Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

1303

L. Accardi W. von Waldenfels (Eds.)

Quantum Probability and Applications III

Proceedings of a Conference held in Oberwolfach, FRG, January 25–31, 1987

Springer-Verlag
Berlin Heidelberg New York London Paris Tokyo

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Mathematics Subject Classification (1980): 43A35, 46LXX, 46M20, 47DXX, 60FXX, 60GXX, 60HXX, 60JXX, 80A05, 81C20, 81K05, 81L05, 81M05, 82A15, 83C40, 83C75

ISBN 3-540-18919-X Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-18919-X Springer-Verlag New York Berlin Heidelberg

Printed in Germany

Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
2146/3140-543210

Introduction

This volume contains the proceedings of the first Quantum Probability meeting held in Oberwolfach which is the fourth of a series begun with the 1982 meeting of Mondragone and continued in Heidelberg ('84) and in Leuven ('85). The main topics discussed during the meeting were: quantum stochastic calculus, mathematical models of quantum noise and their applications to quantum optics, the quantum Feynman-Kac formula, quantum probability and models of quantum statistical mechanics, the notion of conditioning in quantum probability and related problems (dilations, quantum Markov processes), quantum central limit theorems.

We are grateful to the Mathematisches Forschungsinstitut Oberwolfach and to its director Prof. M. Barner for giving us the unique opportunity of scientific collaboration and mutual exchange.

We would like to thank also the speakers and all the participants for their contributions to the vivid and sometimes heated discussions.

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CONTENTS

A NOTE ON MEYER' S NOTE

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(1.) NOTATIONS AND STATEMENT OF THE PROBLEM

Let us denote
- $\Gamma(L^2(\mathbf{R}_+)$ the Boson Fock space over the one-particle space $L^2(\mathbf{R}_+)$
- $\mathcal{E} = \{\psi(f) : f \in L^2(\mathbf{R}_+)\}$ the set of exponential vectors in $\Gamma(L^2(\mathbf{R}_+)$.
- $\Phi = \psi(0)$ the vacuum state in $\Gamma(L^2(\mathbf{R}_+)$.
- $\Gamma(\chi_{[0,t]})$ the orthogonal projector defined by

$$\Gamma(\chi_{[0,t]})\psi(f) = \psi(\chi_{[0,t]} f)$$

- $\Phi_{t]} := \Gamma(\chi_{[0,t]})\Phi$; $\Phi_{[t} := \Gamma(\chi_{[t,\infty)})\Phi$
- W(f) $f \in L^2(\mathbf{R}_+)$ the Weyl operator characterized by the property

$$W(f)\psi(g) = e^{-\frac{\|f\|^2}{2} - <f,g>}\psi(f+h)$$

- A , A^+ , N the annihilation, creation and number (or gauge or conservation) fields defined, on $\mathcal{E}$ by the relations :

$$A(f)\psi(g) =< f,g > \psi(g)$$

$$A^+(f)\psi(g) = \frac{d}{dt}|_{t=0}\ \psi(g+tf)$$

$$N_t\psi(g) = \frac{d}{ds}|_{s=0}\ \psi(e^{s\chi_{[0,t]}}g)$$

we write N(s,t) for $N_t - N_s$.

The W(f) are unitary operators on $\mathcal{H}$ satisfying the CCR

$$W(f)W(g) = e^{-\frac{\|f\|^2}{2} - <f,g>}\psi(f+h)$$

- $f_{t]} = \chi_{[0,t]}f$; $f_{[t} = \chi_{[t,\infty]}f$
- H_0 a complex Hilbert space , called the initial space.
- $\mathcal{H} = H_0 \otimes \Gamma(L^2(\mathbf{R}_+)$
- $\mathcal{H}_{t]} = H_0 \otimes \Gamma(L^2([0,t]) \otimes \Phi_{[t}$
- $\mathcal{B} = \mathcal{B}(\mathcal{H}) = \mathcal{B}\left(H_0 \otimes \Gamma(L^2(\mathbf{R}_+)\right)$
- $\mathcal{B}_{t]} = \mathcal{B}(H_0 \otimes \Gamma(L^2([0,t])) \otimes 1_{t]}$
- $\mathcal{B}_{[t} = \mathcal{B}(1_{H_0} \otimes 1_{t]} \otimes \Gamma(L^2([t,\infty))$
- θ_t the shift on $L^2(\mathbf{R}_+)$.
- $\sigma_t = \Gamma(\theta_t)$ the shift on $\Gamma(L^2(\mathbf{R}_+)$
- $u_t^0 = \imath_0 \otimes \sigma_t$ the free time shift on $\mathcal{B}$. where $\imath_0$ is the identity map on $\mathcal{B}(H_0)$.

