# Discrete Mathematical Structures

## for Computer Scientists and Engineers

M.K. Das

Alpha Science

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## for Computer Scientists and Engineers

**In Memory of**
**Sri Surendra Chandra Das and Smt. Renu Bala Das**
**my parents**

# Preface

The present book is the result of teaching the course *IT-14, Mathematical Foundation for Computer Science* at Institute of Informatics and Communication (IIC), University of Delhi South Campus. The present text is suitable for a one- or two-semester course and covers topics incorporated in the course work for students of Computer Science, Computer Application, Electronics & Communication, Telecommunication Engineering and Informatics and Information Technology at Indian and other Universities. I was guided to write this book because of my interest in the subject of *Discrete Mathematics* and *Theoretical Computer Science*. Keeping this in view, the primary emphasis in the book is to discuss basic issues in *Discrete Mathematical Structures* and also in the area of *Mathematical Theory of Computation*. At IIC, we have students coming from different streams such as Mathematical Sciences, Physical Sciences, Electronics and Engineering. Even though they are quite familiar and conversant with basic algebra, calculus, differential equations, integral transforms etc., it is observed that they find the subject of discrete mathematics as somewhat difficult. Further discussion with many of them, during last several years, suggest that the students in general want to focus on the relevance and practicality of various ideas involved in the area of Discrete Mathematics. This aspect has been taken into account during various stages of this project. It is already known that the discrete mathematics is the mathematics of discrete objects and their binding relationships. Developments in digital technology during the second half of the last century coincided with interest in the subject of discrete mathematics. It is now established that the various practical issues involved in the theory of computation involves basic understanding of some abstract structures and their formulations as discussed in the text. The understanding of such formulations further helps in developing ideas involved in data structures and algorithms and their performance. These ideas along with the concept of finite state machine and automata enables one to visualize and formulate the basic ideas of the theory of computation. In the present text I have tried to concisely present various conceptual ideas involved at various level with examples of their applications. The emphasis is made to show the relevance and practical application of the ideas involved in discrete mathematics. I feel that the present text is self contained and user friendly and hopefully the language is simple for students and amateurs to understand and appreciate the basic ideas of mathematics as used in the theory of computation. Suggestions for improvement are welcome and may be sent to mrinal@iic.ac.in

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I take the responsibility for any error(s) that still remain in this book. I would certainly appreciate receiving any comments and suggestion for further improvements and presentation from readers.

M.K. Das

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# 1

# Elements of Set Theory

A subject called set theory has been identified as even more basic than arithmetic and geometry by mathematicians during the last century. It provides important basic concepts of contemporary mathematics. The ideas of set theory find application in all branches of mathematics as they form a powerful language for reasoning about various mathematical objects. Intuitively we define a *set as collection of objects*. The objects in such a collection are referred to as *set elements.* Even though such a definition lacks rigors, we use it in much the same way as a point and a line are left undefined in geometry. In this chapter, we assume that a set is determined when a plurality (consisting of a number of things) is bunched together into a single entity. For instance a set called Hindi alphabet comprises 49 things called letters. Similarly the real number system could be considered as a set formed collectively from things each of which is a real number.

## 1.1 Basic Notation

In order to make the concept of a set more precise, we distinguish things which are collected together in a set from those which are not. Because of this, the set notation and theory must be precise and unambiguous. It is customary to designate a set by capital letters A, B, P, Q, X, Y etc. When the number of things or objects comprising a set is small, the set is denoted by listing the elements within the braces { }, thus establishing the singular nature of the set. For instance a set of digits in the binary system of enumeration is denoted by $X = \{0, 1\}$. Similarly $A = \{a, e, i, o, u\}$ denotes the set of vowels in English language.

We represent the relationship between a set and its elements by the symbol $\in$ (belongs to) in the following way:

$$i \in V$$

means that the element $i$ is a member of the set named V. Alternatively such a representation also means '$i$ belongs to V' or '$i$ is in V' or '$i$ is an element of V'. If however, an element say $d$ does not belong to a set V above, we write

$$d \notin V.$$

For a large set, the enumeration could be done without actually listing all the elements. For instance, 26 letters of the set English alphabet could be described as:

$$A = \{a, b, c, \ldots, x, y, z\}$$

where three consecutive dots, called ellipsis, mean that the list should be continued in accordance with the pattern indicated. For instance the set of natural numbers could be represented as:

$$N = \{1, 2, 3, \ldots\}.$$

Similarly the set of integers is represented as:

$$Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

Another but convenient notation of writing a set is as:

$$B = \{x: P(x)\},$$

where $x$ is a variable and P($x$) is a descriptive phrase called *predicate*. The variable is mostly indicated by a lower case letter and the predicate is a description of the allowed replacement for that variable. They are usually separated by a colon : or by a vertical bar |. For instance the set B = $\{x: x \text{ is an integer}, x < 0\}$, means that B is a set of $x$ such that $x$ is an integer and $x$ is less than zero. Similarly the set E = $\{x: x^2 - 4 = 0\}$ means that the set E has elements $x$ such that $x^2 - 4 = 0$. Since $x = \pm 2$ are the solutions of the equation $x^2 - 4 = 0$, we have E = $\{2, -2\}$. The two forms are equivalent.

