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Lecture Notes on Elementary Topology and Geometry

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Preface

At the present time, the average undergraduate mathematics major finds mathematics heavily compartmentalized. After the calculus, he takes a course in analysis and a course in algebra. Depending upon his interests (or those of his department), he takes courses in special topics. If he is exposed to topology, it is usually straightforward point set topology; if he is exposed to geometry, it is usually classical differential geometry. The exciting revelations that there is some unity in mathematics, that fields overlap, that techniques of one field have applications in another, are denied the undergraduate. He must wait until he is well into graduate work to see interconnections, presumably because earlier he doesn't know enough.

These notes are an attempt to break up this compartmentalization, at least in topology-geometry. What the student has learned in algebra and advanced calculus are used to prove some fairly deep results relating geometry, topology, and group theory. (De Rham's theorem, the Gauss-Bonnet theorem for surfaces, the functorial relation of fundamental group to covering space, and surfaces of constant curvature as homogeneous spaces are the most noteworthy examples.)

In the first two chapters the bare essentials of elementary point set topology are set forth with some hint of the subject's application to functional analysis. Chapters 3 and 4 treat fundamental groups, covering spaces, and simplicial complexes. For this approach the authors are indebted to E. Spanier. After some preliminaries in Chapter 5 concerning the theory of manifolds, the De Rham theorem (Chapter 6) is proven as in H. Whitney's *Geometric Integration Theory*. In the two final chapters on Riemannian geometry, the authors follow E. Cartan and S. S. Chern. (In order to avoid Lie group theory in the last two chapters, only oriented 2-dimensional manifolds are treated.)

These notes have been used at M.I.T. for a one-year course in topology and geometry, with prerequisites of at least one semester of modern algebra and one semester of advanced calculus "done right." The class consisted of about seventy students, mostly seniors. The ideas for such a course originated in one of the author's tour of duty for the Committee on the Undergraduate Program in Mathematics of the Mathematical Association of America. A program along these lines, but more ambitious, can be found in the CUPM pamphlet "Pregraduate Preparation of Research Mathematicians" (1963). (See Outline III on surface theory, pp. 68–70.) The authors believe, however, that in lecturing to a large class without a textbook, the material in these notes was about as much as could be covered in a year.

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Singer I. M. & Thorpe J. A.

Lecture Notes on Elementary Topology & Geometry.

初等拓扑学和几何学讲义

本书作者都是著名的数学家，60 年代初曾参加美国数学学会关于数学系大学生课程安排委员会的调查工作，他们发现大学的课程中，经典数学教程与近代数学发展不相衔接。为了弥补这个不足，他们在自己教学实践的基础上，并吸收了 E. Spanier 和 H. Whitney 等人所写的书的优点，编写了此书，多次作为麻省理工学院拓扑学和几何学的教本。

本书以最少的预备知识为基础论述拓扑学、几何学和群论的有关内容，如 De Rlam 理论、曲面的高斯-Bonnet 定理、复盖空间的基本群的函数关系、常曲率曲面作为齐次空间等。本书可作为一般数学工作者学习拓扑学、几何学、代数及流形上的微积分的入门书，或大学有关专业的参考书。

全书分 8 章，目次如下：①点集拓扑，②更深入的点集拓扑，③基本群和复盖空间，④单纯复形，⑤流形，⑥同调论和 De Rlam 理论，⑦曲面上的内在黎曼几何，⑧ R^3 中的嵌入流形。书末有文献目录和索引。

Contents

Some point set topology 1

1.1 Naive set theory

We shall accept as primitive (undefined) the concepts of a *set* (collection, family) of objects and the concept of an object *belonging to* a set.

We merely remark that, given a set S and an object x, one can determine if the object belongs to (is an element of) the set, written $x \in S$, or if it does not belong to the set, written $x \notin S$.

Definition. Let A and B be sets. A is a ***subset*** of B, written $A \subset B$, if $x \in A$ implies $x \in B$. A is *equal* to B, written $A = B$, if $A \subset B$ and $B \subset A$.

Notation. The *empty set*, that is, the set with no objects in it, is denoted by $\varnothing$.

Remark

(1) $\varnothing \subset A$ for all sets A.
(2) The empty set $\varnothing$ is unique; that is, any two empty sets are equal. For if $\varnothing_1$ and $\varnothing_2$ are two empty sets, $\varnothing_1 \subset \varnothing_2$ and $\varnothing_2 \subset \varnothing_1$.
(3) $A \subset A$ for all sets A.