The objects described above provide a simple and , for a certain class of models , canonical example of a quantum Markov process (in fact also of a quantum independent increment process in the sense of [2]) and the Feynman-Kac formula allows to perturb such structures by means of unitary cocycles (for the free time shift) giving rise to new quantum processes [1]. In particular, the generator L of the quantum Markovian semigroup canonically associated to the new process is the sum of the generator L_o of the semigroup associated to the original process and of an additional perturbative piece, denoted L_I . The problem with the above class of quantum Markov processes is that the free time shift u_t^o acts trivially on the initial algebra and therefore the corresponding semigroup is zero so that , as remarked in [1] , in this case one is in fact dealing with FK perturbations of the identity semigroup . As a consequence of this one looses one of the most attractive analytical advantages of the classical FK formula, namely the possibility of dealing with perturbations L_I so singular that the operator $L_o + L_I$ is not well defined (a typical example is the possibility of giving a meaning ,via the FK formula , to the formal generator $-\triangle + V$ where $\triangle$ is the Laplacian on $\mathbf{R}^n$ and V is a highly singular potential). By analogy with the classical case we would like to have a time shift v_t^o wich shifts also the initial random variables (observables) and not only the increments. Moreover , in order to be able to apply the quantum FK perturbation technique , the free time shift should be such that the structure of the associated unitary cocycles should be determined quite explicitly , which is rarely the case if this time shift is itself a Feynman-Kac perturbation of u_t^o. This problem was posed by A. Meyer during the Obervolfach meeting and in the following I want to outline a possible general scheme for a solution and illustrate it with an example.

(2.) A POSSIBLE SCHEME FOR A SOLUTION

Let us look for a time shift v_t^o of the form

$$v_t^o = j_t \otimes \sigma_t : \mathcal{B} = \mathcal{B}(\mathcal{H}) = \mathcal{B}\left(H_o \otimes \Gamma(L^2(\mathbf{R}_+)\right) \cong \mathcal{B}(H_o) \otimes \mathcal{B}\left(\Gamma(L^2(\mathbf{R}_+)\right) \longrightarrow \mathcal{B} \tag{2.1}$$

where

$$j_t : \mathcal{B}(H_o) \cong \mathcal{B}(H_o) \otimes 1 \longrightarrow \mathcal{B}_{t]} \tag{2.2}$$

is a $*$-homomorphism . For all $a_o \in \mathcal{B}(H_o)$, $b \in \mathcal{B}(\Gamma(L^2(\mathbf{R}_+))$ one has

$$v_s^o v_t^o(a_o \otimes b) = j_s \otimes \sigma_s(j_t(a_o) \otimes \sigma_t(b)) = (j_s \otimes \sigma_s)(j_t(a_o)) \cdot (j_s \otimes \sigma_s)(1_o \otimes \sigma_t(b)) \tag{2.3}$$

$$= (j_s \otimes \sigma_s)(j_t(a_o)) \cdot (1_{t]} \otimes \sigma_{s+t}(b))$$

and since we want v_t^o to be a 1-parameter semi-group of *-endomorphisms, it follows that, for any $a_o \in \mathcal{B}(H_o)$, $b \in \mathcal{B}(\Gamma(L^2(\mathbf{R}_+))$ the right hand side of (2.3) must be equal to

$$v_{s+t}^o(a_o \otimes b) = j_{s+t}(a_o) \otimes \sigma_{s+t}(b) \tag{2.4}$$

Thus v_s^o will be a 1-parameter semi-group if and only if

$$(j_s \otimes \sigma_s)(j_t(a_o)) = j_{s+t}(a_o) \qquad \forall a_o \in \mathcal{B}(H_o) \tag{2.5}$$

Here we give an example of a j_t satisfying condition (2.5) above. First shall we give the expression of j_t in unbounded form and then shall write the corresponding bounded form .
In the notations of Section (1) choose $H_o = L^2(\mathbf{R})$ with a_o, a_o^+ denoting the usual annihilation and creation operators. Define

$$j_t(a_o^\epsilon) = a_o^\epsilon + X_{[0,t]} \quad ; \quad a_o^\epsilon = a_o \text{ or } a_o^+ \tag{2.6}$$

with $(s,t) \mapsto X_{[0,t]}$ a σ-homogeneous normal additive process, i.e.

$$X_{[0,t]} = X_{[0,t]}^+ \hat{\in} 1_{H_o} \otimes \mathcal{B}\left(\Gamma(L^2([0,t])\right) \quad ; \quad [X_{[0,t]}, X_{[0,t]}^+] = 0 \tag{2.7}$$

$$X_{[r,s]} + X_{[s,t]} = X_{[r,t]} \quad ; \quad r < s < t \tag{2.8}$$

$$\sigma_r(X_{[s,t]}) = X_{[s+r,t+r]} \tag{2.9}$$

$$[X_{[r,s]}, X_{[u,t]}] = 0 \qquad if\ (u,t)\cap(r,s) = \emptyset \tag{2.10}$$

where $[\ .\ ,\ .\ ]$ denotes , as usual , the commutator . In bounded form and under the additional assumption that $X_{[0,t]}$ is self-adjoint, j_t can be defined on the Weyl operators on H_o by :

$$j_t(W_o(z)) = j_t(\exp i(za_o^+ + z^+a_o)) = \exp i(zj_t(a_o^+) + z^+j_t(a_o)) = \tag{2.11}$$

$$= e^{i(za_o^+ + z^+a_o) + (2Rez)X_{[0,t]})} = W_o(z)e^{i(2Rez)X_{[0,t]}}$$

An example of $X_{[0,t]}$ sastisfying the required conditions is the momentum operator $P(\chi_{[0,t]})$. Another example is $X_{[0,t]} = W_t - W_o$, which gives the usual free shift in Wiener space (not only in the increment space cf. Meyer ' s contribution to these proceedings) . Other examples could be obtained using the position or number processes or mixtures of them i.e., Weyl shifts of the form :

$$j_t(W_o(z)) = W_o(z)W(z\chi_{[0,t]}; e^{iz\chi_{[0,t]}}) \tag{1.12}$$

(cf. the remark at the end of this note).

