A SURVEY OF NONLINEAR DYNAMICS

("Chaos Theory")

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PREFACE

This report is intended to give a survey of the whole field of nonlinear dynamics (or "chaos theory," as it is popularly called) in a compressed form. It is slightly expanded from a series of lectures given over the space of a single month in 1989. This young and rapidly growing field is already very extensive, so that this survey cannot be deep or detailed. In particular, no pretense of mathematical rigor is made. But I do insist on stating key definitions or theorems carefully so that the reader need not settle for just a qualitative, intuitive understanding. My intention is to touch on the main ideas so that the reader can see if his or her special discipline fits in anywhere and if so, can get an approximate notion of what new ideas or possibilities nonlinear dynamics brings to that field. The cited literature then allows the reader to proceed further if he or she desires.

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CONTENTS

1. INTRODUCTORY REMARKS

1.1 Linear Versus Nonlinear

A *dynamics* describes the time evolution of a system. As such, the concept is not confined to physics, but occurs in many other fields as well – in related sciences like engineering, chemistry, and biology, but also in ecology, economics, etc. A *nonlinear* dynamics describes the time-evolution via nonlinear equations of motion, which may be ordinary differential equations, partial differential equations, difference equations, iteration of maps, etc. Nonlinear motion equations have been around a long time – since the beginning of science, in fact – so why the sudden blooming of nonlinear dynamics as a new discipline in the last 20 years or so?

The answer is that up to that time nonlinear equations were regarded as not essentially different from linear ones – more complicated and difficult to solve, of course, but nothing that suitably refined linear approximations couldn't handle. Analytic ("closed form") solutions were emphasized in textbooks with the confident expectation that "nonanalytic" solutions, if they existed, formed a small subset of all solutions which didn't greatly add to the understanding of the phenomena. But about 20 years ago it was realized that nonlinear equations are essentially different from linear ones, that they possess properties which can never be captured by linear approximations, that analytic solutions are the exception, not the rule, and that solutions sets may show "deterministic chaos."

Linear equations enjoy by definition the property of superposition. That is, linear combinations of solutions are also solutions: the solutions form a linear, or vector, space. Linear theories are highly structured theories, and one has many helpful theorems at hand. For example, a general solution exists; solutions have only "fixed" singularities, that is, those occurring in the linear equations themselves. But do not get the idea that linear theories are considered *passé* or discredited, now that we are elucidating the mysteries of nonlinear dynamics. Some of the most beautiful and accurate theories in physics are linear. Witness the Maxwell theory of electromagnetism, or quantum mechanics itself, the fundamental theory of the subatomic world. Indeed, today no failure of quantum mechanics is known.

Nonlinear equations are all the rest: all those which are *not* linear. Most of the convenient properties of a linear dynamics mentioned above are lost: there is no superposition and no general solution; analytic solutions are rare or nonexistent; solutions may have singularities not present in the motion equations, and these may depend on the initial conditions, etc. However, interesting new properties show up in compensation. Asymptotic (time $\to \infty$) solutions are often independent of initial conditions and lie on low-dimensional "attractors" in phase space. There is a complicated set of these stable regimes joined by bifurcations of various types. There may exist "chaotic" regimes.

Incidentally, the reader should not worry if some of the statements in these preliminary remarks of chapter 1 seem a bit vague or elusive by reason of undefined terms. For now, it is enough that they carry some sort of intuitive meaning. All important terms and concepts will be defined carefully at the proper places in this report.

1.2 The Goals of Nonlinear Dynamics

The dynamics that will be our main focus of attention in this report will be specified by one or several first order ordinary differential equations in time,

$$\dot{x} = f(x,t) \quad , \qquad \dot{x} \equiv \frac{dx}{dt} \quad , \tag{1-1}$$

or by a *map* $x \longmapsto F(x)$ which is *iterated:*

$$x, \quad F(x), \quad F[F(x)], \quad F[F[F(x)]], \quad \cdots . \tag{1-2}$$

The set of continuous time solutions, or orbits of (1-1) is called a *flow*, while the set of discrete-time orbits (1-2) is sometimes called a *cascade*. Both f and F depend on parameters which can be varied.

