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A Series of Comprehensive Studies in Mathematics

Michio Suzuki

Group Theory II

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Michio Suzuki
Department of Mathematics
University of Illinois
1409 West Green Street
Urbana, IL 61801
U.S.A.

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Preface to the English Edition

This is a translation from the Japanese of the second volume (chapters four through six) of my book "Gunron" (Iwanami Shoten, 1978). After discussing the concept of commutators in the fourth chapter, we turn to a discussion of the methods and theorems pertaining to finite groups. The last chapter is intended as an introduction to the recent progress in the theory of simple groups. For the translation, I have kept the main body of the text unchanged, however I have added a few comments in the last chapter in order to inform the readers of the most recent progress.

I would like to express my appreciation to Kazuko Suzuki for her devoted help in translating this book. Finally, it gives me great pleasure to acknowledge my indebtedness to my wife, Naoko, for her constant support and understanding, and for converting the long and often illegible manuscript into a beautifully typed one. To her, I express my sincere thanks and appreciation.

July, 1985 Michio Suzuki

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List of Notation

(The notation which appeared in the List of Notation in the first volume will be omitted.)

$|G|$ I. p. 6
$|G|_p$ I. p. 101
$[x, y]$ I. p. 119
$D(G)$ I. p. 119
G' I. p. 119
$G^{(i)}$ I. p. 119
$G^{(\infty)}$ I. p. 119
$H \lhd \lhd G$ I. p. 123
$H \times K$ I. p. 127
$H \text{ wr } K$ I. p. 270
$[H, K]$ 2
$[H; x]$ 4
$[x_1, x_2, \ldots, x_n]$ 4
$[H_1, H_2, \ldots, H_n]$ 4
$O^{\pi}(G)$ 8
$C_i(G)$ 13
$Z_i(G)$ 14
$\Phi(G)$ 27
$F(G)$ 28
$O_{\pi}(G)$ 28
$O_P(G)$ 28
$C_{\infty}(G)$ 31
$[H, K; n]$ 33
$Z_{\infty}(G)$ 36
$x \equiv y \pmod N$ 44
$\Omega_n(G)$ 45
$\mho_n(G)$ 45
$M(p^n)$ 54
D_g 54
Q_g 54
S_g 54
$\exp G$ 61
$m(G)$ 64
$E(p^3)$ 67
$\mathcal{A}_P$ 80
$J(P)$ 80
$ZJ(P)$ 81
$O_{\pi',\pi}(G)$ 109
$И(H; \pi)$ 111
$И^*(H; \pi)$ 112
$I(G)$ 120
$i(G)$ 120
$Sr(G)$ 120
$C^*_G(x)$ 121
$\mathrm{Foc}_G(P)$ 141
$G'(p)$ 142
E_{π} 166
C_{π} 166
D_{π} 166
E^n_{π} 166
$\mathrm{proj}_i(H)$ 174
$\mathcal{M}^*_{\pi}(G)$ 177
$\mathcal{M}_{\pi}(G)$ 177
$\mathcal{L}_p(G)$ 189
$\mathcal{L}^*_p(G)$ 189
$\mathcal{L}(G)$ 189
$Qd(p)$ 190
$K^{\infty}(P)$ 227
$K_{\infty}(P)$ 227
$\mathcal{S}$ 248
$(\chi, \theta)_G$ 267
θ^G 270

K^G 282
$gc(H; K)$ 282
$Z^*(G)$ 315
$SCN(P)$ 357
$SCN(p)$ 358
$SCN_k(P)$ 358
$Y \text{ cc } X$ 372
$M * N$ 373
$\mathscr{E}_k(p; H)$ 375
$\mathscr{M}$ 375
$\mathscr{M}(X)$ 375
$\mathscr{U}$ 375
$\mathrm{Syl}_p(G)$ 377
$O(G)$ 392
$E(G)$ 450
$F^*(G)$ 452
$L(G)$ 454
$B(G)$ 457
$O_*(G)$ 461
$C_G^Z(x)$ 466
$И_\theta(H; \pi)$ 472
$И_\theta^*(H; \pi)$ 472
$m_p(X)$ 488
$\mathscr{C}(G)$ 519
$\Gamma_{P,2}(G)$ 520
$И'(H; \pi)$ 591

Chapter 4

Commutators

§1. Commutator Subgroups

The commutator $[x, y] = x^{-1}y^{-1}xy$ of two elements x and y of a group G has been used, without formal development, several times in this book since its initial appearance in Chapter 1, §3. The readers are referred to Definition 3.11 of Chapter 2, p. 110, for the definition, and to proposition (3.12) of Chapter 2, as well as the remarks preceding it, for some of the important properties of the commutator. In this section, we will define the commutator subgroup $[H, K]$ of two arbitrary subgroups H and K of a group G. We will study the properties of the commutator subgroups and also give applications.

