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Heinz Hanßmann

Local and Semi-Local Bifurcations in Hamiltonian Dynamical Systems

Results and Examples

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Preface

Life is in color,
But black and white is more realistic.
Samuel Fuller

The present notes are devoted to the study of bifurcations of invariant tori in Hamiltonian systems. Hamiltonian dynamical systems can be used to model frictionless mechanics, in particular celestial mechanics. We are concerned with the nearly integrable context, where Kolmogorov–Arnol'd–Moser (KAM) theory shows that most motions are quasi-periodic whence the (invariant) closure is a torus. An interesting aspect is that we may encounter torus bifurcations of high co-dimension in a single given Hamiltonian system. Historically, bifurcation theory has first been developed for dissipative dynamical systems, where bifurcations occur only under variation of external parameters.

Bifurcations of equilibria and periodic orbits

The structure of any dynamical system is organized by its invariant subsets, the equilibria, periodic orbits, invariant tori and the stable and unstable manifolds of all these. Invariant subsets form the framework of the dynamics, and one is interested in the properties that are persistent under small perturbations.

The most simple invariant subsets are equilibria, points that remain fixed so that no motion takes place at all. Equilibria are isolated in generic systems, be that within the class of Hamiltonian systems or within the class of all dynamical systems. In the latter case the dynamics is dissipative and an equilibrium may attract all motion that starts in a (sufficiently small) neighbourhood.

Such a dynamically stable equilibrium is also structurally stable in that a small perturbation of the dynamical system does not lead to qualitative changes. If we let the system depend on external parameters, then the equilibrium may lose its dynamical stability under parameter variation or cease to exist. A typical example is the $\mathbb{Z}_2$-symmetric pitchfork bifurcation where an attracting equilibrium loses its stability and gives rise to a pair of two attracting equilibria. Other examples are the saddle-node and the Hopf bifurcation. Such bifurcations have been studied extensively in the literature, cf. [129, 173] and references therein.

The dynamics around equilibria in Hamiltonian systems can be more complicated since it is not generic for a Hamiltonian system to have only hyperbolic equilibria. This also influences possible bifurcations, cf. [61, 43]. For instance, in the Hamiltonian counterpart of the above pitchfork bifurcation it is an elliptic (rather than attracting) equilibrium that loses its stability and gives rise to a pair of two elliptic equilibria. In [254, 78] dynamically stable equilibria are studied for which the nearby dynamics nevertheless changes under variation of external parameters.

Periodic orbits form 1-parameter families in Hamiltonian systems, usually parametrised by the value of the energy. In fact, where continuation with respect ot the energy fails a bifurcation[1] takes place, while other bifurcations are triggered by certain resonances between the Floquet multipliers. For more details see [3, 38] and references therein, and also Chapter 3 of the present notes.

Bifurcation from periodic orbits to invariant tori

In (generic) dissipative systems periodic orbits are isolated and one needs again external parameters μ to encounter bifurcations. One of these is the periodic Hopf [154, 155] or Neĭmark–Sacker [252, 14] bifurcation. Under parameter variation a periodic orbit loses stability as a pair of Floquet multipliers passes at $\pm\exp(\mathrm{i}\nu T)$ through the unit circle, where T denotes the period. In the supercritical case the stability is transferred to an invariant 2-torus that bifurcates off from the periodic orbit, with two frequencies $\omega_1 \approx 1/T$ and $\omega_2 \approx 2\pi\nu$ coming from the internal and normal frequency of the periodic orbit. The subcritical case involves an unstable 2-torus with these frequencies that shrinks down to the periodic orbit and results in a "hard" loss of stability.