What information do we seek in nonlinear dynamics?

a. The geometry, or more often, topology of the flow (or cascade) as a whole: phase portraits, stable and unstable manifolds, various low-dimensional invariant attracting sets if they exist.

b. Bifurcation points, that is, those parameter values at which the flow "changes qualitatively."

c. The characteristics of "chaotic" flows, and the various paths to chaos which the dynamics admits.

What mathematical tools are available for this search? There exist theorems, far fewer than in linear dynamics, which limit the possibilities in nonlinear dynamics. Numerical computation (sometimes called "experimental mathematics") plays a big role in discovering the information listed above and in suggesting and motivating, if not proving, theorems about a particular dynamics.

1.3 "Chaos"

The quotation marks here signal that a consensus has not yet been reached on the precise definition of this term. This accounts for the many apparent contradictions and fruitless controversies in this subject. Following current custom, we shall mean by "chaos" any or all of the following properties: sensitive dependence on initial conditions, broadband power spectra, decaying correlations, or randomness or unpredictability of orbits as measured by positive algorithmic complexity or entropies of the various kinds. These properties are not all independent.

Of these, *sensitive dependence* (on initial conditions understood), abbreviated in this report as SD, is by far the most important ingredient of "chaos." In fact, in its strong, or exponential, form SD is accepted by most as the definition of chaos. The intuitive meaning of SD is the unpredictability - or uncomputability - in principle of some orbits. That is, inevitable errors in initial conditions, *no matter how small*, may get magnified on computation, so that the computed orbit (or some observable function of the orbit) bears no resemblance to the actual orbit (or function thereof). This has nothing to do with noise or perturbations from outside the system. Sensitive dependence is an intrinsic property of the dynamics in some parameter regimes; it is true "deterministic chaos." Obviously, this bears on the ancient philosophical dichotomy between determinism and chance (and seems at first sight to contradict it!).

In a system which displays "chaos," there may be several sequences of regimes leading to "chaotic" behavior, several "paths to chaos," so to say. The universality of these various paths in systems superficially very different (for example, iterating one-dimensional maps and viscous, incompressible fluid flow) is a surprising theoretical and experimental result.

To give the reader a preliminary feeling for sensitive dependence, this perhaps most important concept of nonlinear dynamics, we shall illustrate it on the simple dynamics of a 1D (one-dimensional) map. The other attributes of "chaos" mentioned above will be covered later in the main text. Consider the particular 1D map $F(x) \equiv \mu x(1-x)$ with $0 \le x \le 1$ and $0 < \mu \le 4$, that is, the iteration scheme $x_{n+1} = \mu x_n(1-x_n)$, $n = 0, 1, 2, 3, \cdots$, defining the orbit (1-2). Choose the parameter value $\mu = 4$ and substitute $x_n \equiv \sin^2 \pi\theta_n$, $0 \le \theta_n \le 1$. Then the iteration scheme takes the form $\sin^2 \pi\theta_{n+1} = 4\sin^2 \pi\theta_n \cos^2 \pi\theta_n$, that is,

$$\theta_{n+1} = 2\theta_n (mod 1) \qquad (1-3)$$

where $(mod 1)$ means that any integral part of $2\theta_n$ is chopped off so that the result lies in the interval (0,1). We can actually get an "analytic" solution (!) for this parameter value, namely

$$\theta_n = 2^n \theta_0 (mod 1),$$

where $\theta_0 \in (0,1)$ is the initial value. Now shift the initial point slightly: $\theta_0' = \theta_0 + \epsilon$; then

$$\theta_n' - \theta_n = 2^n \epsilon = \epsilon e^{n \ln 2}$$

(as long as $2^n\epsilon < 1$), that is, exponential separation of the two initially very close orbits with *Lyapunov exponent* $\ln 2$. Obviously, the "error" in the orbit will get big for n large enough, no matter how small ϵ. This is SD (in particular, exponential SD).

