Generalized Functions

& Convergence

GENERALIZED FUNCTIONS and CONVERGENCE

Memorial Volume for Professor Jan Mikusiński

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PREFACE

The Conference on Generalized Functions and Convergence was held in Katowice, Poland, June 13–18, 1988. The conference was organized by the Institute of Mathematics of the Polish Academy of Sciences in collaboration with the Institute of Mathematics of Silesian University and the Polish Mathematical Society. The organizing committee was led by P. Antosik, A. Kamiński and K. Skórnik. The conference was dedicated to the memory of Professor Jan Mikusiński who died a year earlier, on July 27, 1987. A special session devoted to the life and work of the late Professor Mikusiński was held on June 14, 1987. There were 28 participants in the conference and 90 participants of the session, from Poland and other countries.

In this volume the material presented at the conference is collected together with additional articles prepared by those mathematicians who were invited to the conference but they could not take part.

The volume consists of three parts. Part I: Jan Mikusiński (1913–1987) contains biographical material about Professor Jan Mikusiński, i.e. his biography, the list of his publications, two articles by Professors D. Przeworska-Rolewicz and H. Komatsu, presented during the session, and tributes from several mathematicians who agreed to share with us their memories of the late Professor Mikusiński. Part II: Generalized Functions, and Part III: Convergence Theory and Sequential Methods in Functional Analysis and Measure Theory contain mathematical papers presented at the conference or sent for publication in this volume in memory of Professor Jan Mikusiński. It is noteworthy that the theory of generalized functions, convergence structures and sequential methods in functional analysis and measure theory were the chief fields of interest of Professor Mikusiński to whom that volume is dedicated; a long series of international conferences on these topics was initiated by Professor Mikusiński and organized in Poland and other countries.

Many persons contributed in preparing this volume. The editors are very grateful to Drs. J. Burzyk, K. Łoskot, K. Rudnicki, K. Skórnik, Mrs. B. Smołka, and particularly to Dr. J. Uryga for their assistance. With few exceptions, the text of articles was prepared using TEX software in the Institute of Mathematics of the Polish Academy of Sciences in Warsaw. The editors would like to thank Mrs. M. Wolińska for careful and painstaking typesetting of the material.

April 1990
Katowice, Poland

Andrzej Kamiński

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PART I

JAN MIKUSIŃSKI (1913–1987)

LIFE AND WORK OF PROFESSOR JAN MIKUSIŃSKI

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1. Life

Jan Mikusiński, the great Polish mathematician, was born at Stanisławów on April 3, 1913. He completed his secondary education and university studies in Poznań and then, in 1937, started his scientific career at the University of Poznań.

During the war (1939–45) he lived mainly in Cracow. He took part in underground teaching, was arrested by the Nazis and spent some time in prison. Later he took part in Professor Tadeusz Ważewski's secret war-time mathematical seminar, whose participants, in 1943, were the first persons to come in contact with a new theory which is now very well known in the world of mathematics as the Mikusiński operational calculus. The objects of the theory, operators, provide a common generalization of numbers and locally integrable functions on the positive halfline.

The author first called them "hypernumbers" and gave this title to a paper containing the main ideas of the theory. Let us recall the interesting history of this paper. It was first prepared in 1944 and its issuing in the form of a small book was quite extraordinary in war time conditions. The edition was limited to seven copies and the printing was made by the author himself using X-ray films. Only initials were for the name of the author and for the names of the participants in the seminar whom the author thanked in the introduction.

The theory was later developed in numerous papers and books by Jan Mikusiński and his followers, but the first paper "Hypernumbers" had to wait for its next, more standard editions, up to 1983. Then "Hypernumbers" appeared simultaneously in Poland (Studia Mathematica, vol. 77) and in the USA (University of California, Santa Barbara) on the occasion of the 70th birthday of Professor Jan Mikusiński.

