# Geometrical Constructions with compasses only

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ПОПУЛЯРНЫЕ ЛЕКЦИИ ПО МАТЕМАТИКЕ

А. Н. Костовский

# ГЕОМЕТРИЧЕСКИЕ ПОСТРОЕНИЯ ОДНИМ ЦИРКУЛЕМ

ИЗДАТЕЛЬСТВО «НАУКА» МОСКВА

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# GEOMETRICAL CONSTRUCTIONS WITH COMPASSES ONLY

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*На английском языке*

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# PREFACE

The author of this book has often given lectures on the theory of geometrical constructions to participants in mathematical olympiads, which have been organized every year since 1947, for the pupils of secondary schools in the city of Lvov. The first chapter of this work is based on these lectures.

The second chapter describes the investigations made by the author in connection with geometrical constructions carried out by compasses alone with a bounded opening of its legs.

This book is designed for a wide circle of readers. It should help teachers and pupils of senior classes of secondary schools to acquaint themselves in greater detail with geometrical constructions carried out by compasses alone. This work can serve as a teaching aid in school mathematical clubs. It can also be used by students of physical and mathematical departments of universities and teachers' training colleges to deepen their knowledge of elementary mathematics.

The author would like to express his sincere gratitude to professor A. N. Kovan'ko, assistant professors V. F. Rogachenko and I. F. Teslenko, and to experienced teacher B. G. Orach for reading the manuscript and offering a great deal of valuable advice.

# INTRODUCTION

Geometrical constructions form a substantial part of a mathematical education. They represent a powerful tool of geometrical investigations.

The tradition of limiting the tools of geometrical constructions to a ruler and compasses goes back to remote antiquity. The famous geometry of Euclid (3rd century B.C.) was based on geometrical constructions carried out using a pair of compasses and a ruler, the compasses and ruler being regarded as equivalent instruments; it did not matter whether the construction was carried out with a pair of compasses alone, or with a ruler alone, or with both a pair of compasses and a ruler.

It was noted a long time ago that compasses are a more precise tool than rulers. Certain constructions could be carried out by compasses, without using a ruler; for example, dividing a circumference into six equal parts, constructing a point symmetric to a given point with respect to a given straight line, and so on. Attention was drawn to the fact that in engraving thin metal plates, in marking out indexing dials of astronomical instruments only compasses are used. This, probably, was the stimulus for investigations into geometrical constructions that could be carried out by compasses alone.

In 1797 the Italian mathematician Lorenzo Mascheroni, a professor of the University of Pavia, published an extensive tract called *The Geometry of Compasses*, which was later translated into French and German. In the tract he proved the proposition that

*All construction problems solvable by means of a pair of compasses and a ruler can also be solved exactly by a pair of compasses alone.*

This statement was proved in 1890 by Adler in an original way, using inversion. He also proposed a general method of solving geometrical construction problems by means of compasses alone. In 1928 the Danish mathematician Hjelmslev discovered in a bookshop in Copenhagen a book by G. Mohr titled *The Danish Euclid* and published in 1672 in Amsterdam. In the first part of the book there was a complete solution of Mascheroni's problem. Thus, it had been shown a long time before Mascheroni that all geometrical constructions capable of being carried out by a pair of compasses and a ruler can also be carried out by a pair of compasses alone.

The branch of geometry dealing with constructions which can be completed using a pair of compasses alone is called the *geometry of the compasses.*

In 1833 the Swiss geometer Jacob Steiner published a book called *Geometrical Constructions Using a Straight Line and a Fixed Circle*, in which he investigated constructions carried out by a ruler alone. His basic result can be formulated as follows:

*Every construction problem solvable by compasses and a ruler can also be solved by using a ruler alone given a circle with fixed centre and radius in the plane of the drawing.*

Thus, in order to make the ruler equivalent to the compasses, it is sufficient to use a pair of compasses once.

The Russian mathematician Lobachevskii introduced in the early 19th century a new geometry which later became known as non-Euclidean or Lobachevskian geometry. Recently, thanks to the efforts of many scholars, especially Soviet ones, the theory of geometrical constructions in Lobachevskian plane has been rapidly developed.

A. S. Smogorzhevskii, V. F. Rogachenko, K. K. Mokrishchev, among others, have investigated constructions in the Lobachevskian plane without a ruler and shown the possibility of executing constructions similar to those of Mascheroni in the Euclidean plane.