To be more quantitative about the SD, write the initial value θ_0 in binary notation. For example,

$$\theta_0 = 1/2 + 1/4 + 1/16 + 1/128 + \cdots = 0.1101001\cdots. \qquad (1-4)$$

Then iteration algorithm (1-3) amounts to shifting the "decimal point" to the right by one and dropping the digit to the left of this point. For the value (1-4),

$$\theta_0 = .1101001\cdots, \quad \theta_1 = .101001\cdots, \quad \theta_2 = .01001\cdots, \quad \theta_3 = .1001\cdots, \quad \text{etc.}$$

We see that θ_n depends on the $(n+1)$st and higher digits of θ_0, so when n is large, the value of θ_n depends extremely sensitively on the precise value of θ_0. For instance, let θ_0 and θ'_0 differ first in the $(n+1)$st place, where θ_0 has a 0 and θ'_0 has a 1. Then $\theta'_0 - \theta_0 = 2^{-n}$ at most ($<<< 1$ for large n). But $\theta_n = .0\cdots$ and $\theta'_n = .1\cdots$, so that they could differ by as much as 1, or the whole domain (0,1) of the logistic map for $0 < \mu \leq 4$. On a digital computer with capacity 2^N bits, the computed orbit for a given θ_0 has in general no resemblance to the real orbits for times $n \geq N$.

Ex. 1.1 Take $\theta_0 = 1/7$. Then we know that the exact orbit is

$$1/7, \quad 2/7, \quad 4/7, \quad 1/7, \quad 2/7, \quad 4/7, \quad 1/7\cdots, \qquad (1-5)$$

that is, a periodic orbit of period 3. Now perform the iteration (1-3) on a pocket calculator or computer and compare with (1-5) for large n.

2. FUNDAMENTALS OF CONTINUOUS TIME SYSTEMS

2.1 Flows

A system of N first order *ordinary* differential equations in time t,

$$\dot{x} = f(x,t), \quad \dot{x} \equiv \frac{dx}{dt}, \qquad x \in \Re^N, \tag{2-1}$$

defines a *flow*. Here we have taken the flow to be in $\Re^N \equiv$ the set of all real N-tuples $(x_1, x_2, \cdots x_N)$, which is the usual case; the function f, which thus has N components $(f_1, f_2, \cdots f_N)$, maps $\Re^N$ into $\Re^N$, in symbols $f : \Re^N \rightarrow \Re^N$. If $f(x,t) \equiv f(x)$ does not explicitly depend on t, the flow is called *autonomous*. $\Re^N$, or the subset of $\Re^N$ in which the flow is confined, is called the *phase space* of the flow. A solution $x(t) = (x_1(t), x_2(t), \cdots x_N(t))$ of the flow (2-1) with initial value $x_0 \equiv x(0) \equiv (x_1(0), x_2(0), \cdots x_N(0))$ is called the *orbit*. A graph of all orbits or some subset of them in phase space is called a *phase portrait*, and is useful to visualize the flow as a whole in the neighborhood of some interesting point or other structure.

Orbits of an autonomous flow do not intersect! Every point in phase space lies on one and only one orbit. This comes from a beautiful theorem on the uniqueness of orbits, see, say, Guckenheimer and Holmes [20] (Th. 1.0.1), hereafter also GH, which states precisely:

> Let f be C^1 in $\Re^N$. For any open set $U \subset \Re^N$, $\exists$ a time interval $(-c, c)$ such that the orbit $\phi_t(x_0)$ exists and is unique for every $x_0 \in U$. (2 – 2)

For technical mathematical symbols and terms here and hereafter, consult the mathematical Glossary at the end of Guckenheimer and Holmes. We shall use the symbol $\exists$, "there exists," quite often. We shall usually assume the hypotheses of

