

PURE AND APPLIED MATHEMATICS

A Series of Monographs and Textbooks

INTRODUCTION TO FUNCTIONS OF A COMPLEX VARIABLE

J. H. Curtiss

Introduction to Functions of a Complex Variable

J. H. Curtiss

Department of Mathematics
University of Florida
Coral Gables, Florida

MARCEL DEKKER, INC. New York and Basel

Library of Congress Cataloging in Publication Data

Curtiss, John Hamilton, 1909-1977.

Introduction to functions of a complex variable.

(Pure and applied mathematics ; v. 44)

Includes bibliographical references and index.

1. Functions of complex variables. I. Title.

QA331.C84 1978 515'.9 77-25846

ISBN 0-8247-6501-X

COPYRIGHT © 1978 by MARCEL DEKKER, INC. ALL RIGHTS RESERVED

Neither this book nor any part may be reproduced or transmitted in any form or by any means, electronic or mechanical, including photocopying, microfilming, and recording, or by any information storage and retrieval system, without permission in writing from the publisher.

MARCEL DEKKER, INC.

270 Madison Avenue, New York, New York 10016

Current printing (last digit):

10 9 8 7 6 5 4 3 2 1

PRINTED IN THE UNITED STATES OF AMERICA

MONOGRAPHS AND TEXTBOOKS IN PURE AND APPLIED MATHEMATICS

1. *K. Yano*, Integral Formulas in Riemannian Geometry (1970)
2. *S. Kobayashi*, Hyperbolic Manifolds and Holomorphic Mappings (1970)
3. *V. S. Vladimirov*, Equations of Mathematical Physics (A. Jeffrey, editor; A. Littlewood, translator) (1970)
4. *B. N. Pshenichnyi*, Necessary Conditions for an Extremum (L. Neustadt, translation editor; K. Makowski, translator) (1971)
5. *L. Narici, E. Beckenstein, and G. Bachman*, Functional Analysis and Valuation Theory (1971)
6. *D. S. Passman*, Infinite Group Rings (1971)
7. *L. Dornhoff*, Group Representation Theory (in two parts). Part A: Ordinary Representation Theory. Part B: Modular Representation Theory (1971, 1972)
8. *W. Boothby and G. L. Weiss (eds.)*, Symmetric Spaces: Short Courses Presented at Washington University (1972)
9. *Y. Matsushima*, Differentiable Manifolds (E. T. Kobayashi, translator) (1972)
10. *L. E. Ward, Jr.*, Topology: An Outline for a First Course (1972) *out of print*
11. *A. Babakhanian*, Cohomological Methods in Group Theory (1972)
12. *R. Gilmer*, Multiplicative Ideal Theory (1972)
13. *J. Yeh*, Stochastic Processes and the Wiener Integral (1973) *out of print*
14. *J. Barros-Neto*, Introduction to the Theory of Distributions (1973) *out of print*
15. *R. Larsen*, Functional Analysis: An Introduction (1973)
16. *K. Yano and S. Ishihara*, Tangent and Cotangent Bundles: Differential Geometry (1973)
17. *C. Procesi*, Rings with Polynomial Identities (1973)
18. *R. Hermann*, Geometry, Physics, and Systems (1973)
19. *N. R. Wallach*, Harmonic Analysis on Homogeneous Spaces (1973)
20. *J. Dieudonné*, Introduction to the Theory of Formal Groups (1973)