Definition. Let A and B be sets. The *union* $A \cup B$ of A and B is the set of all x such that $x \in A$ or $x \in B$, written

$$A \cup B = [x; x \in A \text{ or } x \in B].$$

The *intersection* $A \cap B$ of A and B is defined by

$$A \cap B = [x; x \in A \text{ and } x \in B].$$

Similarly, if $\mathscr{S}$ is a set (collection) of sets, the union and intersection of all the sets in $\mathscr{S}$ are defined respectively by

$$\bigcup_{S \in \mathscr{S}} S = [x; x \in S \text{ for some } S \in \mathscr{S}],$$

$$\bigcap_{S \in \mathscr{S}} S = [x; x \in S \text{ for every } S \in \mathscr{S}].$$

If $A \subset B$, the *complement* of A in B, denoted A' or $B - A$, is defined by

$$A' = [x \in B; x \notin A].$$

Theorem 1. *Let A, B, C, and S be sets. Then*

(1) $A \cup B = B \cup A$.
(2) $A \cap B = B \cap A$.
(3) $(A \cup B) \cup C = A \cup (B \cup C)$.
(4) $(A \cap B) \cap C = A \cap (B \cap C)$.
(5) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
(6) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
(7) *If* $A \subset S$ *and* $B \subset S$ *then,* $(A \cup B)' = A' \cap B'$.
(8) *If* $A \subset S$ *and* $B \subset S$, *then* $(A \cap B)' = A' \cup B'$.
(9) *If* $\mathscr{S}_1$ *and* $\mathscr{S}_2$ *are two sets (collections) of sets, then*

$$\left(\bigcup_{S \in \mathscr{S}_1} S\right) \cup \left(\bigcup_{S \in \mathscr{S}_2} S\right) = \bigcup_{S \in \mathscr{S}_1 \cup \mathscr{S}_2} S$$

and

$$\left(\bigcap_{S \in \mathscr{S}_1} S\right) \cap \left(\bigcap_{S \in \mathscr{S}_2} S\right) = \bigcap_{S \in \mathscr{S}_1 \cup \mathscr{S}_2} S.$$

(10) *For* $\mathscr{S}_1$ *and* $\mathscr{S}_2$ *as in* (9),

$$\left(\bigcup_{S_1 \in \mathscr{S}_1} S_1\right) \cap \left(\bigcup_{S_2 \in \mathscr{S}_2} S_2\right) = \bigcup_{\substack{S_1 \in \mathscr{S}_1 \\ S_2 \in \mathscr{S}_2}} (S_1 \cap S_2).$$

PROOF. The proof of this theorem is left to the student.

Definition. Let A and B be sets. The *Cartesian product* $A \times B$ of A and B is the set of ordered pairs

$$A \times B = [(a, b); a \in A, b \in B].$$

A *relation* between A and B is a subset R of $A \times B$. a and b are said to be R-related if $(a, b) \in R$.

EXAMPLE. Let $A = B =$ the set of real numbers. Then $A \times B$ is the plane. The order relation $x < y$ is a relation between A and B. This relation is the shaded set of points in Figure 1.1.