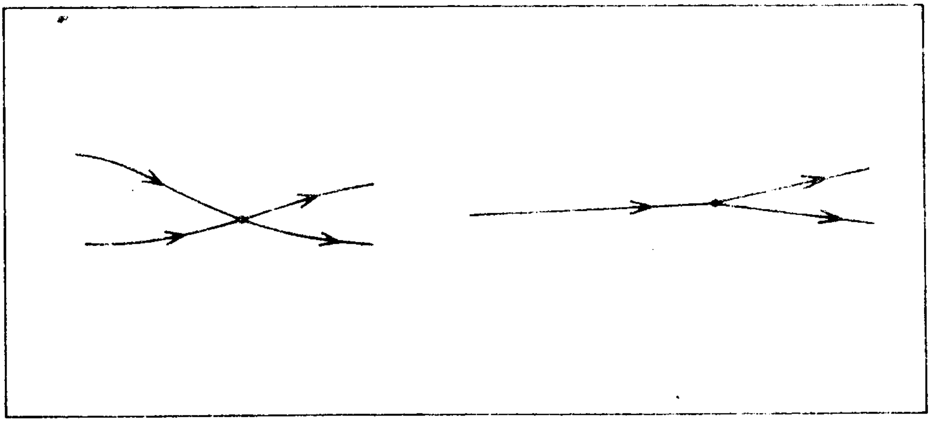

Fig. 2.1 Excluded orbits

this theorem fulfilled for our flows, so that orbits like those shown in Fig. 2.1 are excluded.

The reader might think that restricting our dynamics to autonomous flows (as we shall do) is much too narrow. It seems to rule out higher order motion equations, for example, second order equations such as Newton's laws deliver, all cases with forcing terms, and so on. But this is not so. By enlarging our phase space we can include those cases too. An example will make this clear. Consider the nonautonomous, second order dynamics defined by $\ddot{x}+x = a\cos\omega t$, a harmonically driven linear harmonic oscillator with position coordinate x. Set $x_1 \equiv x$, $x_2 \equiv \dot{x}$, $x_3 \equiv \omega t$. Then we get

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + a\cos x_3, \quad \dot{x}_3 = \omega \quad .$$

But this is just an autonomous flow in $\Re^3$! In particular, even without the forcing term ($a = 0$), the phase space is $\Re^2$, not $\Re^1 \equiv \Re$; phase space is the space of position *and* velocities (or momenta), so it has dimension $2m$ for a configuration space of dimension m. Hence without loss of generality we shall assume all flows autonomous hereafter.

2.2 Linear Stability Analysis

2.2.1 Case of Linear Flows

Consider the *linear* autonomous flow $\dot{x} = Ax$, where A is a real $N \times N$ matrix. We treat the case which *usually* occurs in applications: A can be diagonalized by a similarity transformation

$$T^{-1}AT = \Lambda \quad , \tag{2-3}$$

where Λ is diagonal with the eigenvalues $\lambda_1, \lambda_2, \cdots \lambda_N$ of A on the diagonal. Thus the corresponding eigenvectors $e_1, e_2, \cdots e_N$,

$$Ae_i = \lambda_i e_i \quad , \quad i = 1, 2, \cdots N, \tag{2-4}$$

are *linearly independent* (*span* $\Re^N$), and the columns of T are the components of these eigenvectors. (We prefer to regard A as a linear operator and the e_i as vectors, basis-independent concepts; nevertheless, entirely equivalently one can interpret A as an $N \times N$ matrix, the e_i as $N \times 1$, or column, matrices, and Ae_i as matrix multiplication.) The *secular equation*

$$|A - \lambda \mathbf{1}| \equiv \det(A - \lambda \mathbf{1}) = 0, \qquad (2-5)$$

where $\mathbf{1} \equiv$ unit matrix, determines the eigenvalues λ_i.