(3.) THE SEMI-GROUP ASSOCIATED TO THE CHOICE $X_{[0,t]} = P(\chi_{[0,t]})$

The semi-group P_o^t , associated to the "free" evolution v_t^o is

$$P_o^t = E_{o]}(v_t^o(a_o)) \qquad ; \qquad a_o \in \mathcal{B}(H_o) \tag{3.1}$$

$$E_{o]} = 1_o \otimes < \Phi, (\ .\ )\Phi > :\ \mathcal{B}(H_o) \otimes \mathcal{B} \longrightarrow \mathcal{B}(H_o) \cong \mathcal{B}(H_o) \otimes 1 \tag{3.2}$$

In our case, choosing $b = W_o(z)$ $(z \in \mathbf{C})$ and $X_{[0,t]} = P(\chi_{[0,t]})$, one finds

$$P_o^t(W_o(z)) = E_{o]}(v_t^o(W_o(z))) = E_{o]}\left(W_o(z)e^{i(2Rez)P(\chi_{[0,t]})}\right) = W_o(z)e^{-2(Rez)^2t} \tag{3.3}$$

Hence the Weyl operators are in the domain of the generator of P_o^t and one has:

$$P_o^t = \exp tL \tag{3.4}$$

with

$$L(W_o(z)) = -2(Rez)^2W_o(z) \qquad ; \qquad z \in \mathbf{C}) \tag{3.5}$$

The explicit form of the generator can be obtained with the following semiheuristic, considerations:

$$W_o(z) = \exp i(za_o^+ + z^+a_o) \tag{3.6}$$

therefore

$$\frac{\partial}{\partial a_o^+}W_o(z) = (iz)W_o(z) \qquad ; \qquad \frac{\partial}{\partial a_o}W_o(z) = (iz^+)W_o(z)$$

hence

$$\left(\frac{\partial}{\partial a_o^+} + \frac{\partial}{\partial a_o}\right)W_o(z) = i(2Rez)W_o(z)$$

and therefore

$$\left[\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial a_o^+} + \frac{\partial}{\partial a_o}\right)\right]^2 = L \tag{3.7}$$

Now, from

$$[a_o\ ,\ a_o^+] = 1 \tag{3.8}$$

we deduce

$$[a_o\ ,\ .\ ] = \frac{\partial}{\partial a_o^+} \qquad ; \qquad [a_o^+\ ,\ .\ ] = -\frac{\partial}{\partial a_o} \tag{3.9}$$

In conclusion

$$L = \Big[\frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial a_o^+} + \frac{\partial}{\partial a_o}\Big)\Big]^2 = \Big[\frac{1}{\sqrt{2}}\Big([a_o\ ,\ .\ ] + [a_o^+\ ,\ .\ ]\Big)\Big]^2 = \frac{1}{2}[a_o - a_o^+\ ,\ .\ ]^2 \tag{3.10}$$

and since $\frac{1}{\sqrt{2}}[a_o - a_o^+\ ,\ .\ ]$

$$L = -[p\ ,\ .\ ]^2 = -[p\ ,\ [p\ ,\ .\ ]\ ] \tag{3.11}$$

So the free semigroup is a quasifree semigroup of the type considered by Lindblad in [4]. Notice moreover that, if f is a smooth function and M_f is multiplication by f in $L^2(\mathbf{R})$, then with the identification $p = \frac{1}{i}\frac{\partial}{\partial x}$ one has

$$[p, M_f] = \frac{1}{i}M_{(\frac{\partial}{\partial x}f)} \tag{3.12}$$

hence

$$-[p\ ,\ [p\ ,\ M_f\ ]\ ] = M_{(\frac{\partial^2}{\partial x^2}f)} \tag{3.13}$$

which gives the right answer when we restrict our attention to the classical Wiener process.

(4.) v_t^o- Markovian cocycles : an example

Consider the stochastic differential equation (SDE)

$$dU = \Big((L_o + X_{[0,t]})dA^+ - (L_o^+ + X_{[0,t]})dA + Zdt\Big)U \tag{4.1}$$

the unitarity condition (using the Fock Ito table for dA , dA^+) is

$$Z = iH - \frac{1}{2}\mid L_o^+ + X_{[0,t]}\mid^2 \tag{4.2}$$

with H self-adjoint. By shifting the equation (4.1) , with H = 0 , with the free shift v_s^o, we obtain the equation for $v_s^o(U_t)$ namely

$$dv_s^o(U_t) = \Big((L_o + X_{[0,t+s]})dA_s^+(t) - (L_o^+ + X_{[0,t+s]})dA_s(t) - \frac{1}{2}\mid L_o^+ + X_{[0,t+s]}\mid^2 dt\Big)v_s^o(U_t) \tag{4.3}$$

where we have used the notation

$$dA_s(t) = A(s+t+dt) - A(s+t) \tag{4.4}$$

which means that, by definition

$$\int_0^T Y_{t+s}dA_s(t) := \int_s^{s+T} Y_t dA(t) \tag{4.5}$$

Now, written in integral form, the equations (4.1) and (4.3) look respectively like

$$U_t = 1 + \int_0^t \Big((L_o + X_{[0,r]})dA^+(r) - (L_o^+ + X_{[0,r]})dA(r) - \frac{1}{2}\mid L_o^+ + X_{[0,r]}\mid^2 dr\Big)U_r \tag{4.6}$$

$$v_s^o(U_t) = 1 + \int_0^t \Big((L_o + X_{[0,s+r]})dA^+(s+r) - (L_o^+ + X_{[0,s+r]})dA(s+r) - \frac{1}{2}\mid L_o^+ + X_{[0,s+r]}\mid^2 dr\Big)v_s^o(U_r) \tag{4.7}$$

