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The Theory of Relativity

C. MØLLER

SECOND EDITION

THE THEORY OF RELATIVITY

BY

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PREFACE TO THE SECOND EDITION

AT the time of the appearance of the first edition of this book in 1952, the theory of relativity was generally regarded as a concluded subject, which did not offer any new interesting problems, and the number of physicists working in this field was comparatively small. However, the development of the last ten to fifteen years has completely changed this situation. In the first place, extensive investigations, in particular by a new generation of physicists and mathematicians, have given us a much better understanding of the mathematical structure and the physical contents of the theory. Secondly, the impressive development of experimental techniques and collaboration with astrophysicists have supplied much more accurate experimental tests of the theory and have opened up the possibility of new, exciting applications of the theory in cosmology. In a way, this development is not surprising, for, in contrast with the Newtonian gravitational field that acts only on ponderable matter, the gravitational field in Einstein's theory affects all physical phenomena. Thus, although the influence is usually extremely small, the ever-increasing accuracy of our measuring instruments will probably reveal more and more general relativity effects also in the future.

It was felt imperative to include at least part of the new development in the present edition of the book without changing its character of a textbook for beginners in the field. For this reason, part of the new material is given in the form of exercises or in sections printed in small type, which may be skipped at the first reading. The more advanced student may find these sections the most interesting, although they cover only a small part of the investigations of later years. The method of presentation adopted in the first edition, of combining the abstract four-dimensional description of the phenomena with a more physical description in the three-dimensional physical space, is maintained and even enlarged in the present edition.

The main changes and additions are the following. Chapter 4 contains a new treatment of point mechanics for a particle with varying proper mass. In Chapter 6, the section on the mechanics of elastic continua is now more complete. To Chapter 7 has been added a new section on continuously distributed matter with internal heat conduction, and relativistic thermodynamics has been reformulated. Also the section on electrodynamics in material bodies contains some improvements. In the following part, dealing with the theory of general relativity, Chapter 9 brings some new mathematical tools that are used in the formulation of particle dynamics in Chapter 10. The latter chapter also presents a detailed account of the propagation of light waves and the properties of photons in gravitational fields. In Chapter 11, the problems of the linear and angular momentum as well as the total energy of isolated and radiating insular systems are thoroughly discussed in the light of the latest developments. Finally, Chapter 12 deals with the new experimental verification of the general and of the special theory of relativity up to the end of 1970; likewise the section on cosmological problems has been extended.

In completion of this revised and improved version of my book, I gratefully acknowledge the valuable help from different members of NORDITA's secretariat. I am much indebted to Miss S. Hellmann who, as effectively as twenty years ago, assisted me in preparing the new manuscript and reading the proofs.

COPENHAGEN
July 1971 C.M.

From the

PREFACE TO THE FIRST EDITION

THE present monograph is a somewhat extended version of a course of lectures that I have given at the University of Copenhagen during the last twenty years. Consequently, it is primarily a textbook for students in physics whose mathematical and physical training does not go beyond the methods of non-relativistic mechanics and electrodynamics. The intention has been to give an account of what may be called the classical theory of relativity, in which all quantum effects are disregarded. In view of the paramount importance of quantum phenomena in modern physics, the limitation of the subject to classical phenomena might be considered a serious defect of the book. However, there are several important reasons for such a limitation of the subject. At present, a complete self-consistent relativistic quantum theory does not exist. Moreover, the classical theory of relativity, which by itself gives an admirably precise description of a very extended field of physical phenomena, must be the starting-point for the future development of a consistent relativistic quantum theory. For a student and research worker in this field, an intimate acquaintance with the principles and methods of the classical theory of relativity is, therefore, just as indispensable as is the knowledge of the methods of Newtonian mechanics for a real understanding of ordinary quantum mechanics. Apart from this, the classical theory of relativity is one of the most fascinating and beautiful parts of theoretical physics, on account of its inner consistency and the simplicity and generality of its basic assumptions.

