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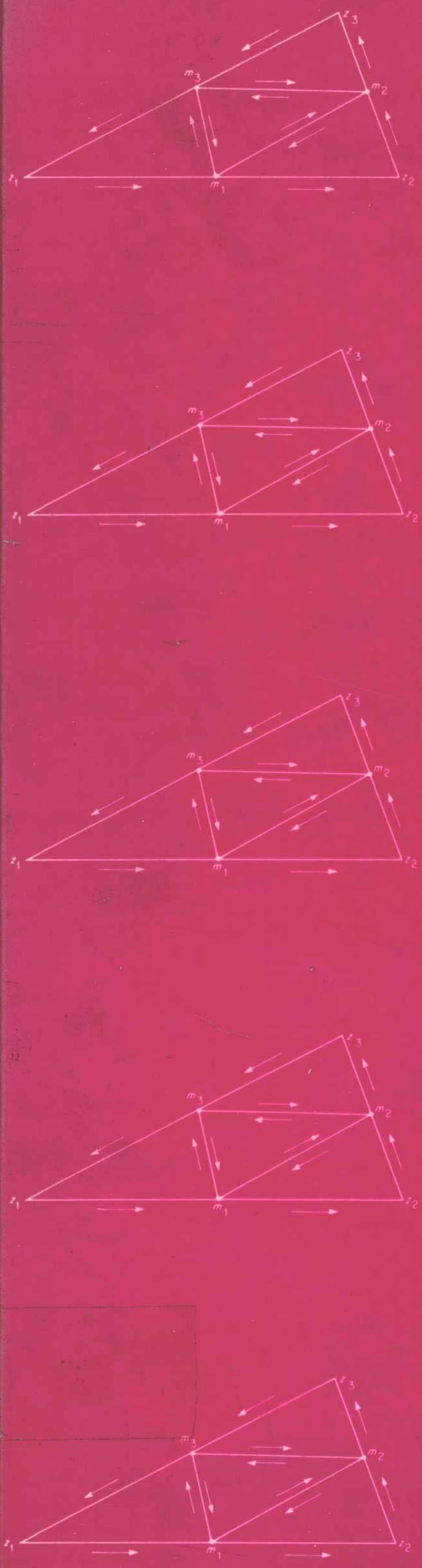

INTRODUCTION TO FUNCTIONS OF A COMPLEX VARIABLE

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MARCEL DEKKER, INC.   New York and Basel

Library of Congress Cataloging in Publication Data

Curtiss, John Hamilton, 1909-1977.
Introduction to functions of a complex variable.

(Pure and applied mathematics ; v. 44)
Includes bibliographical references and index.
1. Functions of complex variables. I. Title.
QA331.C84 1978 515'.9 77-25846
ISBN 0-8247-6501-X

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

Current printing (last digit):
10 9 8 7 6 5 4 3 2 1

PRINTED IN THE UNITED STATES OF AMERICA

MONOGRAPHS AND TEXTBOOKS IN PURE AND APPLIED MATHEMATICS

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To Joseph Leonard Walsh,
in memorium

Foreword

On several occasions, John Curtiss told me of his plans and aspirations for the present volume. It was to be thoroughly modern in approach, featuring a balance of abstract concepts on the one hand and practical applications on the other; relaxed and friendly, but carefully logical, in style; analytically rigorous and independent of geometric intuition, yet with many geometric illustrations and applications; and with a great wealth of examples and exercises. Though intended for an introductory course at the advanced undergraduate and first-year graduate level, it was also to provide doctoral students at any university with a sound preparation for qualifying examinations in complex analysis. For a final, sentimental objective, the book was to expand on his father's brief but still popular Carus Monograph on analytic functions of a complex variable.

John Curtiss was well qualified to write this book. Besides serving terms as Chief of the Applied Mathematics Division of the National Bureau of Standards and as Executive Director of the American Mathematical Society, he held teaching positions at Johns Hopkins University, Cornell University, the Courant Institute of Mathematical Sciences of New York University, and the University of Miami. Much of his original research work involved complex-variable theory; he taught courses on the subject quite often, and his notes for the book developed during many years of experience.

In reviewing the manuscript and in reading proof sheets of the book for typographical errors, I have been pleased to observe that, in my opinion, in each of his aims for the book he has succeeded admirably.

E. F. Beckenbach

Preface

This book is intended for an introductory course in complex analysis at the first-year graduate and advanced undergraduate level. The program of the book will be revealed to some extent to the instructor experienced in the subject by a mere scanning of the table of contents, and no amount of prefatory remarks would be illuminating to most beginners. However, certain highlights of the treatment will be examined here.

It has often been observed that students who enroll in such a course have diverse mathematical backgrounds. Partly for this reason, it is traditional for instructional materials for the course to be self-contained to a considerable degree. This tradition has been observed in the present book. Only some knowledge of the structure of the real number system and some familiarity with elementary calculus are assumed.

Perhaps no other course at a comparable level in the standard mathematics curriculum has been presented with such wide and erratic variations in logical completeness or "rigor." This book falls at the rigorous end of the spectrum. In fact, the intention was to achieve (in some cases to exceed) the same high level of logical completeness which characterizes, for example, the excellent texts on the subject by Ahlfors [2], Heins [8], and Rudin [18]; but with a more relaxed, detailed, and accessible treatment. It goes without saying that to keep the book within reasonable physical dimensions, such a program entailed some sacrifices in the extent of coverage.