The completely general case, when the eigenvectors of A may not span $\Re^N$, so that A is not diagonalizable, is treated in appendix A. A can still be put into a simple, standard form (Jordan canonical form) by a similarity transformation, and the resulting linear stability analysis is not essentially different from the diagonalizable case.

We now define some important subspaces of phase space. Divide the eigenvectors into three subsets,

$$\begin{aligned} &\{u_1, u_2, \cdots u_{N_u}\} \quad \text{such that} \quad Re\lambda_i > 0 \quad , \\ &\{v_1, v_2, \cdots v_{N_s}\} \quad \text{such that} \quad Re\lambda_i < 0 \quad , \qquad (2-6) \\ &\{w_1, w_2, \cdots w_{N_c}\} \quad \text{such that} \quad Re\lambda_i = 0 \quad , \end{aligned}$$

with $N_u + N_s + N_c = N$. Then define

$$\begin{aligned} \text{Unstable subspace} \quad & E^u \equiv span\{u_1, u_2, \cdots u_{N_u}\}, \\ \text{Stable subspace} \quad & E^s \equiv span\{v_1, v_2, \cdots v_{N_s}\}, \qquad (2-7) \\ \text{Center subspace} \quad & E^c \equiv span\{w_1, w_2, \cdots w_{N_c}\}. \end{aligned}$$

The reason for the nomenclature is this: we assert that every orbit based at $x_0 \in E^s$ decays exponentially in t; every orbit based at $x_0 \in E^u$ blows up exponentially in t; and every orbit based at $x_0 \in E^c$ is constant in t, as $t \to +\infty$. We also claim that each subspace is *invariant* (carried into itself) under the flow. Both of these assertions are easily seen by noting that the solution of $\dot{x} = Ax$ is $x(t) = \exp(tA)x_0$. Taking $x_0 \equiv \sum_1^{N_s} c_j \cdot e_j$ in the stable subspace E^s, for example, we see that the orbit is

$$\exp(tA)x_0 = \exp(tA)\sum c_j e_j = \sum c_j \exp(tA) e_j = \sum c_j e^{t\lambda_j} e_j \in E^s \quad , \quad (2-8)$$

Q.E.D. Moreover, since $Re\lambda_j < 0$, the *length* $\|x(t)\| \longrightarrow 0$. Similarly for $x_0 \in E^u$, $\|x(t)\| \longrightarrow +\infty$; for $x_0 \in E^c$, $\|x(t)\| = \|x_0\| =$ const.

A word on the general case: the three subspaces are defined by (2-6) and (2-7), where the vectors are now generalized eigenvectors to the eigenvalues determined by (2-5). One can show that these subspaces are invariant and that every orbit based in E^s, E^u, or E^c decays exponentially, blows up exponentially, or varies algebraically in t as $t \to +\infty$. The only difference is that powers of t times an exponential in t are in general allowed.

Note that the point $x^* = 0$ (the zero *vector*) is a *fixed point*: $\dot{x}^* = 0$ for the linear flow, in fact the only fixed point. The phase portrait of the flow near the fixed point can be constructed, and the subspaces E^u, E^s, and E^c indicated on the same graph.

Ex. 2.1 Take $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & -4 \end{smallmatrix}\right)$. The phase space is $\Re^2 \equiv$ the plane.

a) Find the eigenvalues and eigenvectors. Is A diagonalizable?

b) Find E^u, E^s, and E^c.

c) Draw the phase portrait around $x^* = 0 \equiv (0,0)$, and indicate the three subspaces.

Ex. 2.2 Same question for

$$A = \begin{pmatrix} -1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \text{phase space is } \Re^3.$$

As to part c. of Ex. 2.2, here there is a pair of complex conjugate eigenvalues and eigenvectors: λ_+, e_+ and $\lambda_- = \lambda_+^*, e_- = e_+^*$, where * is complex conjugate. Form two real vectors e_1, e_2 from complex combinations of e_+ and e_-; then e_1, e_2 span the same real two-dimensional subspace of $\Re^3$ as e_+ and e_-. Express the real orbits in terms of e_1 and e_2. You will find spiralling motion.