Soviet scientists have now completely and rigorously formulated a theory for the geometrical constructions in the Lobachevskian plane, a theory as complete as the theory of geometrical constructions in the Euclidean plane.

# 1. CONSTRUCTIONS BY COMPASSES ONLY

## Sec. 1. On the Possibility of Solving Geometrical Construction Problems by Compasses Only. The Basic Theorem

In this section we give the proof of the basic theorem of the geometry of compasses for which purpose it is necessary to examine the solutions of problems on construction by compasses alone.

It is clear that we cannot draw a continuous straight line given by two points using only compasses, although as will be shown later we can construct one, two, and, generally, any number of points on a given straight line*. Thus, the Mohr-Mascheroni theory does not cover the entire construction of a straight line.

In the geometry of compasses, a straight line or a segment is defined by two points and is not considered a continuous straight line (drawn with a ruler). *The construction of a straight line is said to be completed if any two of its points are constructed.*

We introduce the notation.

$(AB)$ is a straight line passing through points $A$ and $B$,

$[AB]$ is a segment $AB$,

$|AB|$ is the distance between the points $A$ and $B$,

$(O, r)$ is a circumference (or a circle) with centre $O$ and radius $r$,

$(A, |BC|)$ is a circumference (or a circle) with centre $A$ and radius $r = |BC|$.

Let us agree to write the phrase "With point $O$ as centre and radius $r$ we describe a circle (or draw an arc)" in the short form: "We describe (or draw) the circle $(O, r)$", or sometimes still shorter: "We describe $(O, r)$". The phrase "The segment $AB$, where $|AB| = a$" we consider to be equivalent to "The segment $a$", and, accordingly, if $|CD| = n|AB|$ we may say that the segment $CD$ is $n$ times as great as the segment $AB$.

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* From the practical point of view, there is no ground to regard a straight line as constructed if some of its points are constructed.

Other symbols and notation used in this book are given in Appendix 1.

**Problem 1.** Construct a point symmetric to a given point $C$ with respect to a given straight line $AB$.

Given $(AB)$ and point $C$. Construct $C_1 = \mathrm{S}_{(AB)}(C)$*.

*Construction.* We describe the circles $(A, |AC|)$ and $(B, |BC|)$ which intersect at point $C_1$ (Fig. 1). The point $C_1$ is the required point.

If the point $C$ lies on the straight line $AB$, then it is symmetric to itself [i.e. $C = \mathrm{S}_{(AB)}(C)$].

*Note.* To verify that three given points $A$, $B$, and $X$ lie on the same straight line, it is necessary to construct any point

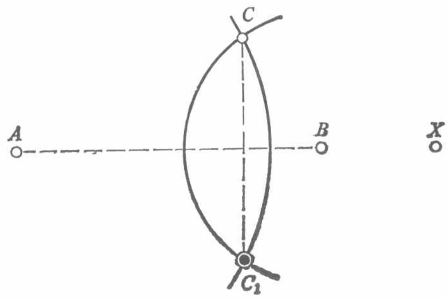



Fig. 1

$C$ outside the straight line $AB$ and then the point $C_1$ symmetric to $C$. Obviously, the point $X$ lies on the straight line $AB$ if and only if $|CX| = |C_1X|$.

**Problem 2.** Construct a segment 2, 3, 4, . . ., and in general $n$ times as great as a given segment $AA_1$ ($n$ is any natural number, $n \in \mathbf{N}$).

Given $|AA_1| = r$. Construct $[AA_n]$, $|AA_n| = n\,|AA_1|$, where $n \in \mathbf{N}$.

*Construction* (1st method). Keeping the opening of the compasses constant and equal to $r$, we describe the circle $(A_1, r)$. Then we construct the point $A_2$ diametrically opposite to the point $A$, for which purpose we describe the circles $(A, r)$, $(B, r)$ and $(C, r)$, at the intersection points of

* See Appendix 1.

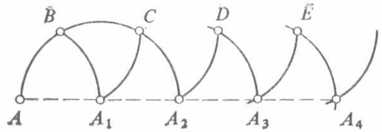



Fig. 2

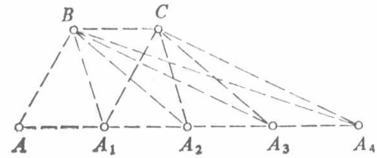



**Fig. 3**

these circles with the circle $(A_1, r)$ we obtain the points $B$, $C$, and $A_2$. The segment $|AA_2| = 2r$ (Fig. 2).