After the war, in 1945, Jan Mikusiński gained his Ph.D. at the Jagiellonian University presenting his thesis "Sur un problème d'interpolation pour les intégrales des équations différentielles linéaires". Then he worked again at the University of Poznań. In 1946, he moved to Lublin and there, at the Maria Skłodowska–Curie University, became associate professor of mathematics.

During the period 1948–1955 Jan Mikusiński was professor of the University of Wrocław and then (1955–1959) of the University of Warsaw, where he qualified as full professor in 1958. At the same time he also worked at the state Institute of Mathematics, established in 1948, which was later transformed into the Institute of Mathematics of the Polish Academy of Sciences.

In 1959 Jan Mikusiński was forced to resign from his professorship of the University of Warsaw because of serious illness but he held a position at the Institute of Mathematics of the Academy, where he was appointed head of the Division of Mathematical Analysis.

In 1960 Professor Mikusiński moved to Katowice where he became one of the creators of the mathematical community in Silesia. In spite of his poor state of health he gave in Katowice a series of lectures which turned into a permanent seminar conducted by him at home. These spontaneous contacts of the Silesian mathematical community with Professor Mikusiński gave the impetus for establishing later a mathematical center in Katowice which is now a part of the Institute of Mathematics of the Polish Academy of Sciences. He was head of this center until his retirement at the end of 1983 and together with his pupils organized an international cooperation in the field of generalized functions. The long series of international conferences on generalized functions and convergence structures was initiated in 1966 in Katowice and organized most frequently in Poland. From 1971 the conferences were held also in other countries. We may recall the most important conferences: Katowice 1966, Srebreno 1971, Rostock 1972, Wisła 1973, Szczyrk 1974, Varna 1975, Oberwolfach 1978, Szczyrk 1979, Moscow 1980, Katowice 1983, Bechyně 1984, Debrecen 1984, Szczyrk 1986, Dubrovnik 1987, Katowice 1988.

Professor Jan Mikusiński was appointed full member of the Polish Academy of Sciences in 1971. He was also a member of the Serbian Academy of Science and Art (1975), a honorary member of the Polish Mathematical Society (1984) and held a honorary doctorate of the University of Rostock (1970). He was honoured with numerous awards and distinctions by the authorities and the mathematical community.

He was invited to many mathematical centers all over the world. He was a visiting professor at the University of Aachen (1961) and the University of Ankara (1963). As an expert of the UNESCO he spent in 1962 three months in Argentina. About three years he spent as a visitor in the USA (Pasadena 1964, Gainesville 1968–70, Santa Barbara 1983, Orlando 1987). In 1980 he gave in Japan a series of lectures in the main academic centers. He also visited many centers in the Soviet Union, Czechoslovakia, Hungary, Bulgaria, Romania, Yugoslavia, East and West Germany, Switzerland, France, Holland, Turkey, Israel, Canada and maintained scientific links with mathematicians in these and other countries.

His interests in mathematics and its applications ranged widely and included real and complex analysis, differential and functional equations, generalized functions, functional analysis, the theory of measure and integration, algebra, geometry, number theory, mechanics, electrical engineering, acoustics, optics, photography, chromatography and music.

Jan Mikusiński is generally recognized as the creator of the algebraic approach in operational calculus and other theories: the elementary theory of distributions, the uniform approach to the Lebesgue and Bochner integrals. He was invited to many mathematical centers in various countries.

He was a great teacher and an efficient organizer of mathematical life.

In spite of poor health, he was a very active researcher and worked to the last days of his life. He published more than 150 papers and several books, the lists of which are presented in this volume.

His books have been published in various languages; the book "Operational Calculus" was published in Polish, English, Russian, German, Hungarian and Japanese and it had numerous editions (see Ref. I, II, IV, V, VII, X, XIII, XIX, XXVII–XXVIII).

2. Operational Calculus*

In the beginning Jan Mikusiński dealt with ordinary differential equations, turning back to the investigations of his teacher, professor Mieczysław Biernacki. He published more than twenty papers on differential equations.