2.2.2 Case of Nonlinear Flows

The linear stability analysis of the flow $\dot{x} = f(x)$, in general nonlinear, now follows easily from that of linear flows. Consider a *fixed point* x^* of the flow, defined by $\dot{x}^* = f(x^*) = 0$. We *linearize* the flow about x^*. Set $x = x^* + u$, where $\|u\|$ is small in some sense, and keep only terms of $O(u)$ in the calculation. Substitute $x = x^* + u$ into the flow equations, expand $f(x^* + u)$ in a power series in u about x^*, and keep only the first two terms. For the $O(u)$ part we get

$$\dot{u} = Df(x^*)u \quad , \qquad (2-9)$$

where $Df(x^*)$ is the *Jacobian matrix* evaluated at the fixed point,

$$[Df(x^*)]_{ij} \equiv \left.\frac{\partial f_i}{\partial x_j}\right|_{x=x^*} , \quad i,j = 1,2,\cdots N. \qquad (2-10)$$

But (2-9) is just a linear flow with $A = Df(x^*)$. So we find the eigenvalues and eigenvectors, invariant subspaces E^u, E^s, E^c, etc.; that is, we perform the *linear* *stability analysis* for this fixed point just as in section 2.2.1. We expect the local nonlinear flow around x^* to be indistinguishable from the linear flow governed by $A \equiv Df(x^*)$. This is true with an important proviso to be made below.

We work out an illustrative example: consider the *van der Pol oscillator* $\ddot{x} + b(x^2 - 1)\dot{x} + x = 0$, $b > 0$. As an autonomous flow in $\Re^2$ it reads $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1 - b(x_1^2 - 1)x_2$, so $f_1 = x_2$, $f_2 = -x_1 - b(x_1^2 - 1)x_2$. The only fixed point is $x^* = (0,0)$. The partial derivatives of f are

$$\frac{\partial f_1}{\partial x_1} = 0, \quad \frac{\partial f_1}{\partial x_2} = 1, \quad \frac{\partial f_2}{\partial x_1} = -1 - 2bx_1x_2, \quad \frac{\partial f_2}{\partial x_2} = -b(x_1^2 - 1),$$

so the Jacobian matrix is

$$Df(x^* = 0) = \begin{pmatrix} 0 & 1 \\ -1 & b \end{pmatrix}. \qquad (2-11)$$

The eigenvalues of this are $\lambda_\pm = b/2 \pm (b^2/4 - 1)^{\frac{1}{2}}$, and the corresponding eigenvectors $e_\pm$ are found to span $\Re^2$. Now $Re\lambda_\pm > 0$, so

$$E^u = span\{e_+, e_-\} = \Re^2, \quad E^s = E^c = 0. \qquad (2-12)$$

All orbits are repelled exponentially from the fixed point **0**.

2.3 Stability Types of Fixed Points

We now have to elucidate the key notion of stability (for a fixed point here, but for more general structures later), and to see what linear stability analysis has to say about it. For the fixed point x^* of a general autonomous flow, which we assume is confined to the open set $U \subset \Re^N$, we have the definitions (GH, p.3):

> The fixed point x^* is *stable* if for every neighborhood $V \subset U$ of x^* there is a neighborhood $V_1 \subset V$ of x^* such that every solution $x(t) = \phi_t(x_0)$ with $x_0 \in V_1$ is defined and $\in V$ for all $t > 0$. (2 – 13a)

> The fixed point x^* is *asymptotically stable* if it is stable and for every neighborhood $V \subset U$ of x^* a neighborhood $V_1 \subset V$ of x^* exists such that $\phi_t(x_0) \to x^*$, $t \to +\infty$, for every $x_0 \in V_1$. (2 – 13b)

> The fixed point x^* is *unstable* if it is not stable. (2 – 13c)