The presentation of the subject in the present volume differs somewhat from the usual one in that the four-dimensional formulation of the theory plays a less dominant role than in most of the current textbooks. Certainly the four-dimensional representation, which is based on the symmetry between the space and time variables revealed by the discovery of the Lorentz transformation, is the most elegant way of expressing the principle of relativity in mathematical language, and it has been of the utmost importance for the rapid development of the general theory of relativity particularly. In the early books on relativity it was, therefore, quite natural to emphasize as strongly as possible this newly discovered similarity between the space and time variables. However, in a textbook of today I think it is useful to stress again the fundamental physical difference between space and time, which was somewhat concealed by the purely formal four-dimensional representation.

In the first three chapters we have, therefore, avoided any reference to the four-dimensional picture, and the kinematics and point mechanics of the special theory of relativity are fully developed by means of the usual three-dimensional vector calculus. But in the following chapters also, where the elegant methods of the four-dimensional tensor calculus are developed and applied, a three-dimensional formulation, which gives a better insight into the physical meaning of the theory, is frequently given.

We have included only those developments of the theory of relativity that can be regarded as safely established, the various attempts at constructing a unified theory of gravitation and electromagnetism falling outside the scope of the present book. Also, the cosmological problems have merely been touched upon, since these problems have been extensively treated by Tolman in this series of monographs. Within these restrictions it is hoped, however, that the reader will find a fairly complete and well-rounded account of one of the most beautiful chapters in the history of science, which for the main part was written by a single man, Albert Einstein.

On completion of this work I gratefully acknowledge the help and advice that I have received from many quarters. First of all, I want to express my deep gratitude to Professor Niels Bohr for his kind interest in my work during all these years and for the constant inspiration derived from many discussions and conversations at his institute. My thanks are due to Professor N. F. Mott and Professor I. N. Sneddon for reading the manuscript and eliminating the worst danicisms. I also wish to thank the staff of the Clarendon Press for their friendly cooperation.

I am indebted to Dr. W. Kohn and Dr. W. J. Swiatecki for many suggestions that have considerably improved the text and, in particular, to mag. scient. J. Lindhard who has been of great help in checking all the equations and reading the proofs. Finally, I am grateful to Miss S. Hellmann for her untiring assistance in the preparation of the manuscript and the proof-reading.

C.M.

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THE FOUNDATIONS OF THE SPECIAL THEORY OF RELATIVITY. HISTORICAL SURVEY

1.1. The relativity principle of mechanics. The Galilean transformation

THE special theory of relativity which was developed in the beginning of the twentieth century, especially through Einstein's work, has its roots far back in the past. In a way, this theory can be regarded as a continuation and completion of the ideas that have been the basis of our description of nature since the times of Galileo and Newton. The basic postulate of this theory, the so-called special principle of relativity,† had already in Galileo's and Huyghens's works played a decisive role in the development of the fundamental laws of mechanics. Also the validity of the principle of relativity for the phenomena of mechanics is a simple consequence of the Newtonian laws of mechanics. Since the laws of mechanics are especially well suited for the illustration of the principle of relativity, we shall start by considering purely mechanical phenomena.

According to Newton's first law, the law of inertia, a material particle when left to itself will continue to move in a straight line with constant velocity. Since one cannot simply speak of motion, but only of motion relative to something else, this statement has a precise meaning only when a certain well-defined system of reference has been established relative to which the velocity of the particle is assumed to be measured. Therefore Newton introduced the notion of the 'absolute space', representing that system of reference relative to which every motion should be measured. Experience shows that the fixed stars as a whole may be regarded as approximately at rest relative to the 'absolute space', for a body sufficiently far away from celestial matter always moves with uniform velocity relative to the fixed stars.

It is, however, obvious that the law of inertia holds also in every other rigid system of reference moving with uniform velocity relative to the absolute system, for a free particle will also be in uniform translatory motion

† When reference is made to the principle of relativity in Chapters 1–7, we always have in view the principle of special relativity as contrasted with the principle of general relativity which is the basis of the general theory of relativity.

with respect to such a system. All systems of reference for which the law of inertia is valid are called systems of inertia. They form a threefold infinity of rigid systems of reference moving in straight lines and with constant velocity relative to each other. One of them is the absolute system which is at rest relative to the fixed stars as a whole; but as regards the validity of the law of inertia, all systems of inertia are completely equivalent.

Now the principle of relativity in mechanics states that the systems of inertia are also completely equivalent with regard to the other laws of mechanics. If this is true all mechanical phenomena will take the same course of development in any system of inertia so that it is impossible from observations of such phenomena to detect a uniform motion of the system as a whole relative to the 'absolute' system. Thus a study of mechanical phenomena alone can never lead to a determination of the absolute system.

