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GENERALIZED INVERSES OF LINEAR OPERATORS

Representation and Approximation

C.W. Groetsch

GENERALIZED INVERSES OF LINEAR OPERATORS

REPRESENTATION AND APPROXIMATION

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You must always invert.

C. G. J. Jacobi

To Sandra, Kurt and Heidi

PREFACE

This monograph stems from lectures which I gave during the years 1973-1975 at the University of Cincinnati and the University of Rhode Island. The aim of these lectures was to present a unified treatment of the approximation theory of generalized inverses of bounded linear operators in Hilbert space. General representation theorems which unify many diverse computational procedures appear here for the first time in book form and a new unified treatment of error bounds is presented. In addition much of the recent literature on approximation methods for generalized inverses of linear operators is surveyed.

Recently several books on the theory and application of matrix generalized inverses have appeared. Particular mention should be made of the excellent volume by Ben-Israel and Greville which contains a chapter on generalized inverses of linear operators in Hilbert space and the recently published book edited by Nashed. However, at the present time, I believe this to be the only self-contained treatment of the computational theory of generalized inverses of bounded linear operators in Hilbert space which is suitable for use as a text in a first or second year graduate seminar. It is also hoped that this text will be useful as supplemental reading in courses on linear operator theory and advanced numerical analysis. With these uses in mind, a list of exercises is provided at the end of each chapter.

The chapters are divided into sections, and equations to which we will have occasion to refer are numbered consecutively in each section. Every proposition has a unique identifying number; the number "a.b.c." refers to proposition number "c" in section "b" of chapter "a". The symbol "#" is used to indicate the end of a proof.

Many friends, colleagues and students have read and commented on various parts of the manuscript. Thanks are particularly due to Betsy Conway, Mohab El-Samaloty, Bart Jacobs, Alan Lazer, John Montgomery, Lew Pakula and Ghasi Verma. I owe a special debt of gratitude to Professor Marvin Marcus for many valuable suggestions on content and style. Finally, I wish to express my appreciation to Linda Patterson for an efficient job of typing. The usual statement concerning the ultimate responsibility for residual errors applies here.

C. W. Groetsch

CONTENTS

CHAPTER I

HILBERT SPACE AND HILBERT SPACE OPERATORS

Hilbert space, since it is the most natural infinite dimensional structure into which our geometric intuition generally carries over, has long been considered as the appropriate vehicle for the study of many linear problems. Although we assume that the reader has some familiarity with Hilbert space and the theory of linear operators, we have included this preliminary chapter in an effort to establish coherent notation and make the presentation reasonably self-contained. It is hoped that this chapter will serve as a brief introduction to basic Hilbert space theory. The results in this chapter are for the most part stated without proof. Proofs of many of the theorems are outlined in the exercises and references are given to standard books on functional analysis and operator theory where more leisurely accounts of the basic theory can be found.

SECTION 1

HILBERT SPACE

We assume that the reader is familiar with the concept of a linear space. By a *normed linear space* we will mean a linear space E endowed with a real-valued function $||\cdot||$ which satisfies the following axioms

$$||x|| \geq 0; \; ||x|| = 0 \text{ if and only if } x = 0,$$
$$||\alpha x|| = |\alpha| \; ||x||,$$
$$||x + y|| \leq ||x|| + ||y||,$$

where x and y are arbitrary elements of E and α is any (real or complex, depending on the context) scalar.

<u>Definition</u>. An <u>inner product space</u> (also called a pre-Hilbert space) is a linear space E endowed with a scalar valued function $(\cdot,\cdot)$, called an <u>inner product</u> for E, which satisfies for any $x,y,z \in E$ and any scalars α and β

$$(x,y) = \overline{(y,x)}$$
$$(\alpha x + \beta y,z) = \alpha(x,z) + \beta(y,z)$$
$$(x,x) \geq 0, \text{ with equality only if } x = 0.$$

The bar denotes complex conjugation (of course, if the scalar field is taken to be the real numbers then the first axiom becomes $(x,y) = (y,x)$). By using Theorem 1.1.1, it is easy to show that any inner product space is also a normed linear space where the norm is defined by

$$||x|| = (x,x)^{1/2}.$$

The next theorem gives an inequality which is very basic in the study of inner product spaces; its proof is outlined in the exercises.

