructions 16.44 to 16.46) and to their colour codings which produce some increasingly well-known beautiful effects. As is commonly done, we study especially the effectively representative case $f(z) = f_c(z) = z^2+c$ and introduce its Mandelbrot set M, the set of points c for which J(f) is connected, or alternatively (a theorem) those c for which $f_c{}^n(c)$ does *not* tend to infinity as n does (see Construction 16.51). We explore the anatomy of M, which has further interesting detail at every level of magnification. Indeed M acts as an encyclopaedia of Julia sets in ways which even include 'snapshots'. Nevertheless M is connected, and we sketch the Douady-Hubbard proof of this. Finally we investigate the Julia set and basins of attraction for Newton's iterative method of solution, starting from any complex number, and shed new light on an old problem.

Chapters 14-16 : a quick trip. Read page 330 and see in Figure 14.1 how frame 2 arises from frame 1 in the same way as frame 1 arises from frame 0. Consider Figure 14.10 and Examples 14.19, 14.21, 14.26. Read pages 355-356 then try the exercise below Figure 14.22 **or** use colour plate 7 to help you understand Figure 14.21. In Chapter 15 look at the following figures and their explanations: 15.4, 15.8, 15.9, 15.12. Apply ten steps of the RIA for a Sierpinski gasket by throwing dice. In Chapter 16, start with the Mandelbrot set of Figure 16.18 in conjunction with page 424 (read Theorem 16.50 rather than the lemma). Take a look at Seahorse valley via colour plates 9-19. For Julia sets look quickly through pages 407-409, then pick up on Figure 16.20 and its preceding explanation. See this exemplified in colour plates 20-24, 27, 28. Note the construction *methods* on pages 422 to 423 and 425. Finally, look at Theorem 16.68 (Newton's method) and its illustration on colour plate 30.

Which chapters depend on which

1-6	Each chapter depends on the previous ones.
7	All depends on Chapter 1; only Section 7.4 depends on Chapter 2.
8	Matrices and vector products from Chapter 7, and Chapters 1,2.
9	As for Chapter 8, plus a little calculus towards the end.
10	Vectors and plane isometries from Chapter 1.
11	Elementary vectors (Section 1.2.1).
12	The definitions and results of Chapter 11; Elementary vectors (Section 1.2.1). Linear functions (Definition 8.4 ff.).
13	Isometries and vectors (Chapter 1). Section 13.4 requires complex numbers (Section 9.1). Topology: the results and definitions of Chapters 11, 12.
14	Vectors and plane isometries from Chapters 1, 2. Matrices (Chapter 7). Certain recapitulated results and the idea of convergence, from Chapter 13.
15	As for Chapter 14, plus iterated function systems (Section 14.1). Continuity (Section 11.4).
16	Complex numbers (Section 9.1). Topology: the results and definitions of Chapters 11, 12 and Section 13.4.1. Note: Sections 16.2.5 to 16.3.2 essentially rely only on results of Chapter 16.

Table of crude chapter dependencies.

A Chapter depends on those it can 'reach' by going down the graph.

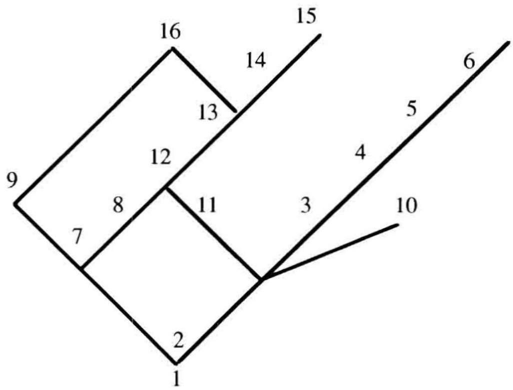

Table of contents

Chapter 1 ISOMETRIES

1.1 Introduction

One practical aim in Part 1 is to equip the reader to build a pattern generating computer engine. The patterns we have in mind come from two main streams. Firstly the *geometrical tradition*, represented for example in the fine Moslem art in the Alhambra at Granada in Spain, but found very widely.

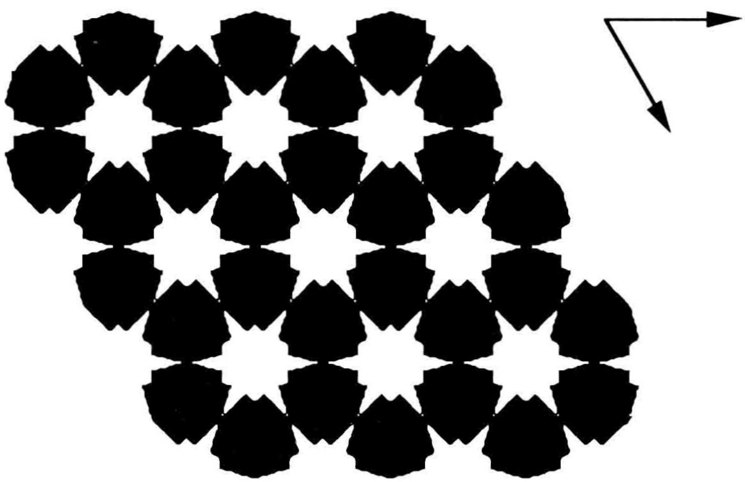

Figure 1.1 Variation on an Islamic theme. For the original, see Critchlow (1976), page 112. The arrows indicate symmetry in two independent directions, and the pattern is considered to continue indefinitely, filling the plane.

Less abundant but still noteworthy are the patterns left by the ancient Romans (Field, 1988). The second type is that for which the Dutch artist M. C. Escher is famous, exemplified in the