We shall see that the fundamental equations of Newtonian mechanics actually are in accordance with the principle of relativity. Let us consider two arbitrary systems of inertia I and I'. In each of these frames of reference we use definite systems of coordinates S and S'. We may, for instance, choose Cartesian coordinates $\mathbf{x} = (x, y, z)$ and $\mathbf{x}' = (x', y', z')$ in I and I' respectively. According to the conceptions of space and time derived from our usual experience, which also form the basis of the Newtonian formulation of the fundamental laws of mechanics, the connection between the coordinate vectors $\mathbf{x}$ and $\mathbf{x}'$ for one and the same space point in the two coordinate systems S and S' is given by

$$\mathbf{x}' = \mathbf{x} - \mathbf{v}t, \tag{1.1a}$$

where $\mathbf{v}$ is a vector denoting velocity and direction of motion of S' relative to S, and t is the time. For the sake of simplicity, it is assumed that the origins of the two systems of coordinates coincide at the time $t = 0$. To the equation (1.1a) may be added the equation

$$t' = t, \tag{1.1b}$$

which states that the parameter describing the time is the same in all systems of inertia. Thus in the Newtonian description of physical phenomena the time is an absolute quantity. The equations (1.1a) and (1.1b) are often referred to as the *Galilean transformation.*

If the directions of the axes of the two systems of coordinates are parallel, and if $\mathbf{v}$ has the direction of the x-axis, we obtain a *special* Galilean transformation which can be written

$$\left.\begin{aligned} &x' = x - vt, \quad y' = y, \quad z' = z \\ &t' = t \end{aligned}\right\}. \tag{1.2}$$

Since the systems of coordinates S and S' are completely equivalent, at any rate as far as kinematics is concerned, and since S obviously moves with the velocity $-\mathbf{v}$ relative to S', the inverse transformations to (1.1) and (1.2) are simply obtained by interchanging the primed and the unprimed variables and simultaneously replacing $\mathbf{v}$ by $-\mathbf{v}$.

Let us now consider an arbitrary motion of a material particle. By differentiation of (1.1*a*) we get

$$\frac{d\mathbf{x}'}{dt'} = \frac{d\mathbf{x}}{dt} - \mathbf{v}$$

or

$$\mathbf{u}' = \mathbf{u} - \mathbf{v}, \tag{1.3}$$

where $\mathbf{u}$ and $\mathbf{u}'$ represent the velocities of the particle in the two systems of inertia. (1.3) is the usual addition theorem of velocities. For a special Galilean transformation (1.2), (1.3) reduces to

$$u'_x = u_x - v, \quad u'_y = u_y, \quad u'_z = u_z. \tag{1.4}$$

When the velocity vector $\mathbf{u}$, and thus also $\mathbf{u}'$, is perpendicular to the z-axis, (1.4) may be written

$$u' \cos \vartheta' = u \cos \vartheta - v,$$

$$u' \sin \vartheta' = u \sin \vartheta,$$

where ϑ and ϑ' are the angles between the x-axis and the directions of $\mathbf{u}$ and $\mathbf{u}'$ respectively. Further, $u = |\mathbf{u}|$, $u' = |\mathbf{u}'|$ denote the magnitudes of the vectors $\mathbf{u}$ and $\mathbf{u}'$. If we now divide one of these equations by the other we obtain

$$\tan \vartheta' = \frac{\sin \vartheta}{\cos \vartheta - v/u}, \tag{1.5}$$

and by summation of the squares of the equations we get

$$u' = u\left(1 - 2\frac{v}{u}\cos\vartheta + \frac{v^2}{u^2}\right)^{\frac{1}{2}}. \tag{1.6}$$

Now let us assume that the material particle with the mass m is acted on by a force $\mathbf{F}$. In the absolute system of coordinates S the particle will then obtain an acceleration given according to Newton's second law by the equation

$$m\frac{d^2\mathbf{x}}{dt^2} = \mathbf{F}. \tag{1.7}$$

From (1.1*a*) and (1.1*b*) it now follows that

$$\frac{d^2\mathbf{x}'}{dt'^2} = \frac{d^2\mathbf{x}}{dt^2}, \tag{1.8}$$

and since in Newtonian mechanics forces and masses are absolute quantities, i.e.