The following proposition gives some of the useful relationships among commutators. (Recall the notation $v^y = y^{-1}vy$ for any elements y and v.)

(1.1) *Let x, y, and u be three elements of a group G. Then, the following identities hold:*

(i) $[x, y]^{-1} = [y, x]$,
(ii) $[xy, u] = [x, u]^y[y, u]$,
(iii) $[u, xy] = [u, y][u, x]^y$.

Proof. These identities are easily verified by using the definition of the commutator. We will verify the third one as an example. We have

$$\begin{aligned}[u, xy] &= u^{-1}(xy)^{-1}u(xy)\\ &= u^{-1}y^{-1}x^{-1}uxy\\ &= u^{-1}y^{-1}(uyy^{-1}u^{-1})x^{-1}uxy\\ &= [u, y]y^{-1}[u, x]y.\end{aligned}$$

This proves identity (iii). Note that the commutator involving x comes after the commutator which involves y in the right side of identity (iii). □

The following definition is basic to the subsequent development of the theory.

Definition 1.2. Let H and K be two subgroups of a group G. The **commutator subgroup** of H and K is defined as the subgroup generated by all the commutators $[h, k]$ where $h \in H$ and $k \in K$. We denote it by $[H, K]$:

$$[H, K] = \langle [h, k] \mid h \in H, k \in K \rangle.$$

If $H = K = G$, the subgroup $[H, K]$ defined here coincides with the commutator subgroup $D(G)$ of G defined in Definition 3.11 of Chapter 2, p. 119. In this chapter, we will call $D(G) = G'$ the **derived group** in order to lessen the probability of confusion. We remark again here that the subgroup $[H, K]$ is not just the totality of the commutators, but the subgroup generated by these commutators.

The following omnibus lemma contains some basic properties of commutator subgroups.

(1.3) *Let H and K be two subgroups of a group G. Then, the following propositions hold.*

(i) *We have* $[H, K] = [K, H]$.
(ii) *For any homomorphism σ of G, we have*

$$[H, K]^{\sigma} = [H^{\sigma}, K^{\sigma}].$$

(iii) $[H, K] \subset H \Leftrightarrow K \subset N_G(H)$.
(iv) *If H_1 and K_1 are subgroups such that $H_1 \subset H$ and $K_1 \subset K$, then we have*

$$[H_1, K_1] \subset [H, K].$$

(v) $[H, K] = \{1\}$ *if and only if H commutes elementwise with K.*

Proof. Clearly, proposition (i) follows from (1.1)(i). Since

$$[h, k]^{\sigma} = [h^{\sigma}, k^{\sigma}]$$

for any homomorphism σ of G (cf. (3.12) of Chapter 2), the second proposition holds. If $[H, K] \subset H$, then for any $h \in H$ and $k \in K$, we have

$$[h, k] = h^{-1}k^{-1}hk \in H.$$

Thus, we get $k^{-1}hk \in H$ and $k^{-1}Hk \subset H$. If we consider k^{-1} in place of k, we obtain $kHk^{-1} \subset H$. So, we have $k^{-1}Hk = H$ and $k \in N_G(H)$. Conversely, if $k \in N_G(H)$, then for any element h of H, we get $[h, k] \in H$. It follows from definition that $[H, K] \subset H$ holds. This proves the equivalence in (iii). The remaining part of (1.3) follows easily from the definitions. □

Theorem 1.4. *If two subgroups H and K are normal subgroups of a group G, then so is the commutator subgroup* $[H, K]$. *Furthermore, we have*

$$[H, K] \subset H \cap K.$$

A similar theorem holds for characteristic, as well as for fully invariant, subgroups.

Proof. Let Ω be an operator domain of G. We prove that if both H and K are Ω-subgroups of G, then the commutator subgroup $[H, K]$ is also an Ω-subgroup. In fact, if $\sigma \in \Omega$, then we have

$$[H, K]^\sigma = [H^\sigma, K^\sigma] \subset [H, K]$$

by (1.3)(ii) and (iv). Thus, $[H, K]$ is Ω-invariant. For $\Omega = \operatorname{Inn} G$, we get the first statement of Theorem 1.4. The last assertions will follow by taking $\Omega = \operatorname{Aut} G$ and $\Omega = \operatorname{End} G$. The containment

$$[H, K] \subset H \cap K$$

is proved easily from (1.3)(iii). □

Theorem 1.5. *Let N be a normal subgroup of a group G, and set* $\bar{G} = G/N$. *Then, for any two subgroups H and K of G, we have*

$$[\bar{H}, \bar{K}] = [H, K]N/N.$$

Proof. Let σ be the canonical homomorphism from G onto $\bar{G}$. Then, we have $\bar{H} = H^\sigma = HN/N$. So, the assertion follows from (1.3)(ii). □

The next theorem is one of the most important properties of commutator subgroups.