The frequency vector $\omega = (\omega_1, \omega_2)$ that in the above description is rather naively attached to the merging invariant tori exemplifies the problems brought by bifurcations to invariant tori. First of all we need non-resonance conditions $2\pi k/T + \ell\nu \neq 0$ for all $k \in \mathbb{Z}$ and $\ell \in \{1, 2, 3, 4\}$. Where these are violated one speaks of a strong resonance as the Floquet multipliers $\pm\exp(\mathrm{i}\nu T) \in \{\pm 1, -\frac{1}{2} \pm \frac{\mathrm{i}}{2}\sqrt{3}, \pm\mathrm{i}\}$ are ℓth order roots of unity, see [272, 173] for more details. While excluding these low order resonances does lead to an invariant 2-torus bifurcating off from the periodic orbit, the motion on that torus need not be quasi-periodic.

For irrational rotation number ω_1/ω_2 the motion is indeed quasi-periodic and fills the invariant torus densely. In case the quotient ω_1/ω_2 is rational (but

[1] For generic Hamiltonian systems this is a periodic centre-saddle bifurcation.

now with denominator $q \geq 5$) we expect phase locking with a finite number of periodic orbits with period $\approx qT$ and all other orbits on the torus heteroclinic between two of these. The invariance (and smoothness) of the torus is guaranteed by normal hyperbolicity, an important property of dissipative systems that does not have the same consequences in the Hamiltonian context.

In the present simple situation it suffices to require that the rotation number ω_1/ω_2 on the invariant torus has non-zero derivative with respect to the bifurcation parameter μ. A more transparent approach is to consider the rotation number as an additional external parameter and it is more convenient to work with both ω_1 and ω_2 as (independent and thus two) additional parameters. In (μ, ω)-space this yields the following description. The bifurcation occurs as μ passes through the bifurcation value $\mu = 0$ and the dynamics on the torus is quasi-periodic except where $\omega = (\omega_0 q, \omega_0 p)$ is a multiple $\omega_0 \in \mathbb{R}$ of an integer vector $(q, p) \in \mathbb{Z}^2$ and thus resonant.

Torus bifurcations in dissipative systems

Bifurcations involving invariant n-tori may similarly be described using external parameters $(\mu, \omega) \in \mathbb{R}^d \times \mathbb{R}^n$. An additional complication is that the flow on an n-torus may be chaotic for $n \geq 3$ and that the torus may be destroyed altogether in the absence of normal hyperbolicity. One therefore excludes resonances $k_1\omega_1 + \ldots + k_n\omega_n = 0$ by means of Diophantine conditions[2]

$$\bigwedge_{k \in \mathbb{Z}^n \setminus \{0\}} \quad | \, k_1\omega_1 + \ldots + k_n\omega_n \, | \geq \frac{\gamma}{|k|^\tau} \tag{0.1}$$

where $\gamma > 0$, $\tau > n - 1$ and $|k| = k_1 + \ldots + k_n$.

A first result along these lines concerns n-tori that bifurcate off from equilibria, cf. [23] and references therein. Here $d = n$ and the parameters μ are used to let n pairs $\mu_j \pm \mathrm{i}\omega_j$ pass through the origin $\mu = 0$ in μ-space. This yields quasi-periodic n-tori for ω in the nowhere dense but measure-theoretically large subset of $\mathbb{R}^n$ defined by (0.1), and also quasi-periodic m-tori where only $m < n$ pairs $\mu_j \pm \mathrm{i}\omega_j$ have crossed the imaginary axis.

Furthermore there are invariant tori of dimension $l > n$. In the simplest case $n = 2$ this has been proved in [32], establishing a quasi-periodic flow on the resulting 3-tori. The procedure in [24] does yield l-tori for general n, but no information on the flow on these tori.

Normal hyperbolicity yields invariant $(n + 1)$-tori bifurcating off from a family of invariant n-tori in [68, 260, 119]. At the bifurcation the invariant n-tori momentarily lose hyperbolicity and the Diophantine conditions (0.1) are needed. As shown in [33, 34] one can similarly use Diophantine conditions

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[2] The $\bigwedge$ at the beginning signifies that the inequalities that follow have to hold true for all non-zero integer vectors.

involving the normal frequency at the bifurcation to establish a quasi-periodic flow on the $(n+1)$-tori. The "gaps" left open where the frequency vector is too well approximated by a resonance are then filled by normal hyperbolicity. On this measure-theoretically small but open and dense collection of $(n+1)$-tori the flow remains unspecified. See also [55, 77] for more details.