Now we describe the circle $(A_2, r)$ which intersects the circle $(C, r)$ at the point $D$. At the intersection point of the circles $(A_2, r)$ and $(D, r)$ we obtain the point $A_3$. The segment $|AA_3| = 3r$, and so on.

Having carried out the above construction $n$ times, we get the segment $|AA_n| = nr$.

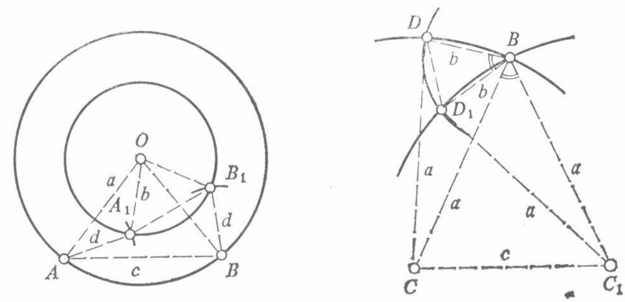
The validity of the result follows from the fact that compasses with an opening equal to the radius of a circle divide its circumference into six equal parts.

*Construction* (2nd method). We take an arbitrary point $B$ outside the straight line $AA_1$ and draw the circles $(A_1, |AB|)$ and $(B, r)$ which intersect at the point $C$ (Fig. 3). If the circles $(A_1, r)$ and $(C, |A_1B|)$ are drawn, they will intersect at the sought-for point $A_2$. The segment $|AA_2| = 2r$. We describe the circles $(A_2, r)$ and $(C, |A_2B|)$ and denote their intersection point by $A_3$. Here $|AA_3| = 3r$, and so on.

The validity of the result follows immediately from the fact that the figures $ABCA_1$, $A_1BCA_2$, $A_2BCA_3$, . . . are parallelograms.

*Note.* It is also easy to construct segments 2, 4, 8, 16, . . . . . ., $2^k$ times as great as the given segment $AA_1$. To this end we describe the circle $(A_1, r)$ and find the point $A_2$ diametrically opposite to the point $A$ ($|AA_2| = 2r$), describe the circle $(A_2, 2r)$ and find the point $A_4$ diametrically opposite to the point $A$ ($|AA_4| = 4r$). The point $A_8$, which is diametrically opposite to the point $A$ on the circle $(A_4, 4r)$, defines $|AA_8| = 8r$, and so on. After $k$ steps we obtain $|AA_{2^k}| = 2^k r$*.

**Problem 3.** Construct a segment whose value $x$ is the extreme term of the proportion $a/b = c/x$, where $a$, $b$, and $c$ are the values of the given segments.




*a* *b*

Fig. 4

Given the segments $a$, $b$, and $c$. Construct a segment $x$ such that $a/b = c/x$.

*Construction* (1st method). We take an arbitrary point $O$ and describe two circles $(O, a)$ and $(O, b)$. With an arbitrary point $A$ on the circle $(O, a)$ as the centre we describe $(A, c)$ and denote the intersection point of these circles by $B$. Now we describe two circles $(A, d)$ and $(B, d)$ of arbitrary radius $d > |a - b|$, which intersect $(O, b)$ at the points $A_1$ and $B_1$. The segment $x = |A_1B_1|$ is the required segment (Fig. 4, *a*).

*Proof.* $\triangle AOA_1 \cong \triangle BOB_1$ (three sides are equal), therefore $\widehat{AOA_1} = \widehat{BOB_1}$ and $\widehat{AOB} = \widehat{A_1OB_1}$. The isosceles triangles $AOB$ and $A_1OB_1$ are similar, hence,

$$a/b = c/|A_1B_1|.$$

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* It is easy to show how to construct segments $m, m^2, m^3, \ldots, m^k$ times as great as the given segment, where $m = 3, 4, 5, \ldots$. For example, for $m = 5$ we construct the segment $|AA_5| = 5|AA_1|$ (Problem 2). Given the segment $AA_5$, we construct the segment $|AA_{25}| = 5|AA_5| = 5^2|AA_1|$ (Problem 2). Then we construct the segment $|AA_{125}| = 5|AA_{25}| = 5^3|AA_1|$, and so on.