Theorem 1.6. *Let H and K be two subgroups of a group G, and let L be the commutator subgroup* $[H, K]$.

(i) *The subgroup H, as well as K, normalizes L; that is, we have*

$$I \lhd \langle H, K \rangle.$$

(ii) *The subgroup LK is the subgroup generated by all the subgroups conjugate to K in* $\langle H, K \rangle$. *The subgroup LK is the smallest normal subgroup of* $\langle H, K \rangle$ *which contains K.*

Proof. (i) We will prove that $H \subset N_G(L)$. It suffices to show that for any element u of H and for any generator $[h, k]$ $(h \in H,\ k \in K)$ of L, we have $[h, k]^u \in L$. By (1.1)(ii), we have

$$[h, k]^u = [hu, k][u, k]^{-1}.$$

Since h and u are elements of H, we have $hu \in H$, and the right side of the above equality belongs to L. Thus, we have $H \subset N_G(L)$. By (1.3)(i), we have $L = [H, K] = [K, H]$. So, similarly, the subgroup K normalizes L. This proves the first part.

(ii) For any element h of H and any element k of K, we have

$$h^{-1}kh = k[k, h] \in KL.$$

Hence, we get $(KL)^h = K^h L^h \subset KL$. This implies that H normalizes KL and that $KL \lhd \langle H, K\rangle$.

Suppose that $K \subset M \lhd \langle H, K\rangle$ for some subgroup M of G. Then, the commutator $[h, k]$ is contained in M, and we have $KL \subset M$. Thus, KL is the smallest normal subgroup of $\langle H, K\rangle$ which contains K. The first part of (ii) follows easily because the subgroup generated by the conjugate subgroups is normal. □

For any element x of G, we define the **commutator subgroup** $[H, x]$ as the subgroup of G which is generated by $[h, x]$ with $h \in H$. Then, a proposition similar to (1.3) holds. Furthermore, the proof of Theorem 1.6(i) shows that we have

(1.6)′ $$H \subset N_G([H, x]).$$

Definition 1.7. Let $x_1, x_2, \ldots, x_n$ $(n \geqq 3)$ be elements of a group G. We define a **higher commutator** of n elements inductively by the formula

$$[x_1, x_2, \ldots, x_n] = [[x_1, \ldots, x_{n-1}], x_n].$$

We will often call it simply the commutator of the n elements $x_1, x_2, \ldots, x_n$. Similarly, a **(higher) commutator subgroup** of n subgroups $H_1, H_2, \ldots, H_n$ is defined by

$$[H_1, H_2, \ldots, H_n] = [[H_1, \ldots, H_{n-1}], H_n].$$

A higher commutator subgroup $[H_1, H_2, \ldots, H_n]$ contains all the higher commutators $[h_1, h_2, \ldots, h_n]$ of elements $h_1, \ldots, h_n$, where $h_i \in H_i$ for each i. We remark here that if $n \geqq 3$, the subgroup generated by these higher commutators of elements is not necessarily equal to the higher commutator subgroup $[H_1, H_2, \ldots, H_n]$ (cf. Exercise 3). We defined the higher commutator by taking the commutator of two elements successively starting from the left end. Thus, in general, we have

$$[x, y, z] \neq [x, [y, z]].$$

We may rewrite identities (1.1)(ii) and (iii) using higher commutators:

(1.1)(ii)′ $$[xy, u] = [x, u][x, u, y][y, u],$$

(1.1)(iii)′ $$[u, xy] = [u, y][u, x][u, x, y].$$

Among the higher commutators of three elements, the following remarkable identity holds.

(1.8) *For three elements x, y, and z of a group, we have*

$$[x, y^{-1}, z]^y[y, z^{-1}, x]^z[z, x^{-1}, y]^x = 1.$$

Proof. Express the higher commutators involved as words of the elements x, y, and z. Then, all the terms will be cancelled. In fact, from

$$[x, y^{-1}] = x^{-1}yxy^{-1},$$

we get

$$[x, y^{-1}, z] = (x^{-1}yxy^{-1})^{-1}z^{-1}(x^{-1}yxy^{-1})z$$
$$= yx^{-1}y^{-1}xz^{-1}x^{-1}yxy^{-1}z.$$

Set $u = xzx^{-1}yx$, $v = yxy^{-1}zy$, and $w = zyz^{-1}xz$. Then, we have

$$[x, y^{-1}, z]^y = u^{-1}v.$$

Similarly, we have $[y, z^{-1}, x]^z = v^{-1}w$ and $[z, x^{-1}, y]^x = w^{-1}u$. It is now clear that (1.8) holds. □

Using formula (1.8) we can prove the following **Three Subgroup Lemma** of P. Hall which, because of its many consequences, may be called one of the most basic results in group theory.