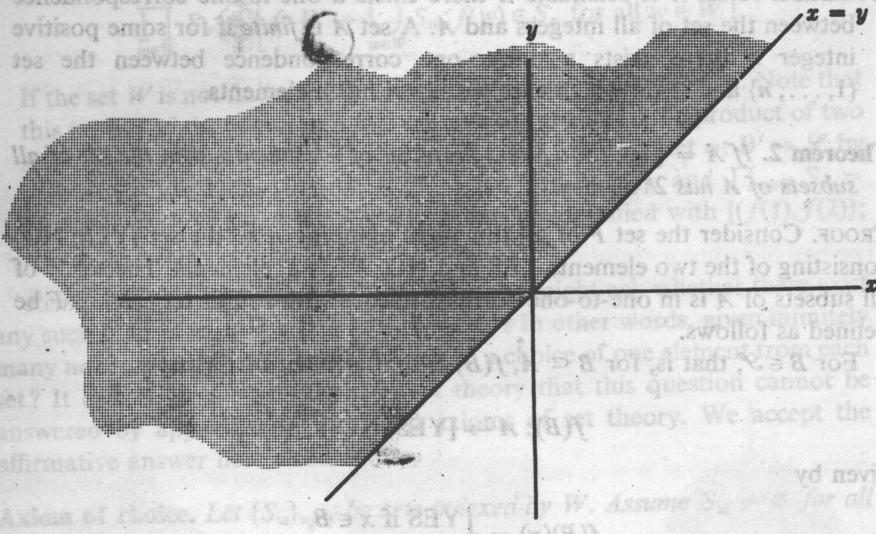

Figure 1.1

Definition. A relation $R \subset A \times A$ is a *partial ordering* if

(1) $(s_1, s_2) \in R$ and $(s_2, s_3) \in R \Rightarrow (s_1, s_3) \in R$ and
(2) $(s_1, s_2) \in R$ and $(s_2, s_1) \in R \Rightarrow s_1 = s_2$.

A relation R is a *simple ordering* if it is a partial ordering, and, in addition:

(3) either $(s_1, s_2) \in R$ or $(s_2, s_1) \in R$ for every pair $s_1, s_2 \in S$.

The order relation for $S =$ real numbers is an example of a simple ordering. In general, we say that S is partially ordered (simply ordered) by R.

Definition. Let A and B be sets. A *function* f mapping A to B, denoted $f: A \to B$, is a relation ($f \subset A \times B$) between A and B satisfying the following properties.

(1) If $a \in A$, then there exists $b \in B$ such that $(a, b) \in f$.
(2) If $(a, b) \in f$ and $(a, b_1) \in f$, then $b = b_1$.

Property (1) says that the function f is defined everywhere on A. Property (2) says that f is a "single-valued" function.

Notation. Let $f: A \to B$. By $f(a) = b$ we mean $(a, b) \in f$.

Definition. Let $f: A \to B$. f is *surjective* (onto) if for each $b \in B$ there exists $a \in A$ such that $f(a) = b$. If f is surjective, we write $f(A) = B$. f is *injective* (one-to-one) if $f(a) = f(a_1) \Rightarrow a = a_1$. If f is both surjective and injective, we say f is a *one-to-one correspondence* between A and B.

Definition. A set A is *countable* if there exists a one-to-one correspondence between the set of all integers and A. A set A is *finite* if for some positive integer n there exists a one-to-one correspondence between the set $\{1, \ldots, n\}$ and A, in which case we say A has n elements.

Theorem 2. *If $A = \{a_1, \ldots, a_n\}$ is a finite set of n elements, then the set of all subsets of A has 2^n elements.*

PROOF. Consider the set F of all functions mapping A to the set {YES, NO} consisting of the two elements YES and NO. F has 2^n elements. The set $\mathscr{S}$ of all subsets of A is in one-to-one correspondence with F. For let $f: \mathscr{S} \to F$ be defined as follows.

For $B \in \mathscr{S}$, that is, for $B \subset A$, $f(B)$ is that element of F (that is,

$$f(B): A \to [\text{YES, NO}])$$

given by

$$f(B)(x) = \begin{cases} \text{YES} & \text{if } x \in B, \\ \text{NO} & \text{if } x \notin B. \end{cases}$$

f is injective because if $f(B) = f(C)$, then $f(B)(x) = f(C)(x)$ for all $x \in A$. Thus $f(B)(x) =$ YES if and only if $f(C)(x) =$ YES; that is, $x \in B$ if and only if $x \in C$. Thus $B = C$. f is surjective because every function $g: A \to$ [YES, NO] determines a $B \subset A$ by

$$B = [x; g(x) = \text{YES}]$$

and $f(B) = g$. □

Notation. Motivated by this proposition, we denote by 2^A the set of all subsets of A. Given two sets A and B, B^A denotes the set of all functions $A \to B$.

Definition. Let $f \in B^A$. The *inverse* f^{-1} of f is the function $2^B \to 2^A$ defined by

$$f^{-1}(B_1) = [a \in A; f(a) \in B_1] \qquad (B_1 \subset B).$$

$f^{-1}(B_1)$ is called the *inverse image* of B_1. Note that

$$f^{-1} \in (2^A)^{2^B}.$$

Notation. Let W be a set, and let $\mathscr{S}$ be a collection of sets. We say $\mathscr{S}$ is *indexed* by W if there is given a surjective function $\varphi: W \to \mathscr{S}$. For $w \in W$, we denote $\varphi(w)$ by S_w and denote the *indexing* of $\mathscr{S}$ by W as $\{S_w\}_{w\in W}$.