So that $v_s^o(U_t)\cdot U_s$ and U_{s+t} satisfy the same SDE (in the t-variable) with the same initial condition i.e. U_s at t = 0. Therefore , if the $X_{[0,t]}$-process is regular enough to assure the existence and uniqueness of the solutions of the above SDE , it will follow that

$$v_s^o(U_t)\cdot U_s = U_{s+t} \tag{4.8}$$

which means that U_t is a v_t^o - Markovian cocycle . The formal unitarity of follows from (4.2) and in many interesting cases the unitarity can be effectively proved . Having the unitary cocycle, we can apply the FK perturbation scheme to the free semigroup associated to v_t^o . Denoting $\mathcal{L}_o$ the generator of this semigroup, a simple calculation shows that the formal generator of the perturbed semigroup

$$P^t(b_o) := E_{o]}\Big(U_t^* \cdot v_t^o(b_o)\cdot U_t\Big) \tag{4.9}$$

will be

$$\mathcal{L}_o + L_o^+ b + bL_o + L_o^+ bL_o \tag{4.10}$$

If the operators $\mathcal{L}_o$, L_o are unbounded, the expression (4.10) will not be in general a well defined operator . However, for $X_{[0,t]}$ as in Section (2) , the operators $L_o + X_{[0,t]}$; $L_o^+ + X_{[0,t]}$ are always well defined and therefore equation (1) makes sense and in some cases the conditions for the existence of a solution of this equation are much weaker than those which allow to realize $\mathcal{L}_o + L_o^+ b + bL_o + L_o^+ bL_o$ as a well defined operator.
Applying the considerations above to the additive functional $X_{[0,t]} = P(\chi_{[0,t]})$, for which the regularity problems mentioned above can be solved with standard techniques, one can produce singular perturbations of the noncommutative Laplacian in full analogy with the classical case.

The physical meaning of the operator $X_{[0,t]}$ in (4.1) can be understood in terms of Barchielli' s analysis [3] : $X_{[0,t]}$ is the input field (the number process in Barchielli 's paper) which interacts with an apparatus described by the operators L_o, Z in (4.1) . The free evolution of the system is given by the time shift v_t^o and equation (4.1) describes the interaction cocycle giving the evolution of the observables of the coupled system (input field + apparatus) according to the scheme proposed in [1] . The advantage of the present approach with respect to [3] is that, due to the v_t^o-cocycle property of the solution of (4.1), the interacting evolution $x \mapsto v_t^o(U_t^+ xU_t)$ will now be a 1-parameter automorphism group, in agreement with the basic principles of quantum physics.

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STOCHASTIC INTEGRATION

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(0.) Introduction.

The programme of developing a "representation free" stochastic calculus was outlined in [1] . In the present note, which is part of a joint work in preparation with J.Quagebeur, we concentrate on the first step of this programme: the definition of stochastic integral. We have chosen to take as our starting point the axiomatic definition of semimartingale adopted by Letta [4] and based on Dellacherie's characterization of the classical semimartingales (cf. [5] and Definition (3.1) below) which has the advantage of looking almost the same in the classical and in quantum case and of not depending on the detailed structure of the Hilbert spaces on which the operators act or of the algebras to which these operators are affiliated. It turns out that in the quantum , as in the classical, case a semimartingale is the most general object for which a stochastic integral with meaningful properties can be defined. However, being at the moment very far from having anything like a quantum version of Dellacherie's theorem, the best we can hope for is to find some sufficient condition, for an additive process to be a semimartingale, which is at the same time easily applicable and sufficiently general to cover all the known cases (and at least some new ones). In Theorem (3.3) such a sufficient condition is proposed; in Section (4.) we show the connection between our notion of stochastic integrals and the Hilbert space valued stochastic integrals in the sense of Metivier and Pellaumail [6] ; in Section (5.) we show that the basic integrators appearing in the varoius quasi-free representation of the CCR or CAR over $L^2(\mathbf{R}_+)$ are semi-martingales in the sense of Definition (3.1); finally, in Section (6.) , we prove an existence and uniqueness theorem for stochastic differential equations.
As mentioned in Remark (3.), after Definition (3.1), the sufficient condition introduced in definition (3.3), is not yet general enough to include all the examples and applications we have in mind. However in the same Remark(3.) we show how this condition has to be modified in order to achieve this goal. It can be proved , but it is not done in this note, that, even with this modification the conclusions of both Theorem (3.3) and Theorem (6.2) continue to hold . In view of this we feel that the present approach is adequate for the development of a representation free notion of stochastic integral.