Notably, these results require the bifurcating n-tori to be in Floquet form, with normal linearization independent of the position on the torus. The skew Hopf bifurcation where this condition is violated is a generic torus bifurcation that has no counterpart for periodic orbits. As shown in [282, 60, 62, 273] one has also in this case quasi-periodic $(n+1)$-tori bifurcating off from n-tori. The gaps left by the necessary Diophantine conditions are again filled by normal hyperbolicity, but to a lesser extent.

From the period doubling bifurcation [223, 173] of periodic orbits one inherits the frequency halving bifurcation of quasi-periodic tori. Under variation of the external parameter μ an invariant n-torus loses stability as a Floquet multiplier passes at -1 through the unit circle. In the supercritical case the stability is transferred to another n-torus that bifurcates off from the initial family of n-tori with the first[3] frequency divided by 2. The subcritical case involves an unstable n-torus with one frequency halved that meets the initial family and results in a "hard" loss of stability.

This situation is clarified in [34]. As μ passes through the bifurcation value $\mu = 0$ a frequency-halving bifurcation takes place for the Diophantine tori satisfying (0.1). By means of normal hyperbolicity the gaps around resonances $k_1\omega_1 + \ldots + k_n\omega_n = 0$ are filled by invariant tori on which the flow need not be conditionally periodic. This leaves small "bubbles" in the complement of Diophantine tori at and near the bifurcation value where normal hyperbolicity is too weak to enforce invariant tori. In [186, 187] this scenario has been obtained along a subordinate curve in the 2-parameter unfolding of a periodic orbit having simultaneously Floquet multipliers -1 and $\pm\exp(\mathrm{i}\nu T) \notin \{\pm 1, -\frac{1}{2} \pm \frac{\mathrm{i}}{2}\sqrt{3}, \pm\mathrm{i}\}$.

The quasi-periodic saddle-node bifurcation is studied in [65] where it appears subordinate to a periodic orbit undergoing a degenerate periodic Hopf bifurcation. The general theory is (again) given in [34], where it appears as the most difficult of the three quasi-periodic bifurcations inherited from generic bifurcations of periodic orbits. For an extension to the degenerate case see [284, 285].

Bifurcations in Hamiltonian systems

Compared to the above rich theory of torus bifurcations in dissipative dynamical systems, there are few results on conservative systems prior to [139] that I am aware of. In [41, 42, 32] invariant tori of dimension 2 and 3 are established in the universal 1-parameter unfolding of a volume-preserving vector

[3] Here a convenient choice of a basis on $\mathbb{T}^n$ is assumed.

field with an equilibrium having eigenvalues $0, \pm i$ or $\pm i\omega_1, \pm i\omega_2$, respectively. In the Hamiltonian case the existence of invariant tori near an elliptic equilibrium is due to the excitation of normal modes and generalizes the Lyapunov centre theorem, see [55] and references therein.

This lack of a bifurcation theory for invariant tori in Hamiltonian systems is all the more surprising as no external parameters are necessary. Indeed, every angular variable on a torus has a conjugate action variable whence n-tori form n-parameter families. The present notes aim to fill this gap in the literature.

In the "integrable" case, when there are sufficiently many symmetries, the situation can be reduced to bifurcations of (relative) equilibria. For this reason we develop the latter theory in a systematic way. From the various families of equilibria one can easily reconstruct the bifurcation scenario of invariant tori in an integrable Hamiltonian system.