Definition. Let $\{S_w\}_{w\in W}$ be indexed by W. The *product* of the sets $\{S_w\}_{w\in W}$ is the set

$$\prod_{w\in W} S_w = \left[f: W \to \bigcup_{w\in W} S_w; f(w) \in S_w \text{ for all } w \in W\right].$$

If the set W is not finite, this product is called an *infinite product*. Note that this notion of the product of sets extends the notion of the product of two sets $S_1 \times S_2$. For let $W = \{1, 2\}$, let $\mathscr{S} = \{S_1, S_2\}$, and let $\varphi: W \to \mathscr{S}$ by $\varphi(j) = S_j$, $j = 1, 2$. Then $S_1 \times S_2 = [(s_1, s_2), s_j \in S_j]$, and $\prod_{w\in W} S_w = [f: \{1, 2\} \to S_1 \cup S_2; f(j) \in S_j]$, which can be identified with $[(f(1), f(2)); f(j) \in S_j]$, which can be identified with $S_1 \times S_2$.

Remark. $\prod_{w\in W} S_w$ is a set of functions. One might ask whether there exist any such functions; that is, is $\prod_{w\in W} S_w \neq \varnothing$? In other words, given infinitely many nonempty sets, is it possible to make a choice of one element from each set? It can be shown in axiomatic set theory that this question cannot be answered by appealing to the usual axioms of set theory. We accept the affirmative answer here as an axiom.

Axiom of choice. *Let $\{S_w\}_{w\in W}$ be sets indexed by W. Assume $S_w \neq \varnothing$ for all $w \in W$. Then*

$$\prod_{w\in W} S_w \neq \varnothing.$$

The axiom of choice is equivalent to several other axioms, one of which is the following.

Maximum principle. *If S is partially ordered by R, and T is a simply ordered subset, then there exists a set M such that the following statements are valid.*

(1) $T \subset M \subset S$.

(2) *M is simply ordered by R.*

(3) *If $M \subset N \subset S$, and N is simply ordered by R, then $M = N$; that is, M is a maximal simply ordered subset containing T.*

1.2 Topological spaces

Definition. A *metric space* is a set S together with a function $\rho: S \times S \to$ the nonnegative real numbers, such that for each $s_1, s_2, s_3 \in S$:

(1) $\rho(s_1, s_2) = 0$ if and only if $s_1 = s_2$.

(2) $\rho(s_1, s_2) = \rho(s_2, s_1)$.

(3) $\rho(s_1, s_3) \leq \rho(s_1, s_2) + \rho(s_2, s_3)$.

The function ρ is called a *metric* on S.

Given a point s_0 in a metric space S and a real number a, the *ball of radius a about* s_0 is defined to be the set

$$B_{s_0}(a) = [s \in S; \rho(s, s_0) < a].$$

EXAMPLE. Let S be the plane, that is, the product of the set of the real numbers with itself. We define three metrics on S as follows.

For $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ two points in S,

$$\rho_1(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$
$$\rho_2(P_1, P_2) = \max\{|x_2 - x_1|, |y_2 - y_1|\},$$
$$\rho_3(P_1, P_2) = |x_2 - x_1| + |y_2 - y_1|.$$

The ball of radius a about the point $0 = (0, 0)$ relative to each of these metrics is indicated by the shaded areas in Figure 1.2. Note that a ball does not necessarily have a circular, or even a smooth, boundary.

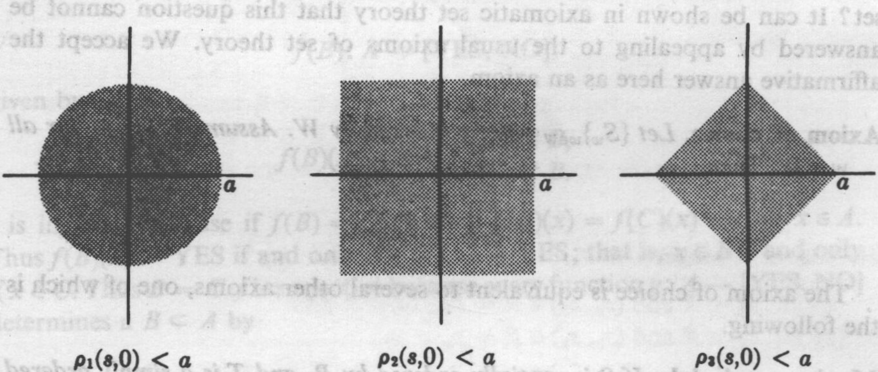

Figure 1.2

Remark. The three metrics defined above provide the plane with three distinct structures as a metric space. Yet for studying certain properties of these spaces, these metrics are equivalent. Thus, if we want to know, for example, whether 0 is a limit point of a set $T \subset S$, we ask whether there is a sequence of points in T which converges to 0; that is, whether a sequence $\{s_n\}$ of points in T exists such that given any $\varepsilon > 0$, there exists an N such that