Theorem 1.1.1. (Schwarz's inequality) For any x and y in an inner product space E

$$|(x,y)| \leq ||x|| \; ||y||$$

with equality holding if and only if x and y are linearly dependent (i.e., there exist scalars α and β, not both equal to zero, such that $\alpha x + \beta y = 0$).

Given points x and y in a normed linear space E the distance between x and y is defined as $d(x,y) = ||x - y||$. It is easy to see that d defines a metric on E which in turn generates a topology for E (the norm topology). Schwarz's inequality shows that in an inner product space the function $(\cdot,\cdot)$ is continuous in the product topology on $E \times E$ induced by the norm topology on E.

The proof of the following theorem is routine.

Theorem 1.1.2. If E is an inner product space then

(a) $||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2)$

(b_1) $(x,y) = (||x + y||^2 - ||x - y||^2)/4$ (real scalars)

(b_2) $(x,y) = (||x + y||^2 - ||x - y||^2 + i||x + iy||^2 - i||x - iy||^2)/4$ (complex scalars)

Identity (a) above is called the parallelogram law owing to its obvious geometrical interpretation. The identities (b) are usually referred to in the literature as the polarization identities.

Definition. A sequence $\{x_n\}$ in a normed linear space is called a Cauchy sequence if given $\varepsilon > 0$ there is a positive integer $N(\varepsilon)$ such that $n,m > N(\varepsilon)$ implies $||x_n - x_m|| < \varepsilon$.

Definition. A normed linear space E is said to be complete (also called a Banach space) if every Cauchy sequence in E converges to some point of E.

Definition. A complete inner product space is called a Hilbert space.

The most important examples of Hilbert spaces are the real Hilbert spaces

$$\ell^2 = \{\{x_i\} : x_i \in R \text{ and } \sum_{i=1}^{\infty} x_i^2 < \infty\}$$

where the inner product of two vectors $x = \{x_i\}$ and $y = \{y_i\}$ is defined by

$$(x,y) = \sum_{i=1}^{\infty} x_i y_i ,$$

and the Hilbert space $L^2[a,b]$ consisting of all (equivalence classes of) square integrable functions on [a,b] with the inner product

$$(f,g) = \int_a^b f(t)g(t)\,dt.$$

Of course, with minor modifications in the above descriptions we could define complex versions of the spaces ℓ^2 and $L^2[a,b]$. It goes without saying that the finite dimensional spaces R^n and $\mathbb{C}^n$ are Hilbert spaces under the usual inner products.

Definition. A scalar valued linear function defined on a linear space E is called a linear functional on E (i.e., $f : E \to F$, where F is the scalar field, is a linear functional if $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$, for $\alpha,\beta \in F$ and $x,y \in E$).

If addition and scalar multiplication are defined on the set of linear functions by

$$(\phi + \psi)(x) = \phi(x) + \psi(x)$$
$$(\alpha\phi)(x) = \alpha\phi(x)$$

we see that this set forms a linear space. The linear space of all (norm) continuous linear functionals on a normed linear space E is denoted by E^* and is called the dual space of E. The space E^* is a normed linear space where the norm is defined for $\phi \in E^*$ by

$$||\phi|| = \sup\{|\phi(x)| : x \in E, ||x|| = 1\}.$$

If the space E is complete then so is E^*. The next theorem shows that Hilbert space enjoys the property of being self-dual (for a proof see the exercises).

Theorem 1.1.3. (Riesz Representation Theorem) If ϕ is a continuous linear functional on a Hilbert space H, then there exists a unique $y \in H$ such that

$$\phi(x) = (x,y)$$

for each $x \in H$.