21. *I. Vaisman*, Cohomology and Differential Forms (1973)
22. *B.-Y. Chen*, Geometry of Submanifolds (1973)
23. *M. Marcus*, Finite Dimensional Multilinear Algebra (in two parts) (1973, 1975)
24. *R. Larsen*, Banach Algebras: An Introduction (1973)
25. *R. O. Kujala and A. L. Vitter (eds.)*, Value Distribution Theory: Part A; Part B. Deficit and Bezout Estimates by Wilhelm Stoll (1973)
26. *K. B. Stolarsky*, Algebraic Numbers and Diophantine Approximation (1974)
27. *A. R. Magid*, The Separable Galois Theory of Commutative Rings (1974)
28. *B. R. McDonald*, Finite Rings with Identity (1974)
29. *J. Satake*, Linear Algebra (S. Koh, T. Akiba, and S. Ihara, translators) (1975)
30. *J. S. Golan*, Localization of Noncommutative Rings (1975)
31. *G. Klambauer*, Mathematical Analysis (1975)
32. *M. K. Agoston*, Algebraic Topology: A First Course (1976)
33. *K. R. Goodearl*, Ring Theory: Nonsingular Rings and Modules (1976)
34. *L. E. Mansfield*, Linear Algebra with Geometric Applications (1976)
35. *N. J. Pullman*, Matrix Theory and its Applications: Selected Topics (1976)
36. *B. R. McDonald*, Geometric Algebra Over Local Rings (1976)
37. *C. W. Groetsch*, Generalized Inverses of Linear Operators: Representation and Approximation (1977)
38. *J. E. Kuczowski and J. L. Gersting*, Abstract Algebra: A First Look (1977)
39. *C. O. Christenson and W. L. Voxman*, Aspects of Topology (1977)
40. *M. Nagata*, Field Theory (1977)
41. *R. L. Long*, Algebraic Number Theory (1977)
42. *W. F. Pfeffer*, Integrals and Measures (1977)
43. *R. L. Wheeden and A. Zygmund*, Measure and Integral: An Introduction to Real Analysis (1977)
44. *J. H. Curtiss*, Introduction to Functions of a Complex Variable (1978)

*To Joseph Leonard Walsh,
in memorium*

Foreword

On several occasions, John Curtiss told me of his plans and aspirations for the present volume. It was to be thoroughly modern in approach, featuring a balance of abstract concepts on the one hand and practical applications on the other; relaxed and friendly, but carefully logical, in style; analytically rigorous and independent of geometric intuition, yet with many geometric illustrations and applications; and with a great wealth of examples and exercises. Though intended for an introductory course at the advanced undergraduate and first-year graduate level, it was also to provide doctoral students at any university with a sound preparation for qualifying examinations in complex analysis. For a final, sentimental objective, the book was to expand on his father's brief but still popular Carus Monograph on analytic functions of a complex variable.

John Curtiss was well qualified to write this book. Besides serving terms as Chief of the Applied Mathematics Division of the National Bureau of Standards and as Executive Director of the American Mathematical Society, he held teaching positions at Johns Hopkins University, Cornell University, the Courant Institute of Mathematical Sciences of New York University, and the University of Miami. Much of his original research work involved complex-variable theory; he taught courses on the subject quite often, and his notes for the book developed during many years of experience.

In reviewing the manuscript and in reading proof sheets of the book for

typographical errors, I have been pleased to observe that, in my opinion, in each of his aims for the book he has succeeded admirably.

E. F. Beckenbach

Preface

This book is intended for an introductory course in complex analysis at the first-year graduate and advanced undergraduate level. The program of the book will be revealed to some extent to the instructor experienced in the subject by a mere scanning of the table of contents, and no amount of prefatory remarks would be illuminating to most beginners. However, certain highlights of the treatment will be examined here.

It has often been observed that students who enroll in such a course have diverse mathematical backgrounds. Partly for this reason, it is traditional for instructional materials for the course to be self-contained to a considerable degree. This tradition has been observed in the present book. Only some knowledge of the structure of the real number system and some familiarity with elementary calculus are assumed.

Perhaps no other course at a comparable level in the standard mathematics curriculum has been presented with such wide and erratic variations in logical completeness or "rigor." This book falls at the rigorous end of the spectrum. In fact, the intention was to achieve (in some cases to exceed) the same high level of logical completeness which characterizes, for example, the excellent texts on the subject by Ahlfors [2], Heins [8], and Rudin [18]; but with a more relaxed, detailed, and accessible treatment. It goes without saying that to keep the book within reasonable physical dimensions, such a program entailed some sacrifices in the extent of coverage.