Looking for simple methods to solve ordinary and partial differential equations, Jan Mikusiński produced an algebraic theory of operational calculus.

The history of operational calculus begins with the research of the English engineer Oliver Heaviside. He considered operators, not precisely defined objects, and performed on them formal calculations, which had no mathematical justification but led to correct results in electromagnetic theory. The justification was found later on the base of the Laplace transform, but this imposed certain restrictions on the growth of the considered functions at infinity. The approach proposed by Jan Mikusiński (see e.g. Ref. XXVII–XXVIII) releases the theory from these limitations and is conceptually simpler, referring to the primary calculus of Heaviside.

The basis for the Mikusiński operational calculus is the Titchmarsh theorem concerning the convolution of continuous functions on the positive halfline.

Let C denote the set of all real- or complex-valued continuous functions on $[0,\infty)$. If $f,g \in C$, then by the *convolution* (*convolutive product*) of f and g,

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* We refer to the list of publications of Professor Jan Mikusiński given in this volume. Note that the books of Jan Mikusiński are denoted by Roman and his papers published in journals by Arabic numerals. The publications of other authors referred to in this article are denoted by Arabic numerals with a bar: $\bar{1}$, $\bar{2}$ etc. (the list of other references is given at the end of the article).

denoted in the operational calculus by the symbol fg of the usual product, we mean the function

$$(fg)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau.$$

The Titchmarsh theorem (in its weaker so-called global version) states that if $fg = 0$ in $[0,\infty)$, then $f = 0$ on $[0,\infty)$ or $g = 0$ on $[0,\infty)$, i.e. C meant as the ring with the usual addition and the convolution is an integral domain. Due to this one may consider formal fractions f/g $(f, g \in C)$, where the division denotes the converse operation to the convolution.

The set of all formal fractions forms a field called the field of Mikusiński operators. This field contains all continuous functions f on $[0,\infty)$, because they may be represented by the fractions fg/g, where g is an arbitrary nontrivial continuous function on $[0,\infty)$. It contains also all elements α of the initial numerical field which can be identified with the fractions $\{\alpha\}/\{1\}$. Here and in the sequel, to distinguish the value $f(t)$ of the function at the point t from the function f itself, the symbol $\{f(t)\}$ is used, i.e. $\{\alpha\}$ and $\{1\}$ are constant functions equal to α and 1 on $[0,\infty)$.

The function $l = \{1\}$ may be interpreted as integral operator, because

$$l\{f(t)\} = \Big\{ \int_0^t f(\tau)\,d\tau \Big\}.$$

On the other hand, the operator $s = 1/l$ plays the role of the differential operator, because $sx = x' + x(0)$ for an arbitrary function $x = \{x(t)\}$ of the class C^1. The latter formula can be easily generalized as follows

$$s^n x = x^{(n)} + x^{(n-1)}(0) + sx^{(n-2)}(0) + \cdots + s^{n-1}x(0), \tag{1}$$

where x is a function of the class C^n.

This identity makes it possible to solve ordinary differential equations with constant coefficients in a strictly algebraic manner. Let us consider a differential equation of the order n:

$$a_n x^{(n)} + a_{n-1}x^{n-1} + \cdots + a_0 x = f \tag{2}$$

with the given function f, the constant coefficients $a_n, a_{n-1}, \ldots, a_0$ $(a_0 \neq 0)$ and the initial conditions:

$$x(0) = c_0,\; x'(0) = c_1,\; \ldots,\; x^{(n-1)}(0) = c_{n-1}. \tag{3}$$

In view of Eq. 1, the problem given by Eq. 2 and 3 reduces to a simple algebraic equation whose solution is of the form

$$x = \frac{b_{n-1}s^{n-1} + \cdots + b_0 + f}{a_n x^n + \cdots + a_0},$$

where

$$b_i = \sum_{k=1}^{n-i} a_{i+k}c_{k-1}.$$

To represent the above solution in the form of a function of the variable t one can use the standard technique of decomposing fractions into simple fractions of the types

$$\frac{1}{(s-\alpha)^p}, \quad \frac{1}{[(s-\alpha)^2+\beta^2]^p}, \quad \frac{s}{[(s-\alpha)^2+\beta^2]^p},$$

for which the functional representations are known.