Definition. A subset S of a linear space E is called a subspace of E if S is itself a linear space. If E is a normed linear space then a subspace S of E is called a closed subspace if S is closed in the norm topology for E.

We note that a closed subspace of a complete linear space is itself complete. We now introduce the concept of convexity, which plays a fundamental role in linear space theory.

Definition. A subset C of a normed linear space is called convex if $tx + (1 - t)y \in C$ for all $x,y \in C$ and all $t \in [0,1]$.

The next theorem is a basic principal in the theory of optimization and best approximation.

Theorem 1.1.4. A closed convex subset C of a Hilbert space contains a unique vector of smallest norm.

Proof. Let $M = \inf\{||x|| : x \in C\}$ and choose a sequence $\{x_n\} \subset C$ such that $\lim_n ||x_n|| = M$. Since C is convex, we have by use of the parallelogram law

$$\begin{aligned} ||x_n - x_m||^2 &= 2(||x_n||^2 + ||x_m||^2) - 4||(x_n + x_m)/2||^2 \\ &\leq 2(||x_n||^2 + ||x_m||^2) - 4M^2 \end{aligned}$$

which converges to 0 as $n,m \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence which therefore has a limit $x \in C$ (since C is closed). It is now easy to see that x is the unique vector in C of minimal norm. Indeed, if y is a vector in C with norm M which is distinct from x, then

$$0 < ||x - y||^2 = 4M - 4||(x + y)/2||^2.$$

Therefore,

$$||(x + y)/2|| < M$$

which is a contradiction. #

The following is an easy consequence of (1.1.4).

Corollary 1.1.5. If C is a closed convex subset of a Hilbert space H, then for each $u \in H$ there is a unique $x \in C$ such that

$$||u - x|| = \inf\{||u - y|| : y \in C\}.$$

Inner product spaces have a richer geometrical structure than general linear spaces owing to the fact that in spaces with an inner product the concept of perpendicularity can be developed.

Definition. Two vectors x and y in an inner product space are said to be orthogonal, denoted $x \perp y$, if $(x,y) = 0$. If S is a subspace of an inner product space E we define its orthogonal complement by

$$S^{\perp} = \{y \in E : x \perp y \text{ for all } x \in S\}.$$

Note that if x and y are orthogonal then they satisfy the Pythagorean property

$$||x + y||^2 = ||x||^2 + ||y||^2.$$

The name is suggestive of the geometrical significance of this identity. It is worthwhile to point out that since the inner product is continuous it follows that for any subspace S the set $S^{\perp}$ is a <u>closed</u> subspace. It is of fundamental importance that in a Hilbert space a closed subspace and its orthogonal complement decompose the space in the following sense.

Theorem 1.1.6. If S is a closed subspace of a Hilbert space H, then H can be written as the direct sum of S and $S^{\perp}$, denoted $H = S \oplus S^{\perp}$, meaning that each $x \in H$ can be written uniquely as $x = x_1 + x_2$, where $x_1 \in S$ and $x_2 \in S^{\perp}$.

Proof. Given $x \in H$ there exists by (1.1.5) a unique vector $x_1 \in S$ such that $||x - x_1||$ is minimal. Let $x_2 = x - x_1$. For any $y \in S$ with $||y|| = 1$ we have

$$\begin{aligned} ||x_2||^2 = ||x - x_1||^2 &\leq ||x - x_1 - (x_2,y)y||^2 \\ &= ||x_2||^2 - |(x_2,y)|^2 \leq ||x_2||^2. \end{aligned}$$

Therefore, $(x_2,y) = 0$, and it follows that $x_2 \in S^{\perp}$. Hence we see that $x = x_1 + x_2 \in S \oplus S^{\perp}$. Uniqueness of the representation is easy to establish. #

SECTION 2

LINEAR OPERATORS

This section sets forth some of the salient features of the theory of linear operators. The results given here will form a foundation for our general development of generalized inverses of bounded linear operators.