The author taught complex analysis for many years, using a variety of successful textbooks. This experience resulted in the compilation of a rather

extensive set of lecture notes, from which this book was distilled. Naturally the lectures were influenced by the textbooks, especially those of Ahlfors [2], Hille [9, 10], Rudin [18], and Saks and Zygmund [19]. It is a pleasure to acknowledge the author's debt to these distinguished mathematicians. Some of the proofs in the last two chapters seem to be new. This applies particularly to the Appendix of Chapter 13.

It has been the author's impression that even some of the more competent mathematics majors find their first rigorous course in complex analysis to be difficult, at least at the outset. Therefore the exposition in this book has been paced so as to start out quite slowly and to gather momentum and depth gradually. In fact, the first five chapters, which include a chapter on elementary general topology, will be viewed by some instructors and readers as only preparatory, to be covered rapidly in the course if at all. But even the well-prepared student may find the frequent appearance of back-references to these earlier chapters helpful.

The central theorem of complex analysis is the Cauchy Integral Theorem, which appears first in Chapter 9. In its most general forms the theorem is deep and difficult to prove. However, for a function analytic in a convex or starlike region a straightforward proof can easily be given. Furthermore, even with the validation of the theorem so restricted, a large number of the famous classical results of complex function theory can be usefully derived. This fact has been exploited in this book, partly with a view, as mentioned above, to keeping things simple as long as possible. All of the theorems and formulas in Chapters 9, 10, and 11 (and these are often considered to be the core of classical complex analysis) are obtained from the Cauchy Integral Theorem for a starlike region. Generalizations of the Cauchy theory to cases in which the paths of integration lie in arbitrary open sets are achieved in Chapter 12 by means of Runge's theorem on approximation by rational functions. The concepts of homology and homotopy in their application to the Cauchy theory are deferred until after the Cauchy Integral Theorem has been established in a very general form. Important as these topological concepts are for a full understanding of complex integration theory, the author has found that it tends to be diversionary to introduce them before the student has become acquainted with some of the basic classical results.

Nowadays in the undergraduate mathematics curriculum a considerable amount of emphasis is given to abstract concepts. This has been duly recognized here by presenting a number of the elementary definitions in reasonably abstract contexts. Also, the reader will find various passages in the earlier chapters where deference is shown to the basics of functional analysis.

A secondary theme relating to the important role of approximation theory in analysis runs through the book, as exemplified by the early introduction of power series in Chapter 4, the extensive exposure given to the Runge

theorem and its polynomial specializations, and the applications of conformal mapping to polynomial approximations in the last chapter.

For better or for worse, the intuitive geometric aspects of the subject are not strongly emphasized. However, the geometric nature of mappings by complex-valued functions is given an early presentation in exercises in Chapter 1; it is touched on in various ways in the sequel (e.g., the Open Mapping Theorem, Section 10.4); and it is studied at some length theoretically in the concluding chapter on conformal mapping.

The principal statements in this book are numbered consecutively in each section with a triple numbering system which indicates at once the particular chapter, section, and statement in the section. The reader should have no difficulty in understanding the code. The statements are variously labeled lemma, proposition, theorem, or corollary. A theorem is a statement of primary importance which occupies a central position in the theory. A proposition is a statement of some interest for itself alone, but which occupies a more peripheral position than a theorem in the mainstream of the exposition. A lemma is a portion of a proof which for convenience in reference has been set out formally.

There are several hundred exercises which are placed at the ends of the sections. Some of them are meant merely for drill, but many of them contain important extensions and developments of the theory. Exercises which contain propositions cited elsewhere as authorities are indicated by boldface numbers.

