# APPLIED GRAPH THEORY

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BY

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# PREFACE

In the past four decades, we have witnessed a steady development of graph theory and its applications which in the last five to ten years have blossomed out into a new period of intense activity. Some measure of this rapid expansion is indicated by the observation that, over a period of only one and a half years, more than 500 new papers on graph theory and its applications were published. The main reason for this accelerated interest in graph theory is its demonstrated applications. Because of their intuitive diagrammatic representation, graphs have been found extremely useful in modeling systems arising in physical science, engineering, social science, and economic problems. The fact is that any system involving a binary relation can be represented by a graph.

As a consequence of this rapid expansion, graph theory is now too extensive a subject for adequate presentation in a volume. Faced with the alternatives of writing a shallow survey of the greater part of the applications of graph theory or of giving a reasonably deep account of a relatively small part which is closely related to the engineering applications, I have chosen the latter. The five key topics that are covered in depth are: foundations of electrical network theory, the directed-graph solutions of linear algebraic equations, topological analysis of linear systems, trees and their generation, and the realization of directed graphs with prescribed degrees. Previously, these results have been found only in widely scattered and incomplete journal articles and institutional reports, some rather unreadable, others virtually unobtainable. In this book, I have tried to present a unified and detailed account of these applications.

An effort has been made to introduce the subject matter in the book as simple as possible. Thus, all unnecessary definitions are avoided in favor of a little longer statement. For example, an edge-disjoint union of circuits may be defined as a *circ*, but I prefer not to do so, since the list of definitions has already been too long. Since the terminology and symbolism currently in use in graph theory are far from standardized, the choice of terms is dictated by their applications in the five key areas covered in the book. Thus, the node is preferred to vertex or point, circuit to cycle, parallel edges to multiple edges, etc. As a result, one saving feature of the book is that many of the terms used have nearly the same meaning as in everyday English and very little conscious effort is required to remember them.

The guide light throughout the book has been mathematical precision. Thus, all the assertions are rigorously proved; many of these proofs are believed to be new and novel. An attempt has been made to present the five key topics in a complete and logical fashion, to indicate the historical background, and to credit to the original contributors as far as I can determine. I have tried to present the material in a concise manner, using discussions and examples to illustrate the concepts and principles involved. The book also contains some of the personal contributions of the author that are not available elsewhere in the literature.

Depending only on chapter 1, each of the remaining five chapters, although they are not completely independent, is virtually self-contained, so that the material may be useful to the persons who are interested in only a single topic.

Chapter 1 establishes the basic vocabulary for describing graphs and provides a number of results that are needed in the subsequent analysis. In order to shorten the monotone of these necessary preliminaries, only the essential terms are introduced; the others are defined when they are needed in the later chapters. Thus, the reader is urged to study the convention of this chapter carefully before proceeding to the other chapters.

Chapters 2, 3, and 4, constituting about two-thirds of the book, discuss the various applications to electrical network theory, which happens to be the major field of interest of the author. As a matter of fact, the most important application of graph theory in the physical science is its use in the formulation and solution of the electrical network problem. Although the techniques discussed may easily be extended to other disciplines, the dominant theme is nevertheless the electrical network theory. In each of these chapters, the reader is assumed to be familiar with the elementary aspects of the subject and the discussions are devoted to those aspects of the theory that are strongly dependent on the theory of graphs.

A special feature of the book is that almost all the results are documented in relationship to the known literature, and all the references which have been cited in the text are listed in the bibliography. Thus, the book is especially suitable for those who wish to continue with the study of special topics and to apply graph theory to other fields.

Although basically intended as a reference text for serious researchers, the book may be used equally well as a text for graduate level courses on network topology and linear systems and circuits. There is little difficulty in fitting the book into a one-semester, or two-quarter course. For example, the first four chapters plus some sections of chapter 5, while treating some of the sections of chapter 3 superficially in the classroom, would serve for this purpose. Some of the later chapters are suitable as topics for advanced seminars. The only prerequisite for this book is really mathematical maturity.

A rich variety of problems has been presented at the end of each chapter. There are 385 problems, some of which are routine applications of results derived in the book. Others, however, require considerable extension of the text material or proof of collateral results, which could easily have been included in the text.

