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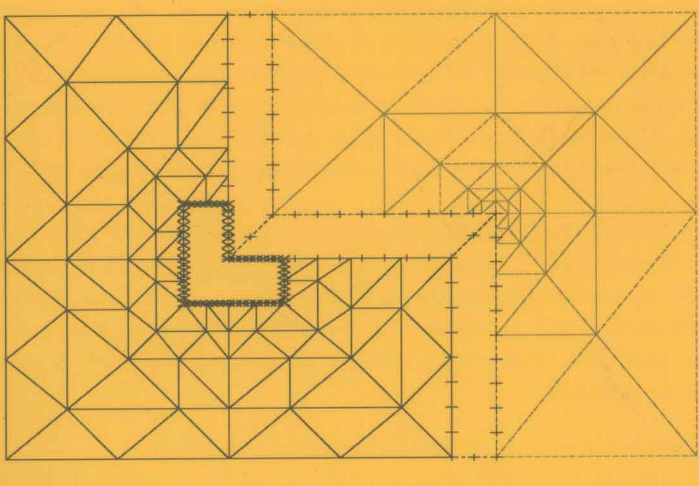
Olaf Steinbach

Stability Estimates for Hybrid Coupled Domain Decomposition Methods



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Contents

Introduction

Domain decomposition methods are a well established tool for an efficient numerical solution of partial differential equations, in particular for the coupling of different

- models, i.e., partial differential equations;
- discretization methods such as finite and boundary element methods;
- finite–dimensional trial spaces and their underlying meshes.

Especially when solving boundary value problems in complicated three–dimensional structures, a decomposition of the complex domain into simpler subdomains seems to be advantageous. Then we can replace the global problem by local subproblems, which are linked together by suitable transmission or coupling conditions. The solution of local boundary value problems defines local Dirichlet–Neumann or Neumann–Dirichlet maps. Hence, in domain decomposition methods we need to find the complete Cauchy data on the skeleton. This results in a variational formulation to find either the Dirichlet or Neumann data on the skeleton, and the remaining data are determined by the local problems and the coupling conditions. By solving local Dirichlet boundary value problems we can define local Dirichlet–Neumann maps involving the Steklov–Poincaré operator acting on the given Dirichlet data and some Newton potential to deal with given volume forces. To describe the Steklov–Poincaré operator locally we can use either a variational formulation within the subdomains or we can use boundary integral equations to find different representations of the Steklov–Poincaré operator. Since all of these definitions are given implicitly, we have to define suitable approximations of the local Steklov–Poincaré operators to be used in practical computations. For this we may use either finite or boundary element methods locally leading to a natural algorithm for the coupling of finite and boundary elements. Moreover, since these approximations are defined locally by solving Dirichlet boundary value problems, the underlying meshes do not need to satisfy any compatibility condition.

The coupling of locally different trial spaces is the main concern of this work. In many situations, for example in case of geometrically singularities or jumping coefficients, one would like to use local trial spaces defined on adaptively refined meshes or of different polynomial degree, or a combination of both. Then the problem arises, how to couple the local trial spaces to get a stable approximation globally.