While integrable systems have received a lot of attention – not to the least because their dynamics *can* be completely understood – it is highly exceptional for a Hamiltonian system to be integrable. Still, one often takes an integrable system as starting point and studies Hamiltonian perturbations away from integrability. Also if explicitly given a non-integrable Hamiltonian system, one of the few methods available is to look for an integrable approximation, e.g. given by normalization, and to consider the former as a perturbation of the latter. By a dictum of Poincaré the problem of studying the effects of small Hamiltonian perturbations of an integrable system is the fundamental problem of dynamics.

KAM theory is a powerful instrument for the investigation of this problem. It states that most[4] of the quasi-periodic motions constituting the integrable dynamics survive the perturbation, provided that this perturbation is sufficiently (and this means *very*) small. In a more geometric language these motions correspond to invariant tori. Under Kolmogorov's non-degeneracy condition one may consider the (internal) frequencies as parameters, and the Diophantine conditions (0.1) bounding the latter away from resonances lead to the Cantor families of tori one is confronted with in the perturbed system.

In its first formulation KAM theory addressed the "maximal" tori, and only later generalizations were formulated and proven for families of invariant tori that derive from hyperbolic and/or elliptic equilibria. For an overview over this still active research area see [55]. The present notes further generalize these results to families of invariant tori that lose (or gain) hyperbolicity during a bifurcation. Such bifurcations are governed by the nonlinear terms of the vector field. In this way singularity theory both governs the bifurcation scenario and helps deciding how these nonlinear terms are dealt with during the KAM-iteration procedure. As a result, the various smooth families

[4] The relative measure of those parametrising internal frequencies for which the torus is destroyed vanishes as the size of the perturbation tends to zero.

of invariant tori of the integrable system get replaced by Cantor families of invariant tori organizing the perturbed dynamics.

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Utrecht, May 2006 *Heinz Hanßmann*

Contents

The sequential order in these notes is from equilibria to invariant tori, which means that the various types of bifurcations appear and re-appear in different chapters. Therefore the following overview on the main Hamiltonian bifurcations of co-dimension one might be helpful.

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1

Introduction

Dynamical systems describe the time evolution of the various states $z \in \mathcal{P}$ in a given state space. When this description includes both (the complete) past and future this leads to a *group action*[1]

$$\begin{array}{rccc} \varphi : & \mathbb{R} \times \mathcal{P} & \longrightarrow & \mathcal{P} \\ & (t, z) & \mapsto & \varphi_t(z) \end{array}$$

of the time axis $\mathbb{R}$ on $\mathcal{P}$, i.e. $\varphi_0 = \mathrm{id}$ (the present) and $\varphi_s \circ \varphi_t = \varphi_{s+t}$ for all times $s, t \in \mathbb{R}$. Immediate consequences are $\varphi_s \circ \varphi_t = \varphi_t \circ \varphi_s$ and $\varphi_t^{-1} = \varphi_{-t}$. In case φ is differentiable one can define the vector field

$$X(z) = \left. \frac{\mathrm{d}}{\mathrm{d}t} \varphi_t(z) \right|_{t=0}$$

on $\mathcal{P}$ and if e.g. $\mathcal{P}$ is a differentiable manifold then φ can be reconstructed from X as its flow. Note that

$$\dot{z} = X(z) \tag{1.1}$$

defines an autonomous ordinary differential equation on $\mathcal{P}$.

Given a state $z \in \mathcal{P}$ the set $\{ \varphi_t(z) \mid t \in \mathbb{R} \}$ is called the orbit of z. Particularly simple orbits are equilibria, $\varphi_t(z) = z$ for all $t \in \mathbb{R}$, and periodic orbits which satisfy $\varphi_T(z) = z$ for some period $T > 0$ and hence $\varphi_{t+T}(z) = \varphi_t(z)$ for all $t \in \mathbb{R}$. All other orbits define injective immersions $t \mapsto \varphi_t(z)$ of $\mathbb{R}$ in $\mathcal{P}$. By definition unions of orbits form sets $M \subseteq \mathcal{P}$ that are invariant under φ, and if M is a differentiable manifold we call M an invariant manifold.