The construction given above is possible for $c < 2a$. If $c \geqslant 2a$ and $b < 2a$, we construct a segment whose value is the extreme term of the proportion $a/c = b/x$. In the case of $c \geqslant 2a$ and $b \geqslant 2a$ we construct the segment $na$ (Problem 2) taking $n$ such that $c < 2na$* (or $b < 2na$). Then we construct a segment $y$ whose value is the extreme term of the proportion $na/b = c/y$. If now we construct a segment $x = ny$ (Problem 2), then we obtain a segment which is the fourth proportional to the segments $a$, $b$, and $c$.

In fact

$$na/b = c/y \text{ or } a/b = c/ny.$$

*Construction* (2nd method). We construct the circles $(C, a)$ and $(C_1, a)$, where $C$ and $C_1$ are the end points of the segment $c$. At the intersection we obtain the point $B$ (Fig. 4,$b$). The circle $(B, b)$ intersects the circles $(C, a)$ and $(C_1, a)$ at points $D$ and $D_1$. The segment $DD_1$ is the required segment.

*Proof*. The isosceles triangles $C_1BD_1$ and $BCD$ are congruent, therefore $\widehat{CBD} = \widehat{C_1BD_1}$. Hence, $\widehat{CBC_1} = \widehat{DBD_1}$. From the fact that the isosceles triangles $CBC_1$ and $DBD_1$ are similar it follows that

$$a/b = c/|DD_1|.$$

For $c \geqslant 2a$ and $b \geqslant 2a$, as well as in the first case, we find the segment $na$ such that $2na > c$ and $2na > b$ and then construct a segment $y$ that is the extreme term of the proportion $na/b = c/y$. The segment $ny$ is the required segment.

**Problem 4.** Bisect the arc $AB$ of the circle $(O, r)$.

We can assume that the centre $O$ of the circle is known; it will be shown below (Problem 13) how to construct the centre of a circle (or arc) using only compasses.

*Construction*. Putting $a = |AB|$, we describe circles $(O, a)$, $(A, r)$ and $(B\ r)$; at the intersection we obtain the points

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* We find a segment $2na > c$ in the following way. We construct the segment $a_1 = 2a$ (Problem 2). We describe a circle $(O_1, c)$ with an arbitrary point $O_1$ as centre and lay off in an arbitrary direction segments $|O_1A_1| = a_1$, $|O_1A_2| = 2a_1$, $|O_1A_3| = 3a_1$, and so on (Problem 2). After a finite number of steps we arrive at the point $A_n$ which lies outside $(O_1, c)$. Obviously, the segment $|O_1A_n| = na_1 = 2na > c$.

$C$ and $D$ (Fig. 5). At the intersection of the circles $(C, |CB|)$ and $(D, |AD|)$ we obtain the point $E$. If now we draw the circles $(C, |OE|)$ and $(D, |OE|)$, then, at their intersection, we obtain the points $X$ and $X_1$. The point $X$ bisects the arc $AB$, while the point $X_1$ bisects the arc which forms, together with the first one, the full circle. (In the case

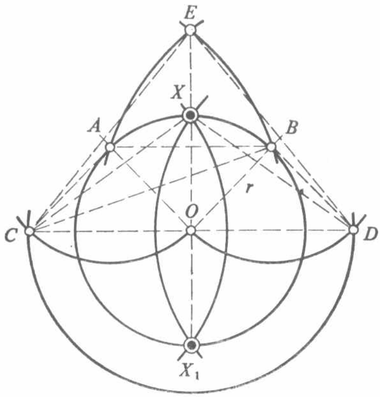


Fig. 5

the whole circle $(O, r)$ is drawn, we can draw only one circle (either $(C, |OE|)$ or $(D, |OE|)$) which defines points $X$ and $X_1$, when intersecting $(O, r)$).

*Proof.* The figures $ABOC$ and $ABDO$ are parallelograms, therefore the points $C$, $O$, and $D$ lie on the same straight line ($[CO] \parallel [AB]$, $[OD] \parallel [AB]$). From the fact that the triangles $CED$ and $CXD$ are isosceles it follows that $\widehat{COE} = \widehat{COX} = 90°$. Thus the segment $OX$ is perpendicular to the chord $AB$. Consequently, in order to prove that the point $X$ bisects the arc $AB$, it is sufficient to show that $|OX| = r$.

