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FINITE ELEMENTS OF NONLINEAR CONTINUA

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PREFACE

This book describes the finite element method and its application to a large class of nonlinear problems in structural and continuum mechanics. Special emphasis is given to the solution of problems in solid mechanics, but the general theory and methods of formulation are sufficiently general to be applied to nonlinear problems in, for example, fluid mechanics, electromagnetism, and partial differential equations. Various numerical methods for the solution of large systems of nonlinear equations are also examined.

My interest in the numerical analysis of nonlinear continua grew from a combination of a long interest in nonlinear mechanics, an appreciation of the great potential of modern digital computers for solving nonlinear problems, and a realization that much of the practical value of modern nonlinear theories of structural and material behavior will ultimately depend upon the availability of means to apply them to specific practical problems. Some years ago, I began to investigate the feasibility of applying the finite element method to the analysis of finite deformations of elastic solids. The surprising success of these early investigations, some of which form the basis for portions of this book, encouraged me to consider expanding the scope to nonlinear continua in general. In subsequent years, I developed and taught a graduate course on finite element applications in nonlinear mechanics at The University of Alabama in Huntsville, in which I attempted to draw together both the fundamentals of continuum mechanics and modern methods of numerical analysis. When these two subjects are brought together, each acquires new meaning and significance. The nonlinear field theories of mechanics then become valuable not only because they provide elegant generalizations of the classical theories, but also because, with the aid of electronic computing techniques, they provide a source for obtaining quantitative information on actual nonlinear phenomena encountered in nature. The finite element concept, with its simplicity and generality, provides the necessary ingredient for bringing these diverse subjects together in a manner which, in retrospect, may appear far more natural than many of the classical treatments of applied mechanics.

In selecting the topics to be covered in this book, it has not been my intention to provide an exhaustive collection of solutions to all kinds of nonlinear structural problems. Rather, the purpose here is to describe a general and physically appealing method for obtaining discrete models of continuous media, and to present a self-contained account of the application of this method to the analysis of representative nonlinear problems in solid mechanics. Once the basic notations are digested, applications to numerous nonlinear problems not examined herein should be straightforward.

So as to make the book self-contained, Chapter I contains an introductory discussion of the general concept of finite elements along with summary discussions of the kinematics of continuous media, the concept of stress, and the fundamental principles of conservation of mass and balance of momentum. Chapter II contains an account of the general theory of finite elements. Here the topological properties of finite element models of general fields are presented in forms valid for spaces of any finite dimension. Various types of finite element models are discussed as well as convergence criteria, and applications to linear and nonlinear differential equations, wave phenomena, and rarefied gas dynamics. This chapter also contains a detailed discussion of conjugate subspaces and the theory of conjugate approximations. Chapter III deals with the mechanics of a typical finite element of a continuous media. It begins with a discussion of appropriate thermodynamical concepts and principles, which is followed by derivations of local and global forms of the principle of conservation of energy for a continuum. These principles are used in conjunction with the theory developed in Chapter II to derive general kinematical equations and equations of motion and heat conduction for a finite element of arbitrary continuous media. A brief survey of the theory of constitutive equations is also included, and forms of constitutive equations cast in terms of discrete models of displacement and temperature fields are presented. In Chapter IV, applications of the finite element method to the analysis of nonlinear elasticity problems are presented. The chapter begins with an account of the theory of finite elastic deformations. Then nonlinear stiffness relations for elastic solids are derived, and solutions to a number of problems are presented. These include the problems of finite deformations of incompressible solids of revolution, stretching and inflation of elastic membranes, and finite plane strain of incompressible elastic solids. Also included in this chapter is a survey of various methods for the solution of large systems of nonlinear equations. Chapter V is devoted to inelastic behavior, with special emphasis on thermomechanically simple materials and materials with memory. General equations of motion and heat conduction for finite elements of such materials are derived. A number of applications of these equations to selected problems are examined, including problems in linear and nonlinear coupled thermoelasticity and nonlinear coupled thermoviscoelasticity.

I have discovered that writing a book is a nonlinear problem, the solution of which requires many iterations. Since the present form of this work varied very little in the last few iterations, I present it with the hope that it provides an approximate solution to the problem at hand. Nonlinear applied mechanics, however, is still in its infancy and is growing more rapidly with each passing day. Thus, the sequence is far from having converged. If this book provides a starting point for further iterations, it will have served a purpose for which it was intended.