$$\rho(s_n, 0) < \varepsilon$$

for all $n > N$. It is not difficult to see that the answer to this question is independent of whichever of the above metrics we use for ρ; that is, given $\varepsilon > 0$, there exists such an N using ρ_1 if and only if there exists such an N using ρ_2, etc. The answer does not depend on the shape of the ball of radius ε, but only on its "fatness" or "openness." For this reason among others, it is convenient to gather together those properties of a metric space that are essential for describing "openness" and to use such properties to define a more abstract structure, a *topological structure*, in which we can still talk about limit points and in which the three metric structures on the plane described above will give the same "open sets."

Definition. A *topological space* is a set S together with a collection $\mathscr{U}$ of subsets of S (that is, $\mathscr{U}$ is a subset of 2^S) satisfying the following conditions:

(1a) $\varnothing \in \mathscr{U}$, $S \in \mathscr{U}$.
(2a) If $U_1, \ldots, U_n \in \mathscr{U}$ then $\bigcap_{i=1}^n U_i \in \mathscr{U}$.
(3a) Arbitrary unions of elements in $\mathscr{U}$ lie in $\mathscr{U}$; that is, if $\tilde{\mathscr{U}} \subset \mathscr{U}$, then $\bigcup_{U \in \tilde{\mathscr{U}}} U \in \mathscr{U}$.

The elements of $\mathscr{U}$ are called *open sets* in S. The collection $\mathscr{U}$ is called a *topology* on S.

Remark. We shall often suppress the $\mathscr{U}$ and simply refer to S as a topological space.

Definition. Let $(S, \mathscr{U})$ be a topological space. A set $A \subset S$ is *closed* if it is the complement of an open set, that is, if $A' \in \mathscr{U}$.

Remark. By taking complements in conditions (1a), (2a), and (3a) above, one sees that the collection $\mathscr{C}$ of closed sets satisfies the following conditions.

(1b) $\varnothing \in \mathscr{C}$, $S \in \mathscr{C}$.
(2b) If $A_1, \ldots, A_n \in \mathscr{C}$, then $\bigcup_{i=1}^n A_i \in \mathscr{C}$.
(3b) Arbitrary intersections of elements in $\mathscr{C}$ lie in $\mathscr{C}$.

Remark. A topology can be described by specifying the collection of closed sets equally as well as specifying the collection of open sets.

Definition. Let $(S, \mathscr{U})$ be a topological space. Let $A \subset S$. A point $s \in S$ is a *limit point* of A if for each $U \in \mathscr{U}$ such that $s \in U$,

$$(U - \{s\}) \cap A \neq \varnothing.$$

Definition. The *closure* of a set $A \subset S$, denoted by $\bar{A}$, is the set

$$\bar{A} = A \cup [s \in S;\ s \text{ is a limit point of } A].$$

Theorem 1. *The closure $\bar{A}$ of a set A is closed.*

Proof. We must show that $\bar{A}'$ is open. For this it suffices to show that for each $s \in \bar{A}'$ there exists an open set U_s with $s \in U_s \subset \bar{A}'$. Then $s \in U_s$ for each s

implies $\bar{A}' \subset \bigcup_{s\in\bar{A}'} U_s$ and $U_s \subset \bar{A}'$ for each s implies $\bigcup_{s\in\bar{A}'} U_s \subset \bar{A}'$. Thus $\bar{A}' = \bigcup_{s\in\bar{A}'} U_s$ is a union of open sets and hence is open.