In [8], a new concept to couple standard finite elements and spectral elements was introduced; this approach finally leads to the Mortar finite element method [9]. Introducing Lagrange multipliers as dual variables, a weak coupling of the primal variables is formulated. Variants of this method are hybrid coupled domain decomposition methods [3]; the hybrid coupling of finite and boundary elements [43]; three–field domain decomposition methods [24]. Since all of these methods can be formulated as saddle point problems, we need to have a certain discrete inf–sup condition to be satisfied [21]. Using a criteria due to Fortin [36], the discrete inf–sup condition is equivalent to the stability of an associated L_2 projection operator in $\widetilde{H}^{1/2}(\Gamma_{ij})$ where Γ_{ij} is the local coupling boundary of the subdomains Ω_i and Ω_j. For globally quasi–uniform meshes, the stability of the L_2 projection operator follows from appropriate error estimates and the use of the inverse inequality. However, for locally quasiuniform meshes such an approach is not applicable. One way out is the use of discrete Sobolev norms [14, 73]. Another possibility is to prove the stability of the L_2 projection operator directly in the scale of Sobolev norms. In [34], the required stability in H^1 was shown for nonuniform triangulations in one and two space dimensions satisfying certain mesh conditions. The analysis is based on decay properties of the L_2 projection and results in conditions which depend on the global behavior of the mesh. In a recent paper [18] we proved the H^1 stability of the L_2 projection onto piecewise linear finite element spaces for arbitrary space dimension. In this case we can formulate explicit local mesh conditions which can be checked easily for a given finite element mesh. This approach can be extended to more general situations, i.e. when using higher order polynomial, dual and biorthogonal basis functions. Biorthogonal basis functions were introduced in [75] to prove the stability of the Mortar finite element method. We will show that biorthogonal basis functions fit in the approach presented here. In the recent literature [7, 13, 49] there is a special emphasis on the numerical analysis of Mortar finite elements in three space dimensions. Using the general approach described in this monograph we are able to design appropriate Lagrange multiplier spaces to be used in hybrid coupled domain decomposition methods. We prove stability estimates for quite general trial spaces assuming only some mild conditions on the underlying mesh. Moreover, by computing some local mesh parameters one is able to control the formulated stability criteria.

Note that the Mortar finite element method is applicable to couple non–matching grids and local trial spaces of different polynomial degree. Another approach to couple locally non-matching grids without the use of Lagrange parameters is based on a domain decomposition formulation with local Dirichlet–Neumann maps. Defining a global trial space on the skeleton, a Galerkin variational problem is formulated for the assembled Steklov–Poincaré operators which correspond to the solution of local Dirichlet boundary value problems. Therefore, an approximation of local Steklov–Poincaré operators is only defined by using local degrees of freedom. When using a boundary element method we need to approximate the conormal derivative, when using a finite element method we need to approximate the solution of a Dirichlet boundary value problem in interior nodes. Since both trial spaces are locally, no compatibility conditions are required. This results in a natural domain decomposition method [68]. For solving a Dirichlet boundary value problem using finite elements we have to extend the given Dirichlet data from the boundary to the domain. In case of nested trial spaces, this can be done by interpolation, otherwise one may use a two–level method globally.

The discretization of several domain decomposition algorithms discussed here leads to linear algebraic systems, where the stiffness matrix is in general positive definite, but either symmetric or block skew–symmetric. Hence, for an efficient iterative solution in parallel, one needs to design special algorithms and almost optimal preconditioners to be used. Note that we will not focus on this topic here, but we refer to [11, 17, 41] for finite element domain decomposition methods; to [14, 38, 70, 74] for multigrid methods for Mortar finite elements; to [27, 51, 60, 69] for boundary element domain decomposition methods; to [16] for positive definite and block skew–symmetric linear systems.

In this work, our main focus is on the formulation and on the stability analysis of hybrid coupled domain decomposition methods. In Chapter 1 we review the definition of Sobolev spaces and give an overview about variational methods for saddle point problems. In particular we discuss a two–fold saddle point formulation. After introducing some notations for finite element spaces we define several L_2 projection operators by Galerkin–Bubnov and Galerkin–Petrov variational problems. To prove the stability of these operators we need to have a bounded projection operator providing local error estimates. For this we recall the definition of quasi interpolation operators from [30].

Based on the equivalence of different stability estimates, and assuming the positive definiteness of a scaled Gram matrix, we prove in Chapter 2 the stability of the L_2 projection in H^s for $s \in (0,1]$. Then we investigate the required positivity assumption on the scaled Gram matrix by computing its minimal eigenvalue. For piecewise linear finite elements this results in an explicit and easily computable formula. When using higher order polynomial basis functions it is in general impossible to compute the minimal eigenvalue of the scaled Gram matrix in an explicit form. However, for a given mesh we can compute the eigenvalues numerically. We will illustrate the applicability of this approach by using Lagrange polynomials and antiderivatives of Legendre polynomials as local basis functions.