I am grateful for the encouragement received from a number of colleagues and students during the preparation of this book. Of particular benefit were the comments and suggestions of Professors H. J. Brauchli and G. Aguirre-Ramirez. My discussions of related topics with Professors T. J. Chung and G. A. Wempner were also rewarding. Early versions of certain chapters of this book were distributed to students, and I sincerely appreciate their assistance and encouragement; among them, I am especially grateful to Messrs. J. E. Key, D. R. Bhandari, W. H. Armstrong, T. Sato, J. W. Poe, and D. A. Kross. For portions of my research which led to some of the ideas discussed in this book, I owe thanks to the support of the National Aeronautics and Space Administration and the U.S. Air Force Office of Scientific Research.

For their patience and diligency in keeping track of the multitude of subscripts, superscripts, and mathematical symbols in my manuscript, the staff at McGraw-Hill, particularly Jack Maisel, have my sincere thanks. It is also a pleasure to acknowledge the assistance of Mrs. D. Wigent, who, with enthusiasm and good humor, did an outstanding job of typing the entire manuscript.

Last, but far from least, I thank the one to whom this book is dedicated; for her continued encouragement, assistance in the proofreading and many personal sacrifices to help me finish this work, I shall always be grateful.

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Preliminary Discussions

1 INTRODUCTION

During the first half of the twentieth century, much of the literature on solid and structural mechanics was concerned with applications of long-standing linear theories to various boundary-value problems. There were notable exceptions, of course, such as the work which led to the rebirth and development of classical plasticity and viscoelasticity, the scattered attempts, some partially successful, at developing unified theories of material behavior, and the large number of studies of geometric nonlinearities by investigators who "retained nonlinear terms." To most of the engineering and scientific community, however, practical applications of solid mechanics meant the solution of linear problems.

The reason for this is easily understood, for the behavior of the majority of practical structures in the past could be adequately described by linear theories. The deformations of most structures under working loads, for example, were often scarcely detectable with the unaided eye, and for small deformations and steady uniform temperatures, the constitutive equations for such common materials as steel and aluminum can be treated as linear, without appreciable error.

This situation has drastically changed. Since 1950, many new materials have been introduced whose response cannot be described by classical linear theories. The thermoviscoelastic response of solid propellants, the postbuckling behavior of flexible structures, the use of highly deformable inflatable structures, and the nonlinear behavior of polymers and synthetic rubbers are only a few of the new problem areas that have encouraged the interest in nonlinear solid mechanics in recent times. The theory of elasticity has since been cast in general form, new nonlinear theories of viscoelasticity and thermoviscoelasticity have been proposed, and guiding principles for deriving constitutive equations for nonlinear materials are now generally accepted. The theme of modern research into nonlinear material behavior has been *generality*, and several theories have been proposed which span the gamut from elastic solids to thermoviscous fluids.

In spite of the advances in nonlinear theories of structural and material behavior, very little quantitative information is available to those who encounter nonlinear phenomena in practical applications. Nonlinear theories lead to nonlinear equations, which immediately render classical methods of analysis inapplicable. In all the work published on nonlinear behavior, only a handful of exact solutions to specific problems can be found; and these, without exception, deal with bodies of the most simple geometric shapes and boundary conditions. Often a "semi-inverse method" is employed, in which the shape of the deformed body is assumed to be known in advance (a situation that one seldom is so fortunate as to encounter in practice), and even in these cases numerical techniques must often be introduced in the final steps of the solution in order to obtain quantitative results.

This scarcity of quantitative information is, in some respects, quite ironic, for concurrent with the recent progress in nonlinear solid mechanics has been the development of the most powerful device for obtaining quantitative data that man has ever known—the digital computer. But, on the one hand, followers of the computational sciences have devoted full attention to new fields such as cybernetics and nonlinear programming, while, on the other hand, most researchers in continuum mechanics have been attracted to the purely theoretical aspects of the subject. In the middle ground lies a fertile and potentially important field: numerical analysis of nonlinear continua. It represents a marriage of modern theories of continuous media and modern methods of numerical analysis, so that, with the aid of electronic computation, quantitative information on the nonlinear behavior of solids and structures can be obtained. A systematic study of a portion of this middle ground is the subject of this book.