A complete understanding of a dynamical system φ is equivalent to finding (and understanding) all solutions of (1.1) whence one often concentrates on the long time behaviour as $t \to \pm\infty$. One approach is to determine all attractors[2]

[1] *Technical terms* are explained in a glossary preceeding the references.

[2] Since there are no attractors in Hamiltonian dynamical systems we do not give a formal definition.

in $\mathcal{P}$, compact invariant subsets A satisfying $\varphi_t(z) \stackrel{t\to+\infty}{\longrightarrow} A$ for all z near A, that are minimal with this property. Such attractors can be equilibria, periodic orbits, invariant manifolds, or even more general invariant sets. If A is an invariant manifold without equilibrium, then the Euler characteristic of A vanishes and the simplest such manifolds are the n-tori T, submanifolds of $\mathcal{P}$ that are diffeomorphic to $\mathbb{T}^n = \mathbb{R}^n/_{\mathbb{Z}^n}$. Where we speak of n-tori we always assume $n \geq 2$ in these notes.

The flow φ on a torus T is parallel or *conditionally periodic* if there is a global chart

$$\begin{array}{ccc} \mathsf{T} & \longrightarrow & \mathbb{T}^n \\ z & \longmapsto & x \end{array}$$

and a frequency vector $\omega \in \mathbb{R}^n$ such that[3]

$$\bigwedge_{x\in\mathbb{T}^n} \bigwedge_{t\in\mathbb{R}} \varphi_t(x) \;=\; x + \omega t \; .$$

In case there are no resonances $\langle k, \omega\rangle = 0\,,\ k \in \mathbb{Z}^n$ every orbit on T is dense. If there are $n-1$ independent resonances then ω is a multiple of an integer vector and all orbits on T are periodic. For $m \leq n-2$ independent resonances the motion is *quasi-periodic* and spins densely around invariant $(n-m)$-tori into which T decomposes. The flow on a given invariant torus may be much more complicated, this is often accompanied by a loss of differentiability. However, if the flow is equivariant with respect to the $\mathbb{T}^n$-action $x \mapsto x + \xi$ then all motions are necessarily conditionally periodic. Our starting point is therefore a family of tori carrying parallel flow, and we hope for persistence under small perturbations for the measure-theoretically large subfamily where the frequency vector satisfies a strong non-resonance condition.

Considering the long time behaviour for $t \to -\infty$ attractors are replaced by repellors and more generally one is interested in "minimal" invariant sets M. Where the dynamics on M itself is understood – for equilibria, periodic orbits and invariant tori with conditionally periodic flow – one concentrates on the dynamics nearby. Equilibria and periodic orbits are (under quite weak conditions) *structurally stable* with respect to small perturbations of the dynamical system, while invariant tori and more complicated, strange invariant sets may disintegrate. This makes it preferable to study parametrised families of such invariant sets.

In applications the equations of motion are known only to finite precision of the coefficients. Giving these coefficients the interpretation of parameters leads to a whole family of dynamical systems. Under variation of the parameters the invariant sets may then bifurcate. Bifurcations of equilibria are fairly well understood, at least for low co-dimension, cf. [129, 173] and references therein. Since these bifurcations concern a small neighbourhood of the equilibrium, we speak of *local bifurcations*. Using a *Poincaré mapping*, periodic orbits can be

[3] We use the same letter φ for the flow in the chart as well.

studied as fixed points of a discrete dynamical system. In addition to the analogues of bifurcations of equilibria, periodic orbits may undergo period doubling bifurcations, cf. [223, 58].