Much of the material in the book was developed in the past six years from the research grants extended to the author by the National Science Foundation, the National Aeronautics and Space Administration, and the Ohio University Research Committee. During this time, I have enjoyed the hospitality of Purdue University which I have had the opportunity to visit. To this I am particularly indebted to Professors L. O. Chua and B. J. Leon for making this visit possible. The writing of this book could not have been possible without the constant encouragement and assistance of Provost R. L. Savage, Dean B. Davison, and Dr. J. C. Gilfert of Ohio University. I wish to express my gratitude to Professor W. Mayeda of University of Illinois and Professor M. E. Van Valkenburg of Princeton University for their invaluable inspiration. Thanks are also due to many friends and colleagues who gave useful suggestions; among them are Professors K. E. Eldridge, G. V. S. Raju, H. C. Chen and F. Y. Chen and my students Dr. S. K. Mark and Mr. H. C. Li. Mr. Li assisted me in plotting the preliminary drawings of all the illustrations. In particular, I would like to single out Professor K. E. Eldridge and Dr. S. K. Mark who kindly read both the manuscript and page proofs critically and made valuable suggestions. Considerable assistance was also contributed by Professor P. M. Lin of Purdue University who gave the complete book a carefully reading. I also wish to thank Dr. C. Korswagen and the North-Holland Publishing Company for their patience and cooperation in all aspects of the production of this book. Finally, I would like to thank my wife, Shiao-Ling, for her careful proofreading of the book and for her infinite patience and understanding, to whom this book is dedicated.

*April, 1971*
*West Lafayette, Indiana*

W.K.C.

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CHAPTER 1

# BASIC THEORY

The chapter establishes the basic vocabulary for describing graphs and provides a number of basic results that are needed in the subsequent analysis, omitting those aspects of graph theory that are unrelated to the applications discussed in this book. Since the terminology and symbolism currently in use in graph theory are far from standardized, the reader is urged to study the conventions of this chapter carefully before proceeding to the other chapters.

## § 1. Introduction

The term "graph" used in this book denotes something quite different from the graphs that one may be familiar with from analytic geometry or function theory. The graphs that we are about to discuss are simple geometrical figures consisting of points (nodes) and lines (edges) which connect some of these points; they are sometimes called "linear graphs". Because of this diagrammatic representation, graphs have been found extremely useful in modeling systems arising in physical science (BUSACKER and SAATY [1965], and HARARY [1967]), engineering (SESHU and REED [1961], and ROBICHAUD et al. [1962]), social science (HARARY and NORMAN [1953], and FLAMENT [1963]), and economic problems (AVONDO-BODINO [1962], and FORD and FULKERSON [1962]). The fact is that any system involving a binary relation can be represented by a graph.

The first paper on graphs was written by the famous Swiss mathematician Leonhard Euler (1707–1783). He started with a famous unsolved problem of his day called the *Königsberg Bridge Problem*. The city of Königsberg (now Kaliningrad) in East Prussia is located on the banks and on two islands of the river Pregel. The various parts of the city were connected by seven bridges as shown in fig. 1.1. The problem was to cross all seven bridges, passing over each one only once. One can see immediately that there are many ways of trying the problem without solving it. EULER [1736] solved the problem by showing that it was impossible, and laid the foundations of graph theory. We mention here only the formulation, rather than the details.

Replace each part of the city by a point and each bridge by a line joining the points corresponding to these parts. The result is a graph as shown in fig. 1.2. Euler then showed that, no matter at which point one begins, one cannot cover the graph completely and come back to the starting point without retracing one's steps.

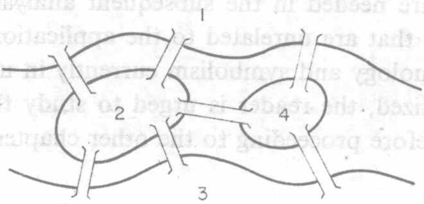


Fig. 1.1. The Königsberg bridge problem.

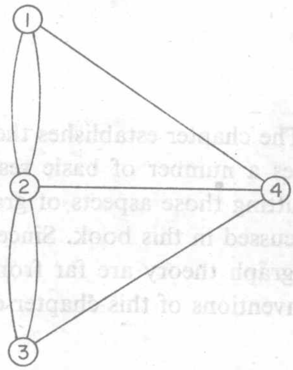


Fig. 1.2. The graph of the Königsberg bridge problem.

The most famous unsolved problem in graph theory is perhaps the celebrated *Four Color Conjecture*. Many centuries ago, makers of maps discovered empirically that in coloring a map of a country, divided into counties, only four distinct colors are required, so that no two adjacent counties should be painted in the same color. At first the problem does not seem to have been taken seriously by the mathematicians until it has withstood every assault by some of the world's most capable mathematicians. Heawood [1890] showed, however, that the conjecture becomes true when "four" is replaced by "five". A counter-example, if ever found, will necessarily be extremely large and complicated, for the conjecture was proved most recently by Ore and Stemple [1970] for all maps with fewer than 40 counties.