2 THE FINITE-ELEMENT CONCEPT

One must often resort to numerical procedures in order to obtain quantitative solutions to nonlinear problems in continuum mechanics. However, regardless of the initial assumptions and the methods used to formulate a problem, if numerical methods are employed in evaluating the results, the continuum is, in effect, approximated by a discrete model in the solution process. This observation suggests a logical alternative to the classical approach, namely, *represent the continuum by a discrete model at the onset.* Then further idealization in either the formulation or the solution may not be necessary. One such approach, based on the idea of piecewise approximating continuous fields, is referred to as the *finite-element method.* Its simplicity and generality make it an attractive candidate for applications to a wide range of nonlinear problems.

Classically, the analysis of continuous systems often began with investigations of the properties of small differential elements of the continuum under investigation. Relationships were established among mean values of various quantities associated with the infinitesimal elements, and partial differential equations or integral equations governing the behavior of the entire domain were obtained by allowing the dimensions of the elements to approach zero as the number of elements became infinitely large.

In contrast to this classical approach, the finite-element method begins with investigations of the properties of elements of finite dimensions. The equations describing the continuum may be employed in order to arrive at the properties of these elements, but the dimensions of the elements remain finite in the analysis, integrations are replaced by finite summations, and the partial differential equations of the continuous media are replaced, for example, by systems of algebraic or ordinary differential equations. The continuum with infinitely many degrees of freedom is thus represented by a discrete model which has finite degrees of freedom. Moreover, if certain completeness conditions are satisfied, then, as the number of finite elements is increased and their dimensions are decreased, the behavior of the discrete system converges to that of the continuous system. A significant feature of this procedure is that, in principle, it is applicable to the analysis of finite deformations of materially nonlinear, anisotropic, nonhomogeneous bodies of any geometrical shape with arbitrary boundary conditions.

2.1 HISTORICAL COMMENT

The idea of representing continuous functions by piecewise approximations is hardly a new one. Rudiments of the ideas of interpolation were supposedly used in ancient Babylonia and Egypt and, hence, preceded the calculus by over two thousand years. Much later, early Oriental mathematicians sought to evaluate the magical number π by determining the approximate area of the unit circle. This they accomplished to accuracies of almost forty significant figures by representing the circle as a collection of a large but finite number of rectangular or polygonal areas, the sum of which was taken as the area of the circle. It was left to Newton and Leibnitz to introduce the ideas of calculus, which have since made possible the formulation of most of the problems of mathematical physics in terms of partial differential and integral equations. Of course, the frequent failure of attempts to apply classical analytical methods to obtain solutions to many of these equations, plus the advent of the digital computer, has led an increasing number of investigators of modern times to consider approximate methods of analysis. It is interesting to note, however, that in many cases these investigators may unknowingly resort to concepts more primitive than those used to obtain the equations they wish to solve.

The practice of representing a structural system by a collection of discrete elements dates back to the early days of aircraft structural analysis, when wings and fuselages, for example, were treated as assemblages of stringers, skins, and shear panels. By representing a plane elastic solid as a collection of discrete elements composed of bars and beams, Hennikoff [1941] introduced his "framework method," a forerunner to the development of general discrete methods of structural mechanics. Topological properties of certain types of discrete systems were examined by Kron [1939]†, who developed systematic procedures for analyzing complex electrical networks and structural systems. Courant [1943]‡ presented an approximate solution to the St. Venant torsion problem in which he approximated the warping function linearly in each of an assemblage of triangular elements and proceeded to formulate the problem using the principle of minimum potential energy. Courant's piecewise application of the Ritz method involves all the basic concepts of the procedure now known as the finite-element method. Similar ideas were used later by Polya [1952]. The hypercircle method, presented in 1947 by Prager and Synge [1947] and discussed at length by Synge [1957]§, can be easily adapted to finite-element applications, and it provided further insight into the approximate solution of certain boundary-value problems in mathematical physics. In 1954, Argyris and his collaborators¶ began a series of papers in which they developed extensively certain generalizations of the linear theory of structures and presented procedures for analyzing complicated, discrete structural configurations in forms easily adapted to the digital computer.

† See also, for example, Kron [1944*a*, 1944*b*, 1953, 1954, 1955].

‡ See also Courant, Fredrichs, and Lewy [1928].

§ Synge [1957] speaks of linear interpolation over triangulated regions; his use of "polyhedral graphs" and "pyramid functions" is clearly in the spirit of the finite-element method.

¶ Argyris [1954, 1955, 1956, 1957], Argyris and Kelsey [1956, 1959, 1960, 1961, 1963], Argyris, Kelsey, and Kamel [1964].