(1.) NOTATIONS

Let $\mathcal{H}$ be a complex separable Hilbert space. We write $\mathcal{B}(\mathcal{H})$ to denote the vector space of all bounded operators on $\mathcal{H}$. Let $\mathcal{A}$ be a Von Neumann sub algebra of $\mathcal{B}(\mathcal{H})$ and let $(\mathcal{A}_{t]})_{t\geq 0}$ be an increasing family of Von Neumann subalgebras of $\mathcal{A}$. We write $\mathcal{A}'$ and $\mathcal{A}'_{t]}$ to denote the commutant of $\mathcal{A}$ and $\mathcal{A}_{t]}$ in $\mathcal{B}(\mathcal{H})$. Let $\mathcal{E}$ be a subset of $\mathcal{H}$ such that the set $\mathcal{A}'\cdot\mathcal{E}$, that we shall denote by $\mathcal{D}$ is dense in $\mathcal{H}$. $\mathcal{L}(\mathcal{D};\mathcal{H})$ will denote the set of pairs (F,F^+) of linear operators on $\mathcal{H}$ with domain containing $\mathcal{D}$ such that

$$<\eta, F\xi>=<F^+\eta,\xi>$$

for all elements η , $\xi\in\mathcal{D}$. The pair (F,F^+) will be denoted $F^{\cdot}$ or, if no confusion can arise, simply F. One easily verifies that $\mathcal{L}(\mathcal{D};\mathcal{H})$ is a vector space. We shall consider two topologies on $\mathcal{L}(\mathcal{D};\mathcal{H})$: the strong-* topology on $\mathcal{D}$, defined by the semi-norms

$$—A \longrightarrow || A\xi || + || A^+\xi || \quad , \quad \xi\in\mathcal{D}$$

and the weak topology on $\mathcal{D}$ defined by the semi-norms

$$A \longrightarrow |< \xi, A\xi >| \quad , \quad \xi \in \mathcal{D}$$

If X is a linear operator on $\mathcal{H}$ we write D(X) to indicate the domain of X. For all $t \geq 0$ we say that an element A of $\mathcal{L}(\mathcal{D};\mathcal{H})$ is affiliated with $\mathcal{A}_{t]}$ and write $A \bar{\in} \mathcal{A}$ if $AA' \supseteq A'A$ for all element A' of $\mathcal{A}'_{t]}$. **A stochastic process** in $\mathcal{H}$ is a family $(F_t)_{t\geq 0}$ of elements of $\mathcal{L}(\mathcal{D};\mathcal{H})$. Two stochastic processes are said to be equivalent if they coincide on $\mathcal{A}' \cdot \mathcal{E}$ The process (F_t) is **strongly-*** (resp. **weakly**) **measurable** if, for all elements $\xi \in \mathcal{D}$ the maps $t \mapsto \| F_t\xi \|$, $\| F_t^+\xi \|$(resp. $t \mapsto < \xi, F_t\xi >$) are measurable with respect to Lebesgue measure . The stochastic process (F_t) is called **adapted** (to the filtration $(\mathcal{A}_{t]})$ if, for all $t \geq 0$, the operators F_t and F_t^+ are affiliated with $\mathcal{A}$. A process is called an elementary predictable process if it can be written in the form

$$\sum_{k=1}^{n} \chi_{(t_k,t_{k+1}]} \otimes F_{t_k}$$

where $0 \leq t_o < t_1 < ... < t_n < \infty$ and F_{t_k} is affiliated with $\mathcal{A}_{t_k]}$ (for all k). If moreover F_{t_k} is an element of $\mathcal{A}_{t_k]}$ then we say that (F_t) is a bounded elementary predictable process.

(2). SIMPLE STOCHASTIC INTEGRALS

DEFINITION (2.1) An **additive process** on $\mathcal{H}$ is a family $(X(s,t))$ $(0 \leq s < t)$ of elements of $\mathcal{L}(\mathcal{D};\mathcal{H})$ such that: (i) for all s,t with $s < t$, $X(s,t)$ is affiliated with $\mathcal{A}_{t]}$. (ii) for all r,s,t with $r < s < t$ we have

$$X(r,t) = X(r,s) + X(s,t)$$

on D. For all additive processes X we shall denote by S(X) the set of all adapted processes (F_t) which can be written in the form

$$F_t = \sum_{k=1}^{n} \chi_{(t_k,t_{k+1}]}(t) F_{t_k} \tag{2.2}$$

where:

$$0 \leq t_o < t_1 < ..... < t_n < \infty \tag{2.3}$$

$$F_{t_k}(\mathcal{D}) \subseteq \bigcap_{t_k \leq r < s} D(X(r,s)) \tag{2.4}$$

$$\bigcup_{t_k \leq r < s} X(r,s)^+ D(X(r,s)^+) \subseteq D(F_{t_k}^+) \tag{2.5}$$

for all integers k with $0 \leq k \leq n$. Given an element F of S(X) one can define the **left stochastic integral**

$$\int dX_s F_s = \sum_{k=1}^{n} X(t_k, t_{k+1}) F_{t_k} \tag{2.6}$$

and the right stochastic integral

$$\int F_s^+ dX_s^+ = \sum_{k=1}^{n} F_{t_k}^+ X(t_k, t_{k+1})^+ \tag{2.7}$$

PROPOSITION (2.2) In the above notations
(i) The left and right stochastic integrals are independent of the representation of F in the form (2.2)
(ii) The pair $(\int dX_s F_s, \int F_s^+ dX_s^+)$ is an element of $\mathcal{L}(\mathcal{D};\mathcal{H})$
(iii) S(X) is a vector space and for all elements F , G of S(X) we have

$$\int dX_s(F_s + G_s) = \int dX_s F_s + \int dX_s G_s \tag{2.8}$$

$$\int (F_s^+ + G_s^+) dX_s^+ = \int F_s^+ dX_s^+ + \int G_s^+ dX_s^+ \tag{2.9}$$