It is equally simple to reduce a system of ordinary differential equations with constant coefficients to a corresponding system of algebraic equations.

The operational calculus also makes it possible to solve certain types of partial differential equations, e.g. the equations of string, heat, telegraphy.

It should be stressed that the method of the operational calculus allows one to find solutions of full generality while the method of the Laplace transform imposes restrictions on the growth of the solution at infinity.

The Mikusiński operators have various applications in areas as distant as electric circuits, statics of beams or chromatography. Simultaneously, operational calculus is an elegant mathematical theory which raises many interesting problems in various fields of mathematics.

The structure of the field of Mikusiński operators is interesting from the algebraic point of view. This is a formally real field in which a linear order can be introduced. Interesting problems are connected with the notion of so-called algebraic derivative, an abstract counterpart of the operation of differentiation for operators.

For classical analysis the connections of operational calculus with the theory of the Laplace transform are of significance. Let us notice that the Laplace transform can easily be extended to operators which are convolution quotients of transformable (in the sense of the Laplace transform) functions. On the other hand, there exist operators which cannot be represented as convolution quotients of transformable functions.

Let us observe that both Mikusiński and Sobolev–Schwartz distributions are generalizations of locally integrable functions and a wide class of generalized functions on $(0,\infty)$ lies in the intersections of the sets of operators and distributions. However distributions whose support coincides with the whole real line cannot be interpreted as operators and conversely, there exist operators, e.g. $\exp(\sqrt{s})$), which are not represented by any distribution (see Ref. 68). It is worth noting that there exists a common generalization of distributions and a wide class of operators, so called regular operators, introduced by T. K. Boehme[12]. Namely, Jan and Piotr Mikusiński[136–137] introduced algebraically (as quotients of some sequences) the notion of Boehmians (see also Ref. 46–47, 16; cf. Ref. 71–72), which embraces most of known generalized functions, e.g. ultradistributions of Beurling[7] and Roumieau[53]. There are also other generalizations of Mikusiński operators (see e.g. Ref. 36–38).

The field of the Mikusiński operators is interesting because of its topological, or rather convergence structure. That is, it can be endowed with the two types of convergence:

(I) $x_n \to x$ if $x_n = f_n/g$, $x = f/g$ ($f_n, f, g \in C$) and $f_n \to f$ almost uniformly as $n \to \infty$;

(II) $x_n \to x$ if $x_n = f_n/g_n$, $x = f/g$ ($f_n, f, g_n, g \in C$) and $f_n \to f$, $g_n \to g$ almost uniformly as $n \to \infty$.

Both types are very useful in the operational calculus but neither of them can be described by a topology.

Many deep results in the operational calculus have been obtained by Professor Mikusiński (see Ref. XXVII–XXVIII) and his numerous collaborators and pupils. The following list of authors of contributions is far from completeness: L. Berg[6], R. Bittner[8], T. K. Boehme[9–13, XXVIII], J. Burzyk[14–15, 17], I. H. Dimovski[20], V. A. Ditkin and A. P. Prudnikov[21], C. Foiaş[25], E. Gesztelyi[27], H. Komatsu[37–38], G. L. Krabbe[42], D. Przeworska-Rolewicz[51–52], C. Ryll-Nardzewski[54–55, 27, 35], B. Stanković[65], R. A. Struble[66–67], K. Urbanik[75], J. Wloka[79], K. Yosida[89].