Further we would like to mention here that the order in which the elements of a set are listed is not important. Thus the representation of the set $\{1, 5, 2\}$ by $\{1, 2, 5\}$, $(2, 5, 1\}$, $\{5, 1, 2\}$ etc., are all equivalent. Moreover the listing in a set might have repeated element(s) and this may be ignored. Thus $\{1, 5, 2, 5, 1\}$ is another representation of $\{1, 5, 2\}$.

### 1.1.1 Subsets

We may write the following definition for the subset of a set:

*Definition*: Let A and B be two sets. If every element of A is also an element of B, then A is a subset of B and is written as $A \subseteq B$.

Here the symbol $\subseteq$ is called the *inclusion* or *subset relation*. Thus $A \subseteq B$ also means 'A is included in B' or 'A is contained in B'. If A is not a subset of B, we write $A \not\subseteq B$. For instance if A = $\{2, 5, 7\}$ and B = $\{5, 7, 2, 3, 9\}$, we find $A \subseteq B$ since for every $x \in A$, we have $x \in B$. Similarly if N, Z, Q and R respectively denote the set of positive integer, set of integers, set of rational numbers and set of real numbers then

$$N \subseteq Z \subseteq Q \subseteq R.$$

In the foregoing definition on subset, observe that the possibility of A = B is not excluded. Excluding such a possibility reduces the set as proper subset of B or $A \subset B$.

Also if A be a set and B = $\{A, \{A\}\}$. We find $A \in B$ and $\{A\} \in B$ and $\{\{A\}\} \subseteq B$. However $A \not\subseteq B$.

***Example 1*** If A = {1, 2, 3} and B = {{1, 2, 3}, 1, 2, 3}. What can be said about the set relationship between A and B?

*Solution* It is readily seen that $A \subseteq B$ and $A \in B$.

### 1.1.2 Equality of Sets

We have the following definition for the equality of sets:

*Definition:* Let A and B be two sets. Then two sets are equal i.e., A = B, if and only if for every $x \in A$, $x \in B$ and vice-versa. In other words if $A \subseteq B$ and $B \subseteq A$ then A = B.

***Example 2*** Consider two sets A = {2, 1, 3, 6} and B = {6, 1, 2, 3}. Since every element of A is also an element of B and vice-versa, we have A = B. Note that the ordering of elements of A or B is unimportant.

## 1.2 Power Set

We define the power set as follows:

*Definition*: Let A be a set consisting of $n$ elements. The set of all subsets of A is called the power set of A. It consists of $2^n$ elements.

***Example 3*** For the set A = $\{a, b, c, d\}$, find the power set $\mathscr{P}(A)$ of A?

*Solution* Here we find four element subset as A = $\{a, b, c, d\}$; three element subsets as $\{a, b, c\}$,$\{b, c, d\}$, $\{a, c, d\}$, $\{a, b, d\}$; two element subsets as $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$; and one element subsets as $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$. Further zero element subset is $\varphi$. Thus in all there are 16 subsets or $2^4$ subsets of A. The power set $\mathscr{P}(A)$ is also denoted as $2^A$.

## 1.3 Complement of a Set

*Definition*: Let A and B be two sets. The set consisting of all elements of B which are not in A is called the complement of A and B and is denoted by B – A.

In the set builder notation, we may write $B - A = \{x: x \in B, x \notin A\}$. To find the complement of A in B, we eliminate in B all elements which belong to A.

For instance Z – N corresponds to a set containing the element zero and all negative integers.

## 1.4 Universal Set

In a set the elements usually belong to some large set called the universal set U. A universal set, for instance, in plane geometry consists of all points in the plane; the elements of the set A = {1, 2, 3} may be treated as belonging to a universal set U = N.

## 1.5 Null Set

We define a null set or an empty set $\phi$ as a set that has no element in it. It is also represented as { }. Thus for any set A, we have $\phi \subseteq A$ as there are no elements of $\phi$ that are not in A. For instance the square of a real number is always positive. Hence $\{x: x \text{ is a real number and } x^2 = -3\}$ is a null set $\phi$.