(iv) For all elements F of S(X), a' of $\mathcal{A}'$ and ξ of $\mathcal{E}$ we have

$$\int dX_s F_s a'\xi = \overline{a'}\int dX_s F_s \xi \quad ; \quad \int F_s^+ dX_s^+ a'\xi = a' \int F_s^+ dX_s^+ \xi \tag{2.10}$$

PROOF. (i) Let F be an element of S(X) and let

$$\sum_{k=1}^{n} \chi_{(t_k, t_{k+1}]}(t) F_{t_k} \quad ; \quad \sum_{h=1}^{m} \chi_{(s_h, s_{h+1}]}(t) F_{s_h}$$

be two representations of F in the form (2.2). We have then

$$\sum_{k=1}^{n} X(t_k, t_{k+1}) F_{t_k} = \sum_{k=1}^{n}\sum_{h=1}^{m} X(t_k \vee s_h, t_{k+1} \wedge s_{h+1}) F_{t_k} = \sum_{h=1}^{m}\sum_{k=1}^{n} X(t_k \vee s_h, t_{k+1} \wedge s_{h+1}) F_{s_h} =$$

$$= \sum_{h=1}^{m} X(s_h, s_{h+1}) F_{s_h}$$

Similar equalities hold for the simple right stochastic integral. (ii) Let F be an element of S(X) which can be written in the form (2.2); we have then (for all $\xi, \eta \in \mathcal{D}$)

$$< \eta, \sum_{h=1}^{m} X(s_h, s_{h+1}) F_{s_h} \xi > = < \sum_{h=1}^{m} F_{s_h}^+ X^+(s_h, s_{h+1})\eta, \xi >$$

(iii) and (iv) are obvious.

NOTATION The pair

$$(\int dX_r F_r \chi_{(s,t]}(r), \int F_r^+ \chi_{(s,t]}(r) dX_r^+)$$

will be also denoted

$$(\int_s^t dX_r F_r, \int_s^t F_r^+ dX_r^+)$$

or simply, when no confusion can arise,

$$\int_s^t dX_r F_r$$

(3.) STOCHASTIC INTEGRALS

We shall denote by $\mathcal{P}_s(X)$ (resp. $\mathcal{P}_w(X)$) the vector space of all processes F such that there exists a sequence $F^{(n)}$ in S(X) with the following properties:

(i) for all t, the operator $F_t^{(n)}$ converges *-strongly (resp. weakly) on $\mathcal{D}$ to F .
(ii) for all $t \geq 0$, $\xi \in \mathcal{E}$ and all integers n one has

$$\sup_{s \leq t} \| F_s^{(n)} \xi \| \leq c_{t.\xi} \qquad \left(resp. \sup_{s \leq t} \| F_s^{(n)} \xi \| \leq c_{t.\xi}\right)$$

where $c_{t.\xi}$ is a constant. We say that a sequence $(F^{(n)})$ $(n \geq 1)$ of elements of $\mathcal{P}_s(X)$ (resp. $\mathcal{P}_w(X)$) converges in $\mathcal{P}_s(X)$ (resp. $\mathcal{P}_w(X)$) to F if the two conditions above are fulfilled.

DEFINITION (3.1) An additive process X is a **strong-*** (resp. **weak**) **semimartingale** if for all sequences $F^{(n)}$ in S(X) converging to zero in $\mathcal{P}_s(X)$ (resp. $\mathcal{P}_w(X)$) and for all $t \geq 0$ the simple stochastic integrals

$$\int dX_r F_r^{(n)}, \int F_r^{(n)+} dX_r^+$$

converge to zero strongly (resp. weakly) in $\mathcal{L}(\mathcal{D};\mathcal{H})$.

REMARK(1.) Identifying, as usual, a scalar (real or complex valued) classical process with the associated multiplication operators on the L^2-space of the process , the content of Dellacherie' s theorem mentioned in the introduction is that a scalar process X is a semimartingale in the sense of Definition (3.1) if and only if it admits a decomposition of the form

$$X = M + A$$

where M is a local martingale and A is the difference of two increasing processes. In fact, in Dellacherie' s formulation of condition (i) above, convergence in probability is substituted for *-strong convergence but, due to condition (ii) and to the fact that for classical processes the *-strong convergence reduces to L^2-convergence, in that case the two conditions are equivalent since a norm bounded sequence in L^2 which converges to zero in probability converges to zero also in L^2.

REMARK(2.) The following example shows that condition (ii) in the definition of convergence in $\mathcal{P}_s(X)$ (resp. $\mathcal{P}_w(X)$) is necessary to have a good notion of stochastic integral. Let

$$X(s,t) = (t-s)\cdot 1$$

and

$$F^{(n)} = 1\cdot n\cdot \chi_{(0,1/n]}$$

Then, for any element ξ of $\mathcal{E}$ we have for all $t > 0$

$$\| F_t^{(n)}\xi \|^2 = n^2\cdot\chi_{(0,1/n]}(t)\cdot \| \xi \|^2 \longrightarrow 0$$

but for all n

$$\| \int_0^1 dX_t F_t^{(n)}\xi \|^2 = \| \xi \|^2$$

REMARK(3.) If we want a larger class of semimartingales we must require that the continuity property expressed in Definition (3.1) hold for a stronger topologyes on a smaller space of integrands . In particular the topologies defined by the seminorms

$$A \mapsto \| CA\xi \| + \| A^+C^+\xi \| \quad ; \quad A \mapsto < \xi, A^+CA\xi >$$

where C is a process arise naturally in several applications. For example, if A is an element of $\mathcal{L}(\mathcal{D};\mathcal{H})$ not bounded and affiliated with $\mathcal{A}_{0]}$, then

$$X(s,t) = (t-s)A$$

is not, in general, an semimartingale in the sense of Definition (3.1) because it may not be true that if $F_t^{(n)}\cdot\xi \longrightarrow 0$ then $AF_t^{(n)}\cdot\xi \longrightarrow 0$ However in the present paper we shall only consider the case $C = 1$ (cf. the remark at the end of the introduction).