Now let $s \in \bar{A}'$. Then s is not a limit point of A, so there exists an open set U_s such that $s \in U_s$ and $(U_s - \{s\}) \cap A = \varnothing$. Furthermore, $s \notin A$ because $s \notin \bar{A}$ and hence, in fact, $U_s \cap A = \varnothing$. Since each element of U_s is contained in an open set, namely U_s itself, whose intersection with A is $\varnothing$, it follows that U_s contains no limit points of A and $U_s \cap \bar{A} = \varnothing$; that is, $U_s \subset \bar{A}'$. □

Theorem 2. *A set A is closed if and only if $A = \bar{A}$.*

Proof. Assume A is closed. Then A' is open. If $s \notin A$, then A' is an open set containing s such that $(A' - \{s\}) \cap A = \varnothing$. Thus s is not a limit point of A. Hence all limit points of A lie in A; that is, $A = \bar{A}$.

Conversely, if $A = \bar{A}$, then A is closed by the previous theorem. □

Definition. A set $\mathscr{B} \subset 2^S$ is a *basis* for a topology on S if the following conditions are satisfied:

(1c) $\varnothing \in \mathscr{B}$.

(2c) $\bigcup_{B\in\mathscr{B}} B = S$.

(3c) If B_1 and $B_2 \in \mathscr{B}$, then $B_1 \cap B_2 = \bigcup_{B\in\tilde{\mathscr{B}}} B$ for some subset $\tilde{\mathscr{B}} \subset \mathscr{B}$.

Theorem 3. *Let S be a set and $\mathscr{B}$ be a basis for a topology on S. Let*

$$\mathscr{U}_{\mathscr{B}} = [U \in 2^S;\ U \text{ is a union of elements of } \mathscr{B}].$$

Then $\mathscr{U}_{\mathscr{B}}$ is a topology on S, the topology generated *by $\mathscr{B}$.*

Proof. We must verify that $\mathscr{U}_{\mathscr{B}}$ satisfies the three open-set axioms for a topology on S.

By (1c) and (2c) in the definition of a basis, both $\varnothing$ and $S \in \mathscr{U}_{\mathscr{B}}$ so that condition (1a) in the definition of a topological space is satisfied.

Suppose $\mathscr{V} \subset \mathscr{U}_{\mathscr{B}}$. Then $\bigcup_{V\in\mathscr{V}} V = \bigcup_{V\in\mathscr{V}} (\bigcup_{B\in\mathscr{B}_V} B) = \bigcup_{B\in\tilde{\mathscr{B}}} B$ where $\mathscr{B}_V \subset \mathscr{B}$ for each $V \in \mathscr{V}$ and $\tilde{\mathscr{B}} = \bigcup_{V\in\mathscr{V}} \mathscr{B}_V \subset \mathscr{B}$. Hence condition (3a) holds.

We prove condition (2a) by induction. We assume that the intersection of k sets in $\mathscr{U}_{\mathscr{B}}$ lies in $\mathscr{U}_{\mathscr{B}}$. (For $k = 1$, the statement is automatically true.) Suppose then $U_1, \ldots, U_{k+1} \in \mathscr{U}_{\mathscr{B}}$. By the inductive hypothesis, $U_1 \cap \cdots \cap U_k \in \mathscr{U}_{\mathscr{B}}$; that is, there exists a subset $\mathscr{B}_1 \subset \mathscr{B}$ such that $U_1 \cap \cdots \cap U_k = \bigcup_{B_1\in\mathscr{B}_1} B_1$. Since $U_{k+1} \in \mathscr{U}_{\mathscr{B}}$, there exists a subset $\mathscr{B}_2 \subset \mathscr{B}$ such that $U_{k+1} = \bigcup_{B_2\in\mathscr{B}_2} B_2$. Hence

$$U_1 \cap \cdots \cap U_{k+1} = \left(\bigcup_{B_1\in\mathscr{B}_1} B_1\right) \cap \left(\bigcup_{B_2\in\mathscr{B}_2} B_2\right) = \bigcup_{\substack{B_1\in\mathscr{B}_1\\ B_2\in\mathscr{B}_2}} (B_1 \cap B_2).$$

But by condition (3c) in the definition of a basis, $B_1 \cap B_2 \in \mathscr{U}_{\mathscr{B}}$.

Hence $U_1 \cap \cdots \cap U_{k+1} \in \mathscr{U}_{\mathscr{B}}$. □

Theorem 4. *Let (S, ρ) be a metric space. Let*

$$\mathscr{B} = [B_s(a);\ s \in S \text{ and } a \text{ is a nonnegative real number}].$$

Then $\mathscr{B}$ is a basis for a topology on S.