It is the author's pleasant duty to give thanks to two friends who were particularly helpful in bringing this book into final form. The first of these is Professor E. F. Beckenbach, who was requested by the publisher to review the first draft of the manuscript. He made many constructive suggestions bearing on both substance and style, and almost all of these were used. The other friend is my colleague Professor Edwin Duda, whose expertise in point-set topology contributed in a significant way to the accuracy of Chapter 5 and of the topological parts of Chapter 13. It goes almost without saying that any errors which may remain are the author's own responsibility. Finally, thanks are due to Mrs. Mamie Cummings of the Mathematics Department of the University of Miami, who overcame the distraction of a multitude of revisions to produce a skillfully typed manuscript.

J. H. Curtiss

Contents

Foreword	v
Preface	vii
1 The Real and Complex Number Fields	1
1.1 Introduction: Set Theoretic Notation	1
1.2 Fields	2
1.3 Definitions Relating to Ordered Fields	5
1.4 Definitions Relating to Boundedness	6
1.5 The Real Number System	7
1.6 The Extended Real Numbers	7
1.7 Functions	8
1.8 The Complex Number Field	10
1.9 Relationships of \mathbf{C} with \mathbf{R}^2 and \mathbf{R}	11
1.10 The Imaginary Unit	12
1.11 Real and Imaginary Parts; Conjugation	12
1.12 Absolute Value	13
1.13 Inequalities Involving Absolute Value	14
1.14 Geometric Representation of Complex Numbers	16
1.15 The Extended Complex Plane and Stereographic Projection	20
1.16 The Linear Fractional Transformations	24

2	Sequences and Series	29
2.1	Basic Definitions	29
2.2	Metric Spaces	30
2.3	Convergence of Sequences	31
2.4	Sequences of Reals	32
2.5	Sequences of Complex Numbers	36
2.6	Infinite Series in \mathbb{C}	38
2.7	Algebraic Operations on Infinite Series	44
3	Sequences and Series of Complex-Valued Functions	49
3.1	Pointwise Convergence	49
3.2	Bounded and Continuous Functions	51
3.3	Uniform Convergence of Sequences of Functions	53
3.4	Infinite Series of Functions	57
4	Introduction to Power Series	63
4.1	Radius of Convergence	63
4.2	Continuity Properties of the Sum Function of a Power Series	68
4.3	Uniqueness of the Power Series Representation	72
5	Some Elementary Topological Concepts	77
5.1	Basic Definitions	77
5.2	Continuity and Functional Limits	80
5.3	Connected Sets	83
5.4	Components	88
5.5	Compactness	91
5.6	Distance Between Sets; Diameter	96
6	Complex Differential Calculus	101
6.1	The Derivative of a Complex-valued Function	101
6.2	Analytic Functions	104
6.3	The Cauchy-Riemann Equations	107
6.4	Analyticity of Power Series	113
6.5	Functions Locally Representable by Power Series	116
7	The Exponential and Related Functions	121
7.1	The Exponential Function	121
7.2	The Trigonometric Functions	123

Contents

7.3	The Mapping Given by $w = e^{iy}$	126
7.4	The Periods of e^z and of the Trigonometric Functions	130
7.5	Logarithm and Argument	132
7.6	The General Exponential Function	136
7.7	Continuity of Arg and Log; Analyticity of Log	137
8	Complex Line Integrals	141
8.1	The Riemann Integral of a Complex-valued Function	141
8.2	Complex Line Integrals	143
8.3	Integration over Paths	146
8.4	A Special Type of Complex Line Integral	152
8.5	Primitives and Integration over Paths	157
9	Introduction to the Cauchy Theory	163
9.1	General Remarks	163
9.2	The Cauchy-Goursat Theorem for a Triangle	164
9.3	Some Extensions of the Cauchy Integral Theorem	167
9.4	The Cauchy Integral Formula and the LPS Property of Analytic Functions	170
9.5	The Gauss Mean Value Theorem and the Cauchy Estimates	179
9.6	Liouville's Theorem and the Fundamental Theorem of Algebra	180
9.7	The Maximum Modulus Principle	184
9.8	Schwarz's Lemma	187
9.9	Almost Uniform Convergence of Sequences of Analytic Functions	191
9.10	Introduction to the Laplace Transform	195
9.11	Evaluation of Improper Real Integrals by Using the Cauchy Integral Theorem	201
10	Zeros and Isolated Singularities of Analytic Functions	207
10.1	A Factorization of Analytic Functions	207
10.2	Integrals Related to the Set of Zeros of an Analytic Function	210
10.3	Rouché's Theorem with Applications	212
10.4	The Open Mapping Theorem	215
10.5	Isolated Singularities	219
10.6	Laurent Series	227
10.7	Zeros and Singularities at ∞	235