We can now define the strong-* integral with respect to a semimartingale X . Let F be an element of $\mathcal{P}_s(X)$ and $(F^{(n)}\cdot)$ be a sequence of elements of S(X) converging to F in $\mathcal{P}_s(X)$. For all elements ξ of $\mathcal{E}$ (hence also for all $\eta \in \mathcal{D}$ and all $t \geq 0$, the sequences $(F^{(n)}\cdot\xi)$ are Cauchy in $\mathcal{H}$. Moreover in view of the property of the semimartingale X the limits are independent of the particular sequence. One can therefore define

$$\int_0^t dX_s F_s \xi = \lim_{n\to\infty} \int_0^t dX_s F_s^{(n)}\xi$$

$$\int_0^t F_s^+ dX_s^+ \xi = \lim_{n\to\infty} \int_0^t F_s^{(n)+} dX_s^+ \xi$$

Similarly one defines the weak sthocastic integral. The following elementary properties of the stochastic integral are easily checked :

PROPOSITION (3.2) Let X be a weak semimartingale , then :

(i) For any element F of $\mathcal{P}_w(X)$ and for all $t \geq 0$, the pair $(\int_0^t dX_s F_s \ , \ \int_0^t F_s^+ dX_s^+)$ is an element of $\mathcal{L}(\mathcal{D};\mathcal{H})$.

(ii) $\mathcal{P}_w(X)$ is a vector space and for all elements F , G of $\mathcal{P}_w(X)$ the relations (2.8) and (2.9) hold for all $t \geq 0$.

(iii) For all elements F of $\mathcal{P}_w(X)$, a' of A' and ξ of $\mathcal{E}$ and for all $t \geq 0$ the relations (2.10) hold.

Moreover the same statements hold for $\mathcal{P}_s(X)$ when X is a strong semimartingale .

PROOF. (i) Let $F^{(n)\cdot}$ a sequence of elements of S(X) converging to $F^\cdot$ in $\mathcal{P}_s(X)$ (resp. $\mathcal{P}_s(X)$). Then for all elements $\xi, \eta \in \mathcal{D}$ using (2.8) (ii) we have

$$< \int_0^t F_s^+ dX_s^+ \eta, \xi >= \lim_{n\to\infty} < \int_0^t F_s^{(n)+} dX_s^+ \eta, \xi >= \lim_{n\to\infty} < \eta, \int_0^t dX_s F_s^{(n)} \xi >=< \eta, \int_0^t dX_s F_s \xi >$$

The other statements can be proved in a similar way.

DEFINITION (3.3) An additive process X is called a regular semimartingale for the set $\mathcal{E}$. if it satisfies the following condition : for all elements $\xi \in \mathcal{E}$ there exist two positive funtions $g_\xi \in L^1_{loc}(\mathbf{R}_+)$ such that, for all elements F of S(X) and all $t \geq 0$ we have:

$$\| \int_0^t dX_s F_s \xi \|^2 \leq c_{t.\xi} \cdot \int_0^t \| F_s \xi \|^2 \, g_\xi(s) ds \tag{3.2a}$$

$$\| \int_0^t F_s^+ dX_s^+ \xi \|^2 \leq c_{t.\xi} \cdot \int_0^t \| F_s^+ \xi \|^2 \, g_\xi^+(s) ds \tag{3.2b}$$

where $c_{t.\xi}$ is a positive constant.

THEOREM (3.4) Any regular semi-martingale is a strong-* semimartingale .

PROOF. Let $(F^{(n)\cdot})_{n\geq 1}$ be a sequence in S(X) converging to zero in $\mathcal{P}_s(X)$ then, for all $t \geq 0$ we have

$$\lim_{n\to\infty} \| F_s^{(n)\cdot} \xi \| = 0$$

$$\sup_{s\leq t} \| F_s \xi \| \leq c'_{t.\xi}$$

where $c'_{t.\xi}$ is a positive constant. But the conditions (3.5) combined with Lebesgue's theorem implies then that

$$\lim_{n\to\infty} \int_0^t dX_s F_s^{(n)} \xi = \lim_{n\to\infty} \int_0^t F_s^{(n)+} dX_s^+ \xi = 0$$

If X is a regular semimartingale then ,as shown by the following theorem , we can extend the stochastic integral (with respect to X) to a class of processes larger than $\mathcal{P}_s(X)$.