Proof

(1) $B_s(0) = \varnothing$ for any $s \in S$, so $\varnothing \in \mathscr{B}$.
(2) For any $a > 0$, $S = \bigcup_{s \in S} B_s(a)$, so $S = \bigcup_{B \in \mathscr{B}} B$.
(3) Let $s_1, s_2 \in S$, let $a_1, a_2 > 0$, and let $T = B_{s_1}(a_1) \cap B_{s_2}(a_2)$. We may assume $T \neq \varnothing$.

To show that T is a union of elements of $\mathscr{B}$, it suffices to show that for each $s \in T$ there exists $a_s > 0$ such that $B_s(a_s) \subset T$. For then $T \subset \bigcup_{s \in T} B_s(a_s) \subset T$. The first inclusion follows because $s \in B_s(a_s)$ for each $s \in T$, and the second because $B_s(a_s) \subset T$ for each s. Thus $T = \bigcup_{s \in T} B_s(a_s)$ is a union of elements of $\mathscr{B}$.

Now for $s \in T$, let $b_j = \rho(s, s_j)$ for $j = 1, 2$. Then $b_j < a_j$ since $s \in B_{s_j}(a_j)$. Let $a_s = \min\{a_1 - b_1, a_2 - b_2\}$. Then $a_s > 0$, and we claim that $B_s(a_s) \subset T$. For suppose $t \in B_s(a_s)$. Then

$$\rho(t, s_j) \le \rho(t, s) + \rho(s, s_j) < a_s + b_j \le a_j - b_j + b_j = a_j;$$

so $t \in B_{s_j}(a_j)$, $j = 1, 2$. □

Corollary. *A metric space has the structure of a topological space in which the open sets are unions of balls.*

Definition. Let S be a set, and let $\mathscr{B}_1$ and $\mathscr{B}_2$ be bases for topologies on S. $\mathscr{B}_1$ and $\mathscr{B}_2$ are *equivalent* if they generate the same topology; that is, if $\mathscr{U}_{\mathscr{B}_1} = \mathscr{U}_{\mathscr{B}_2}$.

Theorem 5. *Let S be a set, and let $\mathscr{B}_1$ and $\mathscr{B}_2$ be bases for topologies on S. Then $\mathscr{B}_1$ and $\mathscr{B}_2$ are equivalent if and only if*

(1) *for each $s \in S$ and $B_1 \in \mathscr{B}_1$ with $s \in B_1$, there exists $B_2 \in \mathscr{B}_2$ such that $s \in B_2 \subset B_1$, and*
(2) *for each $s \in S$ and $B_2 \in \mathscr{B}_2$ with $s \in B_2$, there exists $B_1 \in \mathscr{B}_1$ such that $s \in B_1 \subset B_2$.*

Proof. Suppose $\mathscr{B}_1$ and $\mathscr{B}_2$ are equivalent, and let $s \in B_1 \in \mathscr{B}_1$. Then $B_1 \in \mathscr{U}_{\mathscr{B}_1} = \mathscr{U}_{\mathscr{B}_2}$, so $B_1 = \bigcup_{B_2 \in \tilde{\mathscr{B}}_2} B_2$ for some subset $\tilde{\mathscr{B}}_2 \subset \mathscr{B}_2$. Hence $s \in B_2 \subset B_1$ for some $B_2 \in \tilde{\mathscr{B}}_2 \subset \mathscr{B}_2$. Thus (1) is proved, and (2) is proved similarly.

Conversely, suppose (1) and (2) are satisfied. We first show that $\mathscr{U}_{\mathscr{B}_1} \subset \mathscr{U}_{\mathscr{B}_2}$. Let $B \in \mathscr{B}_1$. By (1) for each $s \in B$ there exists $B_s \in \mathscr{B}_2$ such that $s \in B_s \subset B$. Now $B \subset \bigcup_{s \in B} B_s \subset B$, so $B = \bigcup_{s \in B} B_s \in \mathscr{U}_{\mathscr{B}_2}$. Thus $\mathscr{U}_{\mathscr{B}_1} \subset \mathscr{U}_{\mathscr{B}_2}$. Similarly, using (2), $\mathscr{U}_{\mathscr{B}_2} \subset \mathscr{U}_{\mathscr{B}_1}$, and so $\mathscr{U}_{\mathscr{B}_1} = \mathscr{U}_{\mathscr{B}_2}$. □