For a family of invariant n-tori with conditionally periodic flow the frequency vector ω varies in general with the parameter; let us therefore now consider $\omega \in \mathbb{R}^n$ itself as the parameter. Clearly both the resonant and the non-resonant tori are dense in the family. Under an arbitrary small perturbation (breaking the $\mathbb{T}^n$-symmetry that forces the toral flows to be conditionally periodic) the situation changes drastically. Using KAM-techniques one can formulate conditions under which most invariant tori survive the perturbation, together with their quasi-periodic flow; the families of tori are parametrised over a Cantor set of large n-dimensional (Hausdorff)-measure, see [159, 56, 55]. Within the gaps of the Cantor set completely new dynamical phenomena emerge; the dynamics on the torus may cease to be conditionally periodic[4] even in case there are circumstances like *normal hyperbolicity* that force the torus to persist. Note that the union of the gaps of a Cantor set is open and dense in $\mathbb{R}^n$. This is an exemplary instance of coexisting complementary sets, one of which is measure-theoretically large and the other topologically large, cf. [231].

It turns out that the bifurcations of equilibria and periodic orbits have quasi-periodic counterparts, see [34, 284] and references therein. In the integrable case where the perturbation respects the $\mathbb{T}^n$-action this is an immediate consequence of the behaviour of the reduced system obtained after reducing the torus symmetry. In the nearly integrable case where the torus symmetry is broken by a small perturbation one can use KAM theory to show that the bifurcation persists on Cantor sets. Notably the bifurcating torus has to be in *Floquet form*. In the same way the higher topological complexity of periodic orbits leads to period doubling bifurcations, tori that are not in Floquet form can bifurcate in a skew Hopf bifurcation, see [282, 60].

Bifurcations of invariant tori have a *semi-local* character, they concern a neighbourhood of the invariant torus which need not be confined to a small region of $\mathcal{P}$. Exceptions are bifurcations subordinate to local bifurcations and these were in fact the motivating examples for the above results. In contrast, global bifurcations lead to new interactions of different parts of $\mathcal{P}$ not present before or after the bifurcation. Examples are *connection bifurcations* involving heteroclinic orbits (these also exist subordinate to local or semi-local bifurcations).

The quasi-periodic persistence results in [159, 56, 55] are formulated and proven in terms of Lie algebras of vector fields and this allowed for a generalization to volume-preserving, Hamiltonian and reversible dynamical systems,

[4] For instance, if $\omega \in \omega_0 \cdot \mathbb{Z}^n$ only finitely many periodic orbits are expected to survive and the perturbed flow may consist of asymptotic motions between these. The structural stability of surviving periodic orbits is in turn the reason why a simple resonant frequency vector opens a whole gap of the Cantor set.

see also [216]. We will henceforth speak of dissipative systems when there is no such structure preserved. A dynamical system is Hamiltonian if the vector fields derives[5] from a single "Hamiltonian" function by means of a *Poisson structure*, a bilinear and alternating composition on $\mathcal{A} \subseteq C(\mathcal{P})$ that satisfies the Jacobi identity and Leibniz' rule. An important feature of integrable Hamiltonian systems is that the torus symmetry yields conjugate actions by Noether's theorem. Accordingly, invariant n-tori in integrable Hamiltonian systems with d degrees of freedom, $d \geq n$, occur as "intrinsic" n-parameter families, without the need for external parameters.

In particular, periodic orbits form 1-parameter families, or 2-dimensional cylinders (while equilibria remain in general as isolated as in the dissipative case). Thus, periodic orbits in (single) integrable Hamiltonian systems may undergo co-dimension one bifurcations, without the need of an external parameter. The ensuing possibilities were analysed in [205, 207], see also [208, 38, 232, 227, 228]. This yields transparent explanations for common phenomena like the gyroscopic stabilization of a sleeping top, cf. [13, 84, 81, 147].

Interestingly, results on bifurcations of invariant n-tori (which form n-parameter families in a Hamiltonian system) were first derived in the dissipative context (where external parameters are needed), see again [34] and references therein. Our aim is to detail the Hamiltonian part of the theory, extending the results in [139, 50] to more general bifurcations. At the same time we seize the occasion to put the well-known results on Hamiltonian bifurcations of equilibria, which are scattered throughout the literature, into a systematic framework. See also [75, 76, 45, 44] for recent progress concerning torus bifurcations in the reversible context.