The problem can easily be transformed into a problem in graph theory because every map yields a graph in which the counties including the exterior region are represented by the points, and two points are joined by a line if and only if their counties have a common boundary.

The most important application of graph theory in the physical science, from our point of view, is its use in the formulation and solution of the electrical network problem by Kirchhoff [1847]. His contributions will be treated in great detail in this book; chapters 2 and 4 contain most of his contributions to electrical network theory.

While many of the examples of the graphs arising in applications are geometric, the essential structure in the context of graph theory is combinatorial in nature. In the following sections, we shall introduce the concept of abstract graphs. Aside from stripping the incidental geometric features away from the essential combinatorial characteristics of a graph, the concept enlarges the prospects of applications.

## § 2. Basic concepts of abstract graphs

Like every mathematical theory, we have to begin with a long list of definitions, since we must have a few words to talk about, and in the interest of precision these have to be formally defined. Fortunately, many of these terms that we will define have nearly the same intuitive meaning as in everyday English and so very little conscious effort is required to remember them. In order to relieve the monotony of these necessary preliminaries, we will use diagrams to illustrate our points.

### 2.1. *General definitions*

DEFINITION 1.1: *Abstract graph.* An *abstract graph* $G(V, E)$, or simply a *graph* $G$, consists of a set $V$ of elements called *nodes* together with a set $E$ of *unordered* pairs of the form $(i, j)$ or $(j, i)$, $i, j$ in $V$, called the *edges* of $G$; the nodes $i$ and $j$ are called the *endpoints* of $(i, j)$.

Other names commonly used for a node are *vertex, point, junction, 0-simplex, 0-cell*, and *element*; and for edges *line, branch, arc, 1-simplex*, and *element*. We say that the edge $(i, j)$ is *connected* between the nodes $i$ and $j$, and that $(i, j)$ is *incident* with the nodes $i$ and $j$ or conversely that $i$ and $j$ are *incident* with $(i, j)$. In the applications, a graph is usually represented equivalently by a *geometric diagram* in which the nodes are indicated by small circles or dots, while any two of them, $i$ and $j$, are joined by a continuous curve, or even a straight line, between $i$ and $j$ if and only if $(i, j)$ is in $E$. This definition of a graph is sufficient for many problems in which graphs make their appearance. However, for our purpose, it is desirable to enlarge the graph concept somewhat.

We extend the graph concept by permitting a pair of nodes to be connected by several distinct edges as indicated by the symbols $(i, j)_1$, $(i, j)_2, \ldots, (i, j)_k$; they are called the *parallel edges* of $G$ if $k \geqq 2$. If no particular edge is specified, $(i, j)$ denotes any one, but otherwise fixed, of the parallel edges connected between $i$ and $j$. We also admit edges for which the two endpoints are identical. Such an edge $(i, i)$ shall be called a *self-loop*. If there are two or more self-loops at a node of $G$, they are also referred to as the parallel edges of $G$. In the geometric diagram the parallel edges may be represented by continuous lines connected between the same pair of nodes, and a self-loop $(i, i)$ may be introduced as a circular arc returning to the node $i$ and passing through no other nodes.

As an illustration, consider the graph $G(V, E)$ in which

$$V=\{1, 2, 3, 4, 5, 6, 7\},$$

$$E = \{(1, 1), (1, 2), (1, 4), (4, 4)_1, (4, 4)_2, (4, 3), (2, 3)_1, (2, 3)_2, (6, 7)_1, (6, 7)_2, (6, 7)_3\}.$$

The corresponding geometric graph is as shown in fig. 1.3 in which we have a self-loop at node 1, two self-loops at node 4, two parallel edges connected between the nodes 2 and 3, and three parallel edges between the nodes 6 and 7. We emphasize that in a graph the order of the nodes $i$ and $j$ in $(i, j)$ is immaterial. In fact we consider $(i, j)=(j, i)$, e.g., $(1, 2)=(2, 1)$ and $(6, 7)_2=(7, 6)_2$.

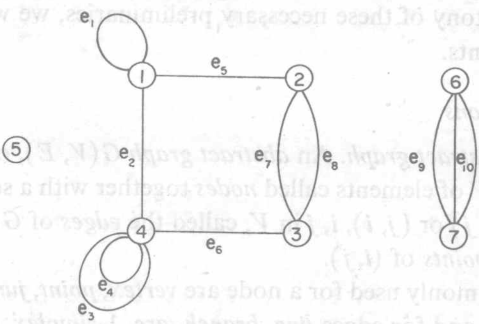

Fig. 1.3. A geometric graph.