THEOREM (3.5) Let X be an additive process satisfying condition (3.4) and let $(F_t)_{t\geq 0}$ be a measurable adapted process such that, for all $\xi \in \mathcal{E}$ and all $t \geq 0$

$$\int_0^t \left(\| F_s \xi \|^2 g_\xi(s) + \| F_s^+ \xi \|^2 g_\xi^+(s) \right) ds < \infty \tag{3.3}$$

$$F_t(\mathcal{D}) \subseteq \bigcap_{s>r\geq t} D(X(r,s)) \tag{3.4}$$

$$\bigcup_{s>r\geq t} [X^+(r,s)\xi] \subseteq D(F_t^+) \tag{3.5}$$

then we can define the stochastic integral of F with respect to X . Moreover, for all element $\xi \in \mathcal{E}$ the inequalities (3.5) hold.

PROOF. Suppose first that for all element $\xi \in \mathcal{E}$ the functions $s \mapsto F_s \xi$ are continuous and consider then the sequence of elements of S(X)

$$F_t^{(n)} = \sum_k \chi_{(\frac{k}{n}, \frac{k+1}{n}]}(t) F_{\frac{k}{n}}$$

Then we can show that $F^{(n)}$ converges to F in $P_s(X)$. In fact for all $\xi \in \mathcal{E}$, all $t \geq 0$ and all $\epsilon > 0$ there exists a $\delta > 0$ such that, if

$$| r - s | < \delta \qquad ; \qquad 0 \leq r, s \leq t$$

then

$$\| F_r \xi - F_s \xi \| < \epsilon$$

Thus for all n such that $1/n < \delta$ we have

$$\| F_s^{(n)} \xi - F_s \xi \| < \epsilon \qquad ; \qquad \sup_{s\leq t} \| F_r^{(n)} \xi \| \leq \sup_{s\leq t} \| F_s \xi \| = c'_{t,\xi}$$

Now let $(F_t)_{t\geq 0}$ be a measurable adapted process satisfying conditions (3.3), (3.4), (3.5) and let $(\phi_n)_{n\geq 1}$ be the sequence of positive measurable functions

$$\phi_n(t) = n\chi_{(0,\frac{1}{n}]}(t)$$

Let us consider the processes

$$F_s^{(n)} \xi = \int_0^s \phi_n(u) F_{s-u} du$$

which is strongly continuous on $\mathcal{E}$ and adapted. Then, for all $\xi \in \mathcal{E}$,

$$\| F_s^{(n)} \xi - F_s \xi \|^2 = \| \int_0^t \phi_n(u) [F_{s-u} \xi - F_s \xi] du \|^2$$

and therefore

$$\int_0^t \| F_s^{(n)} \xi - F_s \xi \|^2 g_\xi(s) ds \leq \int_0^t g_\xi(s) ds \int_0^s \phi_n(s-u) \| F_u \xi - F_s \xi \|^2 du$$

$$= \int_0^t g_\xi(s) ds \int_0^s \phi_n(u) \| F_{s-u} \xi - F_s \xi \|^2 du$$

$$= \int_0^t \phi_n(u) \int_u^t \| F_{s-u}\xi - F_s\xi \|^2 g_\xi(s)ds$$

$$= n \int_0^{\frac{1}{n}} du \int_u^t \| F_{s-u}\xi - F_s\xi \|^2 g_\xi(s)ds$$

$$\leq n \int_0^{\frac{1}{n}} du \int_0^t \| F_{s-u}\xi - F_s\xi \|^2 g_\xi(s)ds \longrightarrow 0$$

as $n \to \infty$. Similarly

$$\int_0^t \| F_s^{(n)+}\xi - F_s^+\xi \|^2 g_\xi^+(s)ds \leq n \int_0^{\frac{1}{n}} du \int_0^t \| F_{s-u}^+\xi - F_s^+\xi \|^2 g_\xi^+(s)ds$$

And so

$$\lim_{n\to\infty} \int_0^t \| F_s^{(n)\cdot}\xi - F_s^{\cdot}\xi \|^2 g_\xi^+(s)ds = 0 \tag{3.6}$$

Therefore the sequences

$$\int_0^t dX_s F_s^{(n)}\xi \qquad , \qquad \int_0^t F_s^{(n)+} dX_s^+\xi$$

are Cauchy in $\mathcal{H}$. Moreover these limits are the same for any sequence satisfying (3.6) so we can define

$$\int_0^t dX_s F_s \xi = \lim_{n\to\infty} \int_0^t dX_s F_s^{(n)}\xi$$

$$\int_0^t F_s^+ dX_s^+ \xi = \lim_{n\to\infty} \int_0^t F_s^{(n)+} dX_s^+\xi$$

And we have moreover

$$\| \left(\int_0^t dX_s F_s \xi \right) \|^2 \leq c_{t.\xi} \int_0^t \| F_s^{\cdot}\xi \|^2 g_\xi(s)ds$$

(4.) CLASSICAL STOCHASTIC INTEGRALS

In this section we shall show that, under quite general conditions, the definition of quantum stochastic integral of the preceeding section includes the classical one . We shall deal with the following objects:

- H - separable Hilbert space
- $(\Omega, \mathcal{F}, P)$ a probability space
- $(\mathcal{F}_{t]}$ $(t \geq 0)$ a filtration
- (x_t) $(t \geq 0)$ a locally integrable H-valued semimartingale

We will suppose, for simplicity, that x has a decomposition of the form

$$x_t = m_t + \int_0^t b(s)ds \tag{4.1}$$

where: - $b : \mathbf{R}_+ \times \Omega \to H$ is adapted and , for all $t \geq 0$:

$$\int_0^t E(\| b(s) \|^2 ds < \infty$$

- m is a locally square integrable martingale with quadratic variation $< m >$ of the form