3. Distributions

As mentioned in section 2, distributions introduced by S. L. Sobolev[64] and L. Schwartz[56] stand for another generalization of functions (defined in an arbitrary open set in R^m). The theory, developed in Schwartz's book (Ref. 57), is based on notions and deep results of functional analysis and thus is rather difficult for engineers who want to apply it. Therefore various approaches to the theory of distributions and other generalized functions appeared later (see e.g. Ref. 28, 39, 40, 58, 62, 73, 76, 36, 63, 66, 71, 72, 80, 81).

Professor Mikusiński already in 1948 (see Ref. 11) pointed out a simpler method of defining this notion. Using this method and the formal definitions given in Ref. 49, he developed together with R. Sikorski the so-called sequential theory of distributions, presented in Ref. 63 and 86 (see also Ref. VIII, IX, XII, XIV, XVI, XVII, XX) and later extended in collaboration with P. Antosik (see Ref. XXI–XXII).

The definition is based on the idea of approximating distributions by continuous or smooth (i.e. of the class C^∞) functions.

A sequence $\{\phi_n\}$ of smooth (real-valued) functions in R^1 is said to be *fundamental* if for every finite open interval I in R^1 there exist a nonnegative integer k and a sequence $\{\phi_n\}$ of smooth functions so that $\Phi_n^{(k)} = \phi_n$ in I and $\{\Phi_n\}$ uniformly converges in I. Two fundamental sequences $\{\phi_n\}$ and $\{\psi_n\}$ are equivalent ($\{\phi_n\} \sim \{\psi_n\}$) if the sequence $\phi_1, \psi_1, \phi_2, \psi_2, \ldots$ is fundamental. The relation is an equivalence relation and equivalence classes are called *distributions in* R^1 (an analogous definition can be formulated for distributions in an arbitrary open set in R^m). Smooth, continuous and locally integrable functions are easily identified with respective distributions.

The definition is elementary and understandable for everybody who knows the concepts of derivative and uniform convergence in the case of usual functions. An advantage of the sequential approach to the theory of distributions is easiness of extending to distributions many operations which are defined for smooth functions.

We say that an operation A, which to every system $(\varphi_1, \ldots, \varphi_k)$ of smooth functions in $\mathbb{R}^1$ assigns a smooth function in $\mathbb{R}^1$ (or a number), is *regular* if for arbitrary fundamental sequences $\{\varphi_{1n}\}, \ldots, \{\varphi_{kn}\}$ of smooth functions in $\mathbb{R}^1$ the sequence $\{A(\varphi_{1n}, \ldots, \varphi_{kn}\}$ is fundamental (convergent). If now $f_1, \ldots, f_k$ are arbitrary distributions in $\mathbb{R}^1$ and $\{\varphi_{1n}\}, \ldots, \{\varphi_{kn}\}$ the corresponding fundamental sequences, i.e. $f_1 = [\varphi_{1n}], \ldots, f_k = [\varphi_{kn}]$, then we define

$$A(f_1, \ldots, f_k) = [A(\varphi_{1n}, \ldots, \varphi_{kn})]$$

and the definition does not depend on fundamental sequences representing the distributions $f_1, \ldots, f_k$. All formulae involving regular operations, which hold true for smooth functions, are extended automatically to distributions. An example of a regular operation is the differentiation (of a given order j):

$$A(f) = f^{(j)},$$

which can be performed for an arbitrary distribution f. It is well known that every distribution is locally (i.e. on an arbitrary finite open interval in $\mathbb{R}^1$) a (distributional) derivative of a finite order of a continuous function. In the sequential approach it is a simple consequence of the definitions given above and properties of the uniform convergence. Using this theorem one can define the convergence of sequences of distributions in a natural way. It is also easy to show that the sequence $\{f * \delta_n\}$ is distributionally convergent to f (i.e. fundamental for f) for an arbitrary distribution f in $\mathbb{R}^1$, where $\{\delta_n\}$ is a so-called delta-sequence, i.e. a sequence of nonnegative smooth functions such that $\int \delta_n = 1$ and $\delta_n(x) = 0$ for $|x| \geq \alpha_n$ with $\alpha_n \to 0$ as $n \to \infty$. The convolution $f * w$ of a distribution f with a fixed smooth function ω of bounded support is meant here as a regular operation $A(f) = f * \omega$, which for smooth functions φ is defined in the known way:

$$A(\varphi)(x) = (\varphi * \omega)(x) = \int_{-\infty}^{\infty} \varphi(x - t)\omega(t)\, dt.$$