In Chapter 3 we introduce the Dirichlet–Neumann map and define the Steklov–Poincaré operator and the Newton potential by solving related Dirichlet boundary value problems. A first approach is based on a variational formulation using the Dirichlet bilinear form, the second one is based on a symmetric representation of the Steklov–Poincaré operator by using boundary integral operators. Using these representations we obtain results on the mapping properties of the Steklov–Poincaré operator. Since the Steklov–Poincaré is defined via the solution of a Dirichlet boundary value problem, we have to introduce suitable approximations. According to the definitions we use either a finite element method in the domain or a boundary element method on the boundary. Both lead to stable approximations of the Steklov–Poincaré operator. Applying the same ideas we can approximate the Newton potential and therefore we obtain an approximate Dirichlet–Neumann map.

This approximate Dirichlet–Neumann map is used in Chapter 4 for the numerical solution of mixed boundary value problems. The trial space for the unknown Dirichlet data on the boundary is in general independent of the trial space used to approximate the Steklov–Poincaré operator. In a first approach we eliminate the Neumann data while in a second approach we keep the Neumann data as an unknown function in the variational formulation. This is then equivalent to variational formulations using Lagrange multipliers [4, 15]. When using a compatible trial space to approximate the Steklov–Poincaré operator by finite elements, this discrete Steklov–Poincaré operator coincides with the Schur complement of the standard finite element method.

In Chapter 5 we use the same ideas to formulate hybrid coupled domain decomposition methods. Using a global trial space on the skeleton and eliminating the Neumann data by a weak coupling condition across the local interfaces, this gives a variational formulation of the assembled Steklov–Poincaré operators. To approximate the local Steklov–Poincaré operators, we can use trial spaces, which are independent of the trial space on the skeleton. Especially when using finite element approximations of the local Steklov–Poincaré operators, we obtain a method, which includes the coupling of non–matching meshes in a natural way. For a more practical approach we can formulate this method as a two–level algorithm consisting of a global coarse grid space and local fine grid spaces. To be more flexible, one may introduce an additional trial space for the primal variable locally. This leads to a three field domain decomposition method [24] which can be analyzed as a two–fold saddle point problem. To ensure stability, we have to define appropriate trial spaces satisfying the stability conditions as formulated in Chapter 2. When using a strong coupling of the local Neumann data, i.e. using a formulation with Lagrange parameters, we obtain a Mortar finite element method. Using the theory on saddle point problems we can ensure stability and convergence, when the trial spaces are chosen in an appropriate way. To illustrate the applicability of the proposed natural domain decomposition method we describe then a simple numerical experiment. We consider two model problems with jumping coefficients requiring a heterogeneous discretization within the subdomains. Finally, to describe a more practical situation, we consider a three–dimensional problem from linear elastostatics, where the domain is non Lipschitz. A domain decomposition leads to local subproblems where the substructures are Lipschitz domains. The local Steklov–Poincaré operators are then discretized by a symmetric Galerkin boundary element method.

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1 Preliminaries

In this chapter we summarize some results which are needed frequently in the succeeding chapters. In particular, we give a brief introduction to Sobolev spaces, for a detailed presentation, see for example [1, 52, 53]. Following [21] we then describe abstract results for the variational solution of saddle point problems, see also [57]. Then we summarize some basic definitions and properties of general finite element spaces and their underlying triangulations. Using Galerkin–Bubnov and Galerkin–Petrov variational formulations we introduce L_2 projection operators onto finite element spaces. To prove the stability of these operators in a scale of Sobolev spaces we need to have projection operators which are stable and admit local error estimates. Following [30] we introduce quasi–interpolation operators which satisfy both of these requirements.

1.1 Sobolev Spaces

Let $\Omega \subset \mathbb{R}^n$ with $n = 2$ or $n = 3$ be a bounded Lipschitz domain with boundary $\Gamma := \partial\Omega$. For $k \in \mathbb{N}_0$ we define the norm

$$||u||_{H^k(\Omega)} := \left\{ \sum_{|\alpha|\leq k} ||D^\alpha u||^2_{L_2(\Omega)} \right\}^{1/2} \tag{1.1}$$

while for $0 < s \in \mathbb{R}$, $s \notin \mathbb{N}$, we define

$$||u||_{H^s(\Omega)} := \left\{ ||u||^2_{H^{[s]}(\Omega)} + |u|^2_{H^s(\Omega)} \right\}^{1/2} \tag{1.2}$$

using the Sobolev–Slobodeckii norm

$$|u|_{H^s(\Omega)} := \left\{ \sum_{|\alpha|=[s]} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2s}} dx dy \right\}^{1/2} . \tag{1.3}$$