$$\mathbf{F}' = \mathbf{F}, \quad m' = m, \tag{1.9}$$

we obtain

$$m' \frac{d^2\mathbf{x}'}{dt'^2} = \mathbf{F}'. \tag{1.10}$$

Thus we see that the second law of Newton is valid in every system of inertia in accordance with the principle of relativity. This can be expressed more accurately by stating that the Newtonian fundamental equations are invariant under Galilean transformations. As is well known, this invariance does not hold for more general transformations leading to accelerated systems of reference. If one wants to treat mechanical phenomena in such systems, one has to introduce extra fictitious forces, e.g. centrifugal forces and Coriolis forces, which depend only on the acceleration of the frame of reference and therefore are in no causal relationship with the physical properties of other terrestrial systems. It was just this difference between the uniformly moving and the accelerated systems of reference which led Newton to the conception of absolute space.

1.2. The special principle of relativity

As already mentioned, the validity of the principle of relativity in mechanics prevents a unique determination of the absolute system of reference from studies of mechanical phenomena alone. Now the basic assumption of the special theory of relativity is that *the special principle of relativity is valid for all physical laws.*† According to this theory, all physical phenomena should have the same course of development in all systems of inertia, and observers installed in different systems of inertia should thus, as a result of their experiments, arrive at the establishment of the same laws of nature.

If this is so, the notion of absolute space obviously loses its meaning, since any system of inertia with equally good reason can claim to be the absolute system of reference. Of course nobody can prevent us from calling one definite system of inertia, for example the one which is at rest relative to the fixed stars, the absolute system and expressing all laws of nature in coordinates of this system. Such a procedure is, however, extremely unsatisfactory in view of the arbitrariness in the choice of the absolute system. It is, furthermore, very inconvenient to proceed in this manner. The physical experiments from which the laws of nature are derived are

† With the exception of the laws of gravitation, which find their natural place in the general theory of relativity.

usually not performed in a system of reference that is at rest relative to the fixed stars. On account of its motion around the sun the earth will in the course of a year represent widely different systems of inertia if we disregard the small acceleration of the earth in this motion. The transformation to the coordinates of the absolute system is therefore rather complicated.

The validity of the principle of relativity for all physical phenomena makes such a transformation unnecessary, since the system of inertia in which the earth is at rest at the moment considered is equivalent to any other system of inertia. This obviously leads to an enormous simplification in our description of nature.

However, this simplification has to be paid for, as we shall see, by an abandonment of our usual notions of time and space. The extension of the principle of relativity to electromagnetic phenomena means, as mentioned before, that physicists who have established their laboratories in two different systems of inertia will, as a result of their experiments, be led independently to Maxwell's fundamental equations of electrodynamics. These equations contain a universal constant c, which can be determined by means of purely electromagnetic measurements, and is very closely equal to 3×10^8 m/s (Weber and Kohlrausch 1856). On the other hand, it is a simple consequence of Maxwell's equations that electromagnetic waves in empty space propagate with the velocity c, independently of the way in which they are created. Since light waves, according to Maxwell's theory of light, are special electromagnetic waves, the velocity with which light is propagated *in vacuo* must also be independent of the state of motion of the light source and equal to the constant c. If Maxwell's equations in accordance with the relativity principle are valid in any system of inertia, the velocity of light must have the same constant value c in all systems of inertia, independently of the motion of the light source. This is obviously in conflict with the usual kinematical concepts according to which we should expect, for instance, to find a lower velocity of light in S' than in S if the relative motion of S' with respect to S has the same direction as the direction of propagation of the light ray.

Consequently the acceptance of the relativity principle must necessarily lead to a revision of our ordinary concepts of space and time. Before taking such a radical step one would naturally want to be sure that it is really necessary. This question can only be settled as a result of experiment. Optical experiments are especially suited to this purpose in view of the high accuracy obtainable with optical instruments. In the following sections we shall therefore give a short historical survey of the numerous optical experiments which have been performed in an attempt to detect effects depending on the motion of the apparatus with respect to an absolute space. These experiments all gave negative results and finally led to a general acceptance of the principle of relativity.