It should be noted that among operations important in practice there are regular and not regular ones. For instance, the two-argument operations of product $A(\varphi, \psi) = \varphi \cdot \psi$ and the convolution $A(\varphi, \psi) = \varphi * \psi$ are not regular operations and they cannot be defined for arbitrary distributions.

Professor Mikusiński pointed out a general method of defining irregular operations on distributions by using delta-sequences (see Ref. 82, 83 and Ref. XXI, p. 256–257).

Let us assume that an operation A is feasible for arbitrary smooth functions $\varphi_1, \ldots, \varphi_k$ and let $f_1, \ldots, f_k$ be arbitrary distributions in $\mathbf{R}^1$. We say that $A(f_1, \ldots, f_k)$ *exists* if for an arbitrary delta-sequence $\{\delta_n\}$ the sequence

$$\{A(f_1 * \delta_n, \ldots, f_k * \delta_n)\}$$

is fundamental; then we define

$$A(f_1, \ldots, f_k) = [A(f_1 * \delta_n, \ldots, f_k * \delta_n)].$$

If $A(f_1, \ldots, f_k)$ exists, then the distribution does not depend on the choice of delta-sequence $\{\delta_n\}$. If A is a regular operation then, of course, A exists and coincides with the earlier defined result of the regular operation. If A is irregular, it need not exist for all distributions, but the definition embraces not only earlier known cases, but also new ones. For instance, for the operation of the product $A(f_1, f_2) = f_1 \cdot f_2$, the definition can be expressed in the form

$$f_1 \cdot f_2 = \lim_{n \to \infty} (f_1 * \delta_n)(f_2 * \delta_n)$$

and it exists for a wide class of pairs of distributions; some important products of distributions exist in this sense while they do not exist in the classical sense of L. Schwartz[57]. For example, the significant in physics product $1/x \cdot \delta(x)$ exists and equals, according to expectations of physicists, $-1/2 \cdot \delta(x)$ (see Ref. 96 and XXIII, p. 249). The generalizations of this formula can be found in Ref. $\overline{23}$ and $\overline{30}$.

Another definition of the product (cf. Ref. 87):

$$f \cdot g = \lim_{n \to \infty} (f * \delta_{1n}) \cdot (g * \delta_{2n}), \tag{5}$$

where the limit is assumed to exist in the distributional sense for arbitrary delta-sequences $\{\delta_{1n}\}$ and $\{\delta_{2n}\}$, is known in the literature under the name of the Mikusiński product. R. Shiraishi and M. Itano[60] showed that the definition given in Eq. 5 is equivalent to the definition of Y. Hirata and H. Ogata[29], also formulated by means of delta-sequences. This definition is, however, less general than the definition given earlier in Eq. 4, because the product $1/x \cdot \delta(x)$ does not exist in the sense of Eq. 5 (cf. Ref. $\overline{33}$). The properties of both Mikusiński's products, their applications and connections with other definitions of the product of distributions have been studied by many authors (see e.g. Ref. $\overline{1}$, $\overline{18}$, $\overline{31}$, $\overline{34}$, $\overline{50}$, $\overline{60}$, $\overline{61}$, $\overline{74}$, $\overline{76}$ (p. 114), $\overline{77}$, $\overline{78}$).

The Mikusiński product of distributions in the sense of Eq. 5 is closely related to the notion of the Łojasiewicz value of a distribution at a point (see Ref. $\overline{31}$, $\overline{60}$, $\overline{61}$), which was defined and characterized in Ref. $\overline{44}$, but which may also be defined as an irregular operation by means of delta-sequences (see Ref. $\overline{80}$).