Theorem 1.9. *Let H, K, and L be three subgroups of a group G. If*

$$[H, K, L] = [K, L, H] = \{1\},$$

then we have $[L, H, K] = \{1\}$.

Proof. Let x, y, and z be arbitrary elements such that $x \in H$, $y \in K$, and $z \in L$. Then, by assumption, we have

$$[x, y^{-1}, z] = [y, z^{-1}, x] = 1.$$

Hence, by (1.8), we get

$$[z, x^{-1}, y]^x = 1 = [z, x^{-1}, y].$$

By definition, the commutator subgroup $[L, H]$ is generated by the commutators of the form $[z, x^{-1}]$. The above identity shows that any element y of K commutes with any element of the form $[z, x^{-1}]$ where $z \in L$ and $x \in H$. Thus, K centralizes $[L, H]$, and we have $[L, H, K] = \{1\}$ by (1.3)(v). □

Corollary. *Let N be a normal subgroup of a group G, and let H, K, and L be arbitrary subgroups of G. If N contains two of the following three higher commutator subgroups*

$$[H, K, L], \quad [K, L, H], \quad \textit{and} \quad [L, H, K],$$

then N contains all three commutator subgroups.

Proof. Let $\bar{G} = G/N$. By assumption, two of the following three higher commutator subgroups

$$[\bar{H}, \bar{K}, \bar{L}], \quad [\bar{K}, \bar{L}, \bar{H}], \quad \text{and} \quad [\bar{L}, \bar{H}, \bar{K}]$$

are $\{1\}$. By Theorem 1.9, all these higher commutator subgroups are $\{1\}$. This proves the Corollary. □

The concept of commutator subgroups can be generalized to groups with an operator group.

Definition 1.10. Let A be an operator group which acts on a group G (cf. Definition 8.1 of Chapter 1, p. 66). We define

$$[G, A] = \langle g^{-1}g^a \mid g \in G, a \in A \rangle$$

and call it the **commutator subgroup** of G and A.

If a group A acts on a group G, we can form their semidirect product S by the method discussed in Chapter 1, §8. We may consider both G and A as subgroups of S. Then, the given action of A on G is realized by conjugation in S. Thus, the commutator subgroup $[G, A]$ of Definition 1.10

coincides with the commutator subgroup of the two subgroups G and A in the semidirect product S. This explains the choice of the notation and the terminology in Definition 1.10, and assures us that there will be no distinction between the two possible meanings of $[G, A]$ when A may be considered both as a subgroup of S and, at the same time, as an operator group of G. Furthermore, the theorems which have been proved so far about commutator subgroups also hold for commutator subgroups of operator groups. In particular, we have the following result.

(1.11) *Let A be an operator group which acts on a group G. Then, the commutator subgroup $[G, A]$ is an A-invariant normal subgroup of G.*

Proof. This is a special case of Theorem 1.6(i). □

Although the concept of commutator subgroups has not been defined formally until this section, several results of Chapter 2 concern commutator subgroups. Thus, the subgroup denoted as A_0 in the proof of (5.17) of Chapter 2, p. 156, is the commutator subgroup $[A, Q]$, and (5.17) of Chapter 2 gives the direct sum decomposition

$$A = C_A(Q) + [A, Q]$$

for the abelian group A with a regular operator group Q. The Corollary of Theorem 8.13 of Chapter 2, p. 239, shows that if a π'-group Q acts on a π-group G, then we have

$$G = C_G(Q)[G, Q].$$

The proof given there for this corollary is a special case of the proof of (1.11). The commutator subgroup appeared also in (9.2) of Chapter 2, p. 246.

The following lemma gives a useful characterization of the commutator subgroup of a group with operators.

(1.12) *Let A be an operator group which acts on a group G. Let S be the semidirect product of G and A with respect to the given action of A on G, and consider G and A as subgroups of S. Then, we have*

$$[G, A] = G \cap N,$$

where N is the smallest normal subgroup such that $A \subset N \lhd S$.

Proof. Let N be as defined above. Then, by Theorem 1.6(ii), we have

$$N = [G, A]A.$$