A graph $G(V, E)$ is said to be *finite* if both $V$ and $E$ are finite. In this book, we only consider finite graphs. Infinite graphs have some very interesting properties. For interested readers, we refer to KÖNIG [1950] and ORE [1962].

DEFINITION 1.2: *Subgraph*. A *subgraph* of a graph $G(V, E)$ is a graph $G_s(V_s, E_s)$ in which $V_s$ and $E_s$ are subsets of $V$ and $E$, respectively. If $V_s$ or $E_s$ is a proper subset, the subgraph is called a *proper subgraph* of $G$. If $V_s = V$, the subgraph is referred to as a *spanning subgraph* of $G$. If $V_s$ or $E_s$ is empty, the subgraph is called the *null graph*. The null graph is considered as a subgraph of every graph, and is denoted by the symbol $\emptyset$.

DEFINITION 1.3: *Isolated node*. A node not incident with any edge is called an *isolated node*.

In fig. 1.3, for example, the node 5 is an isolated node. Some examples of subgraphs are presented in fig. 1.4. Fig. 1.4(a) is a spanning subgraph since it contains all the nodes of the given graph. Figs. 1.4(b) and (c) are examples of proper subgraphs. A graph itself is also its subgraph.

We say that two subgraphs are *edge-disjoint* if they have no edges in common, and *node-disjoint* if they have no nodes in common. Clearly, two subgraphs are node-disjoint only if they are edge-disjoint, but the converse is not valid in general. For example, in fig. 1.3 the subgraphs (1, 2) and (3, 4) are node-disjoint, and thus they are also edge-disjoint. On the other hand, the subgraphs as shown in figs. 1.4(b) and (c) are edge-disjoint but they are not node-disjoint.

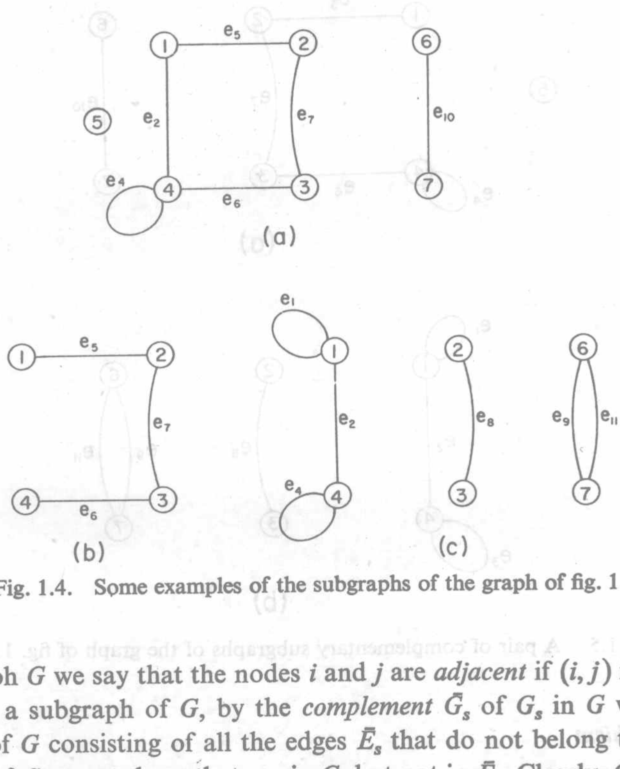


Fig. 1.4. Some examples of the subgraphs of the graph of fig. 1.3.

In a graph $G$ we say that the nodes $i$ and $j$ are *adjacent* if $(i, j)$ is an edge of $G$. If $G_s$ is a subgraph of $G$, by the *complement* $\bar{G}_s$ of $G_s$ in $G$ we mean the subgraph of $G$ consisting of all the edges $\bar{E}_s$ that do not belong to $G_s$ and all the nodes of $G$ except those that are in $G_s$ but not in $\bar{E}_s$. Clearly, $G_s$ and $\bar{G}_s$ are edge-disjoint but not necessarily node-disjoint, and their node sets may not be complementary. Thus, the complement of the null graph in $G$ is the graph $G$ itself, and the complement of $G$ in $G$ is the null graph. We also say that $G_s$ and $\bar{G}_s$ are *complementary subgraphs* of $G$. For example, figs. 1.5(a) and (b) are complementary subgraphs of the graph as shown in fig. 1.3.