Since $ABOC$ is a parallelogram, we have

$$|AO|^2 + |BC|^2 = 2|OB|^2 + 2|AB|^2$$

or

$$r^2 + |BC|^2 = 2r^2 + 2a^2,$$

so that

$$|BC|^2 = 2a^2 + r^2.$$

Since the triangle $COE$ is right-angled, we have

$$|CE|^2 = |BC|^2 = |OC|^2 + |OE|^2,$$

whence

$$2a^2 + r^2 = a^2 + |OE|^2$$

and

$$|OE|^2 = a^2 + r^2.$$

Finally, using the right-angled triangle $COX$, we obtain

$$|OX| = \sqrt{|CX|^2 - |OC|^2} = \sqrt{|OE|^2 - |OC|^2}$$
$$= \sqrt{a^2 + r^2 - a^2} = r.$$

This construction is also valid when the given arc $AB$ is a semicircle ($\smile{AB} = 180°$). Here the points $A$ and $B$ lie on the segment $CD$ and the circles $(A, r)$ and $(B, r)$ touch the circle $(O, a)$ at the points $C$ and $D$, respectively. Because draftsman's instruments (compasses) are imperfect, it is difficult to indicate the position of the points $C$ and $D$ exactly. In this case ($\smile{AB} = 180°$) it is necessary to bisect the arc $A_1B_1$ ($\smile{A_1B_1} \neq 180°$) such that $\smile{AA_1} = \smile{BB_1} > 0$ and $\smile{AA_1} + \smile{A_1B_1} + \smile{B_1B} = \smile{AB}$. Obviously, the point bisecting the arc $A_1B_1$ will also bisect the arc $AB$.

As we have already pointed out, in the geometry of the compasses a straight line is regarded to be constructed as soon as any two of its points are defined. In the subsequent discussion (Problems 22, 23, 24, and others) we are going to construct one, two, and, in general, any number of points on the given straight line using compasses alone. This can be done in the following way.

**Problem 5.** Construct one or several points on a straight line, defined by two points $A$ and $B$.

Given $(AB)$. Construct $X \in (AB)$, $X_1 \in (AB)$, . . . .

*Construction.* We take an arbitrary point $C$ outside the straight line $AB$ (Fig. 6) and construct a point $C_1$ symmetric to $C$ with respect to $AB$ (Problem 1). We describe the circles $(C, r)$ and $(C_1, r)$ of arbitrary radius $r$. At their intersection we obtain the required points $X$ and $X_1$, which lie on the straight line $AB$. Varying the radius $r$, it is possible to construct any number of points on the given straight line: $X'$, $X_1'$, etc.

*Proof*. The point $C_1$ is symmetric to the point $C$, therefore the straight line $AB$ passes through the midpoint of the segment $CC_1$ at right angles. Therefore the straight line $AB$

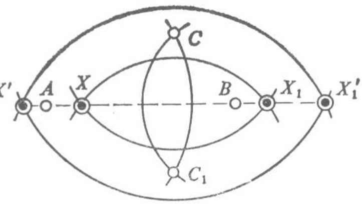


**Fig. 6**

is a set of points equidistant from the points $C$ and $C_1$. By virtue of construction $|CX| = |C_1X| = r$ and $|CX_1| = |C_1X_1| = r$, hence, $X \in (AB)$ and $X_1 \in (AB)$.

**Problem 6.** Construct the intersection points of the circle $(O, r)$ and the straight line given by two points $A$ and $B$.

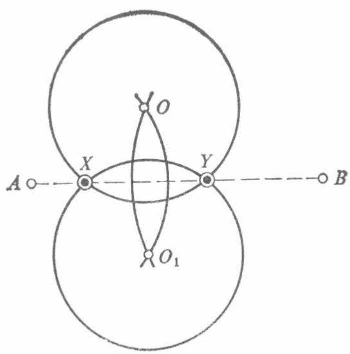


**Fig. 7**

Given $(O, r)$ and $(AB)$. Construct $\{X; Y\} = (O, r) \cap \cap (AB)$.

*Construction* when the centre $O$ does not lie on the straight line $AB$* (Fig. 7).

We construct the point $O_1$ symmetric to the centre $O$ of the given circle with respect to the straight line $AB$ (Prob-

* With the help of compasses alone it is easy to check whether three given points lie on one straight line or not (see note to Problem 1).