1.1 Hamiltonian systems

A Hamiltonian system is defined by a Hamiltonian function on a phase space. The latter is a *symplectic manifold*, or, more generally a *Poisson space*, where the Hamiltonian H determines the vector field

$$X_H \; : \; \dot{z} \; = \; \{z, H\} \; .$$

If all solutions of X_H exist for all times, the flow φ^H is a group action

$$\begin{array}{rccc} \varphi^H : & \mathbb{R} \times \mathcal{P} & \longrightarrow & \mathcal{P} \\ & (t, z) & \mapsto & \varphi_t^H(z) \end{array} \qquad (1.2)$$

on the phase space $\mathcal{P}$ – in case there are orbits that leave $\mathcal{P}$ in finite time (1.2) is only a *local group action*.

Despite this simple construction where a single real valued function defines a whole vector field, the study of Hamiltonian systems is a highly non-trivial

[5] Similar to gradient vector fields defined by means of a Riemannian structure.

task. The first systems that were successfully treated were integrable and the study of Hamiltonian systems still starts with the search for the integrals of motion. Since $\{H, H\} = 0$ the Hamiltonian is always[6] an integral of motion, whence all systems with one *degree of freedom* are integrable.

However, already in two degrees of freedom integrable systems are the exception rather than the rule, cf. [239, 117, 26]. This led to the so-called *ergodic* hypothesis that the flow of a Hamiltonian system is "in general" ergodic on the *energy shell*. That this hypothesis does not hold for *generic* Hamiltonian systems, see [191], is one of the consequences of KAM theory.

KAM theory deals with small perturbations of integrable systems and may in fact be thought of as a theory on the integrable systems themselves. Indeed, in applications the special circumstances that render a Hamiltonian system integrable may not be satisfied with absolute precision and only properties that remain valid under the ensuing small perturbations have physical relevance.

An integrable Hamiltonian system with, say, compact energy shells gives the phase space $\mathcal{P}$ the structure of a *ramified torus bundle*. The regular fibres of this bundle are the maximal invariant tori of the system. The singular fibres define a whole hierarchy of lower dimensional tori, in case of (dynamically) unstable tori together with their *(un)stable manifolds*. In this way there are two types of "least degenerate" singular fibres: the *elliptic* tori with one *normal frequency* and the *hyperbolic* tori T with stable and unstable manifolds of the form $\mathsf{T} \times \mathbb{R}$. These two types of singular fibres determine the distribution of the regular fibres. Different families of maximal tori are separated by (un)stable manifolds of hyperbolic tori and may shrink down to elliptic tori.

On the next level of the hierarchy of singular fibres of the ramified torus bundle we can distinguish four or five different types. Lowering the dimension of the torus once more we are led to elliptic tori with two normal frequencies, to *hypo-elliptic* tori and to hyperbolic tori with four *Floquet exponents*. For these latter we might want to distinguish between the focus-focus case of a quartet $\pm\Re \pm i\Im$ of complex exponents and the saddle-saddle case of two pairs of real exponents. This decision would relegate hyperbolic tori with a double pair of real exponents to the next level of the hierarchy of singular fibres. We can do the same with elliptic tori with two resonant normal frequencies. Where the two normal frequencies are in $1{:}-1$ resonance, the torus may undergo a quasi-periodic *Hamiltonian Hopf bifurcation* and we *always* relegate these elliptic tori to the third level of the hierarchy of singular fibres of the ramified torus bundle.

The last type of second level singular fibres consists of invariant tori (and their (un)stable manifolds) of the same dimension as the first level tori, but with *parabolic* normal behaviour. Such tori may for instance undergo a quasi-periodic *centre-saddle bifurcation*. We see that the kth level singular fibres determine the distribution of the $(k-1)$th level singular fibres (where we could abuse language and address the regular fibres as 0th level singular fibres).

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[6] Our Hamiltonians are autonomous, there is no explicit time dependence.