In practical applications, it is sometimes convenient to represent the edges of a graph by letters $e_i$. In this way, a subgraph having no isolated nodes may be expressed by the "product" or by juxtaposition of its edge-designation symbols. For example, in fig. 1.3 the edges of the graph are also represented by the letters $e_i$: $e_1=(1, 1)$, $e_2=(1, 4)=(4, 1), \ldots$, and $e_{11}=(6, 7)_3=(7, 6)_3$. The subgraphs of figs. 1.4(b) and (c) may be denoted by the products of their edge-designation symbols as $e_5e_6e_7$ and $e_1e_2e_4e_8e_9e_{11}$, respectively. Of course, we can also use this technique to represent subgraphs having isolated nodes, but then an ambiguity involving the null graph will arise.

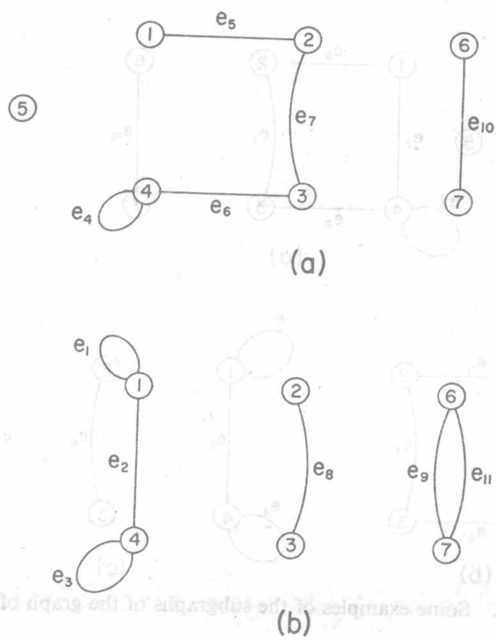


Fig. 1.5. A pair of complementary subgraphs of the graph of fig. 1.3.

## 2.2. *Isomorphism*

In the preceding section, we have already pointed out that in drawing the geometric diagram of a graph we have great freedom in the choice of the location of the nodes and in the form of the lines joining them. This may make the diagrams of the same graph look entirely different. In such cases we would like to have a precise way of saying that two graphs are really the same even though they are drawn or labeled differently. The next definition provides the terminology necessary for this purpose.

DEFINITION 1.4: *Isomorphism*. Two graphs $G_1$ and $G_2$ are said to be *isomorphic*, denoted by $G_1 = G_2$, if there exist a one-to-one correspondence between the elements of their node sets and a one-to-one correspondence between the elements of their edge sets and such that the corresponding edges are incident with the corresponding nodes.

In other words, in two isomorphic graphs the corresponding nodes are connected by the edges in one if, and only if, they are also connected by the same number of edges in the other. Definition 1.4 as stated places two requirements on isomorphism of two graphs. First, they must have the same number of nodes and edges. Second, the incidence relationships must be preserved. The latter is usually difficult to establish.

As an illustration, consider the graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ as shown in fig. 1.6. These two graphs look quite different, but they are isomorphic. The

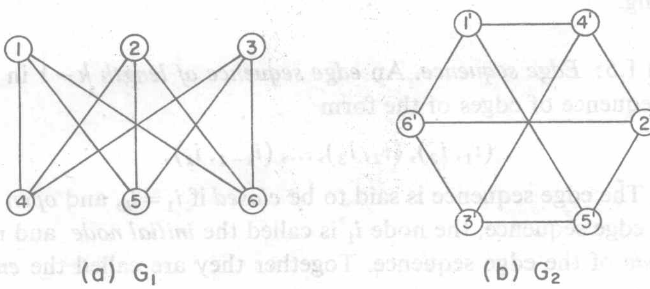


Fig. 1.6. Two isomorphic graphs.

isomorphism of these two graphs can be established by considering the nodes $i$ of $V_1$ and $i'$ of $V_2$, $i = 1, 2, 3, 4, 5, 6$, as the corresponding elements of their node sets. It is easy to verify that the corresponding edges are incident with the corresponding nodes. In other words, the incidence relationships are preserved.

As another example, the two graphs given in fig. 1.7 are not isomorphic even though there exists a one-to-one correspondence between their node sets which preserves adjacency. The reason for this is that they do not contain the same

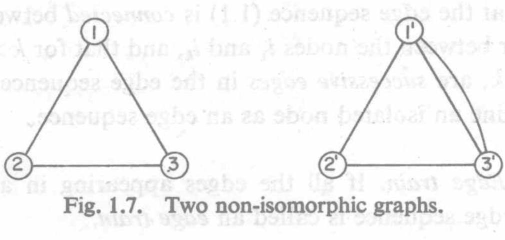


Fig. 1.7. Two non-isomorphic graphs.