## 1.6 Basic Set Operations

Addition and multiplications are examples of binary operations on numbers. These operations could be defined in many other contexts and are not restricted to numbers only. The union, intersection, and complement are some basic operations on sets. Based on these operations, set theory offers a well defined mathematical structure and becomes an indispensable tool in solving quite complicated combinatorial problems.

The basic idea, of a Venn diagram, could be used to understand and visualize the basic set operations and set relationships. In Venn diagram, the largest set under consideration, U-the universal set is portrayed by a rectangle while its various subsets are represented by circular regions inside it. The Venn diagram (Fig. 1.1) shows the set A separating U into two disjoint pieces A and $\bar{A}$ (or $A^c$)—the complement of A (shaded region). In the following section, we will define the basic operation on sets and their representation in Venn diagram.

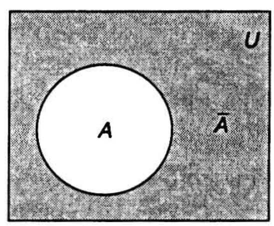


**Fig. 1.1** *Venn diagram*

*Definition*: Let A and B be two sets. The set of all elements $\in$ A or $\in$ B (or in both A and B) is called the union of A and B and is abbreviated as $A \cup B$.

Alternatively $A \cup B = \{x: x \in A \text{ or } x \in B\}$. The Venn diagram of union of A and B is shown below (Fig. 1.2).

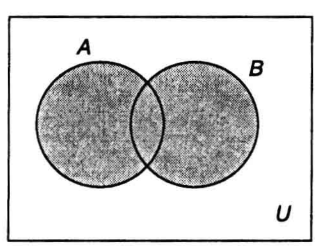


**Fig. 1.2** *The shaded region showing A ∪ B*

The union of $n$ sets $A_i$'s, $i = 1, n$ written as:

$$A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i$$

represents the set of elements that belongs to one or more of the sets $A_i$.

*Definition*: Let A and B be two sets. The set of all elements which are both in A and B defines the intersection of set A and B and is symbolically written as $A \cap B$.

Alternatively $A \cap B = \{x: x \in A \text{ and } x \in B\}$. The Venn diagram of intersection of two sets A and B is shown below (Fig. 1.3).

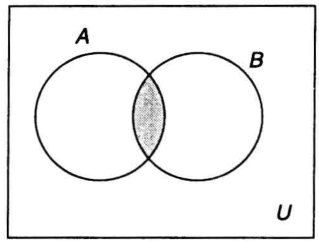


**Fig. 1.3** *The shaded region A ∩ B*

The intersection of $n$ sets $A_i$'s, $i = 1, n$ written as:

$$A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n = \bigcap_{i=1}^{n} A_i$$

represents the set of elements which belongs to all of the sets $A_i$.

The following theorem provides the relation between the union and intersection:

*Theorem*: Let A and B be two sets. If U is the universal set then $\phi \subseteq A \cap B \subseteq A \subseteq A \cup B \subseteq U$.

*Proof*: The null set is a subset of any set. If $x \in A \cap B$ then by definition, $x \in A$ and $x \in B$. If $x \in A$, then it is in at least one of the sets A and B. Hence $x \in A \cup B$. Since A and B $\in$ U, $A \cup B \subseteq U$.

***Example 5*** Let A = {1, 2, 3, 4, 5}, B = {2, 4, 5, 6, 7} and C = {1, 15, 17}. Find $A \cup B$, $A \cap B$, $A \cap C$ and $B \cap C$?

*Soution* Using the definition of basic set operations, we find that

$$A \cup B = \{1, 2, 3, 4, 5, 6, 7\} = B \cup A,$$
$$A \cap B = \{2, 4, 5\} = B \cap A,$$
$$A \cap C = \{1\},$$

$B \cap C = \{\ \} = \phi$ {If two sets are such that no element is common then the two sets are said to be disjoint or nonintersecting}.

***Example 6*** If A = {2, 4, 6, 8, 10, ...}, B = {1, 3, 5, 7, 9, ....}then $A \cup B = N$ and $A \cap B = \phi$.

***Example 7*** For any set A, $A \cup A = A$ and $A \cap A = A$. Therefore, using the foregoing theorem, it is readily seen that $\phi \in A$.

***Example 8*** The Venn diagram for union and intersection of any three sets could be drawn as (Fig. 1.4).

The complement of set A with respect to a set B and vice-versa, defined earlier could be represented by the following Venn diagrams (Fig. 1.5).