11	Residues and Rational Functions	247
11.1	The Residue Theorem	247
11.2	A General Cauchy Integral Theorem for Rational Functions	250
11.3	The Principle of the Argument and Jensen's Formula	252
11.4	Evaluation of Definite Integrals by Residues	259
12	Approximation of Analytic Functions by Rational Functions, and Generalizations of the Cauchy Theory	269
12.1	General Remarks	269
12.2	A Cauchy-type Integral Representation of a Function Analytic on a Compact Set	270
12.3	Approximation of a Rational Function by Another Rational Function	275
12.4	Approximation of a Function Analytic on a Compact Set by a Rational Function with Poles on a Prescribed Set	280
12.5	Almost Uniform Approximation of a Function Analytic on an Open Set by Rational Functions	285
12.6	Generalized Formulations in the Cauchy Theory	287
12.7	Homology, Homotopy, and a Generalization of the Complex Line Integral	292
12.8	Characterizations of a Simply Connected Region	297
13	Conformal Mapping	301
13.1	The Conformal Property of the Mapping Given by an Analytic Function	301
13.2	Conformal Equivalences and a Review of Some Relevant Results in Earlier Chapters	305
13.3	The Stieltjes-Vitali Convergence Theorem and Normal Families	308
13.4	The Riemann Mapping Theorem	312
13.5	Level Curves	319
13.6	Introduction to the Classes \mathcal{S} and \mathcal{U}	326
13.7	Continuation of Analytic Functions and Conformal Equivalences Onto and Across Boundaries	336
13.8	Applications of Conformal Mapping to the Approximation of Analytic Functions by Polynomials	345
13.9	Continuation of Section 13.8; Interpolation in the Fekete Points and in the Fejér Points	353

Contents

	xv
Appendix to Chapter 13	366
Some Notation Listed by Chapter and Section Where They First Appear	379
References	387
Index	389

The Real and Complex Number Fields

1.1 Introduction: Set Theoretic Notation

The basic language of mathematics nowadays is that of set theory. It is safe to assume that the reader is acquainted with the fundamental intuitive ideas of set theory. The purpose here is to summarize the basic notation and terminology to be used in this book.

Given a set E , we write $x \in E$ to express the statement “ x is an element (or member) of E .” If x is not an element of E , we write $x \notin E$. A set is completely determined by its members, so if sets E and F have the property that $x \in E$ if and only if $x \in F$, then E and F are the same set, and we write $E = F$.

If the sets E and F have the property that $x \in E$ implies $x \in F$, then E is called a *subset* of F , and we write $E \subset F$ or $F \supset E$. In this situation, we shall also say that E is *contained* in F , or more colloquially, E *lies in* F .

There are essentially two ways of specifying a set. The first consists of listing its elements or displaying a partial list which indicates a pattern; the notation used is typified by $\{a, b, c\}$ and $\{1, 2, 3, \dots\}$. This is sometimes called the census method. The other way, and the one most frequently employed in mathematics, consists of announcing a condition which the elements must satisfy, or a property or properties which each element must have. A set of elements with property P is denoted by $\{x: P(x)\}$. For example, if X is the set of positive real integers, then the subset $\{1, 2, \dots, 10\}$ can be denoted by $\{x: x \in X, x \leq 10\}$.

In using the census method for specifying a set, it is understood that any permutation of the elements in the list still represents the same set; for ex-