The author taught complex analysis for many years, using a variety of successful textbooks. This experience resulted in the compilation of a rather extensive set of lecture notes, from which this book was distilled. Naturally the lectures were influenced by the textbooks, especially those of Ahlfors [2], Hille [9, 10], Rudin [18], and Saks and Zygmund [19]. It is a pleasure to acknowledge the author's debt to these distinguished mathematicians. Some of the proofs in the last two chapters seem to be new. This applies particularly to the Appendix of Chapter 13.

It has been the author's impression that even some of the more competent mathematics majors find their first rigorous course in complex analysis to be difficult, at least at the outset. Therefore the exposition in this book has been paced so as to start out quite slowly and to gather momentum and depth gradually. In fact, the first five chapters, which include a chapter on elementary general topology, will be viewed by some instructors and readers as only preparatory, to be covered rapidly in the course if at all. But even the well-prepared student may find the frequent appearance of back-references to these earlier chapters helpful.

The central theorem of complex analysis is the Cauchy Integral Theorem, which appears first in Chapter 9. In its most general forms the theorem is deep and difficult to prove. However, for a function analytic in a convex or starlike region a straightforward proof can easily be given. Furthermore, even with the validation of the theorem so restricted, a large number of the famous classical results of complex function theory can be usefully derived. This fact has been exploited in this book, partly with a view, as mentioned above, to keeping things simple as long as possible. All of the theorems and formulas in Chapters 9, 10, and 11 (and these are often considered to be the core of classical complex analysis) are obtained from the Cauchy Integral Theorem for a starlike region. Generalizations of the Cauchy theory to cases in which the paths of integration lie in arbitrary open sets are achieved in Chapter 12 by means of Runge's theorem on approximation by rational functions. The concepts of homology and homotopy in their application to the Cauchy theory are deferred until after the Cauchy Integral Theorem has been established in a very general form. Important as these topological concepts are for a full understanding of complex integration theory, the author has found that it tends to be diversionary to introduce them before the student has become acquainted with some of the basic classical results.

Nowadays in the undergraduate mathematics curriculum a considerable amount of emphasis is given to abstract concepts. This has been duly recognized here by presenting a number of the elementary definitions in reasonably abstract contexts. Also, the reader will find various passages in the earlier chapters where deference is shown to the basics of functional analysis.

A secondary theme relating to the important role of approximation theory in analysis runs through the book, as exemplified by the early introduction of power series in Chapter 4, the extensive exposure given to the Runge theorem and its polynomial specializations, and the applications of conformal mapping to polynomial approximations in the last chapter.

For better or for worse, the intuitive geometric aspects of the subject are not strongly emphasized. However, the geometric nature of mappings by complex-valued functions is given an early presentation in exercises in Chapter 1; it is touched on in various ways in the sequel (e.g., the Open Mapping Theorem, Section 10.4); and it is studied at some length theoretically in the concluding chapter on conformal mapping.

The principal statements in this book are numbered consecutively in each section with a triple numbering system which indicates at once the particular chapter, section, and statement in the section. The reader should have no difficulty in understanding the code. The statements are variously labeled lemma, proposition, theorem, or corollary. A theorem is a statement of primary importance which occupies a central position in the theory. A proposition is a statement of some interest for itself alone, but which occupies a more peripheral position than a theorem in the mainstream of the exposition. A lemma is a portion of a proof which for convenience in reference has been set out formally.

There are several hundred exercises which are placed at the ends of the sections. Some of them are meant merely for drill, but many of them contain important extensions and developments of the theory. Exercises which contain propositions cited elsewhere as authorities are indicated by boldface numbers.

It is the author's pleasant duty to give thanks to two friends who were particularly helpful in bringing this book into final form. The first of these is Professor E. F. Beckenbach, who was requested by the publisher to review the first draft of the manuscript. He made many constructive suggestions bearing on both substance and style, and almost all of these were used. The other friend is my colleague Professor Edwin Duda, whose expertise in point-set topology contributed in a significant way to the accuracy of Chapter 5 and of the topological parts of Chapter 13. It goes almost without saying that any errors which may remain are the author's own responsibility. Finally, thanks are due to Mrs. Mamie Cummings of the Mathematics Department of the University of Miami, who overcame the distraction of a multitude of revisions to produce a skillfully typed manuscript.

J. H. Curtiss

Contents

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The Real and Complex Number Fields

1.1 Introduction: Set Theoretic Notation

The basic language of mathematics nowadays is that of set theory. It is safe to assume that the reader is acquainted with the fundamental intuitive ideas of set theory. The purpose here is to summarize the basic notation and terminology to be used in this book.

Given a set E, we write $x \in E$ to express the statement "x is an element (or member) of E." If x is not an element of E, we write $x \notin E$. A set is completely determined by its members, so if sets E and F have the property that $x \in E$ if and only if $x \in F$, then E and F are the same set, and we write $E = F$.

If the sets E and F have the property that $x \in E$ implies $x \in F$, then E is called a *subset* of F, and we write $E \subset F$ or $F \supset E$. In this situation, we shall also say that E is *contained* in F, or more colloquially, *E lies in F.*

There are essentially two ways of specifying a set. The first consists of listing its elements or displaying a partial list which indicates a pattern; the notation used is typified by $\{a, b, c\}$ and $\{1, 2, 3, \ldots\}$. This is sometimes called the census method. The other way, and the one most frequently employed in mathematics, consists of announcing a condition which the elements must satisfy, or a property or properties which each element must have. A set of elements with property P is denoted by $\{x\colon P(x)\}$. For example, if X is the set of positive real integers, then the subset $\{1, 2, \ldots, 10\}$ can be denoted by $\{x\colon x \in X, x \leq 10\}$.

In using the census method for specifying a set, it is understood that any permutation of the elements in the list still represents the same set; for ex-