For $s \geq 0$ we introduce the Sobolev spaces

$$H^s(\Omega) := \overline{C^\infty(\Omega)}^{||\cdot||_{H^s(\Omega)}},$$
$$H_0^s(\Omega) := \overline{C_0^\infty(\Omega)}^{||\cdot||_{H^s(\Omega)}},$$
$$\widetilde{H}^s(\Omega) := \overline{C_0^\infty(\Omega)}^{||\cdot||_{H^s(\mathbb{R}^n)}}.$$

Note that for $s \geq 0$ we have the embedding

$$\widetilde{H}^s(\Omega) \subseteq H_0^s(\Omega). \tag{1.4}$$

In fact, see for example [52, Theorem 3.33],

$$\widetilde{H}^s(\Omega) = H_0^s(\Omega) \quad \text{provided } s \notin \left\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\right\}.$$

For $s < 0$ the Sobolev spaces are defined by duality with respect to the $L_2(\Omega)$ inner product,

$$H^s(\Omega) := [\widetilde{H}^{-s}(\Omega)]^*, \quad \widetilde{H}^s(\Omega) := [H^{-s}(\Omega)]^* \tag{1.5}$$

with norms

$$||f||_{H^s(\Omega)} := \sup_{0 \neq v \in \widetilde{H}^{-s}(\Omega)} \frac{|\langle f, v\rangle_{L_2(\Omega)}|}{||v||_{H^{-s}(\Omega)}},$$
$$||f||_{\widetilde{H}^s(\Omega)} := \sup_{0 \neq v \in H^{-s}(\Omega)} \frac{|\langle f, v\rangle_{L_2(\Omega)}|}{||v||_{H^{-s}(\Omega)}}.$$

Let $f \in \widetilde{H}^s(\Omega)$ be given for some $s < 0$. Then,

$$\begin{aligned} ||f||_{H^s(\Omega)} &= \sup_{0 \neq v \in \widetilde{H}^{-s}(\Omega)} \frac{|\langle f, v\rangle_{L_2(\Omega)}|}{||v||_{H^{-s}(\Omega)}} \\ &\leq \sup_{0 \neq v \in H^{-s}(\Omega)} \frac{|\langle f, v\rangle_{L_2(\Omega)}|}{||v||_{H^{-s}(\Omega)}} = ||f||_{\widetilde{H}^s(\Omega)} \end{aligned} \tag{1.6}$$

and therefore $f \in H^s(\Omega)$. Hence we have the embedding $\widetilde{H}^s(\Omega) \subseteq H^s(\Omega)$ for all $s \in \mathbb{R}$.

In a similar way as above we can define Sobolev spaces on the closed boundary $\Gamma := \partial\Omega$. In particular, we are interested in the case $s \in (0, 1)$ where the norm of the Sobolev space $H^s(\Gamma)$ is given by

$$||u||_{H^s(\Gamma)} := \left\{||u||^2_{L_2(\Gamma)} + |u|^2_{H^s(\Gamma)}\right\}^{1/2} \tag{1.7}$$

with

$$|u|^2_{H^s(\Gamma)} = \int_\Gamma \int_\Gamma \frac{|u(x)-u(y)|^2}{|x-y|^{n-1+2s}} ds_x ds_y . \tag{1.8}$$

For $s = 1$ we use the covariant derivatives to define

$$||u||_{H^1(\Gamma)} := \left\{ ||u||^2_{L_2(\Gamma)} + \sum_{|\alpha|=1} ||D^\alpha u||^2_{L_2(\Gamma)} \right\} . \tag{1.9}$$

Note that for $s > 1$ we need stronger assumptions on Ω to define Sobolev spaces $H^s(\Gamma)$, see for example [52]. In particular, if Ω is $C^{k-1,1}$ for $k \geq 0$, then $H^s(\Gamma)$ is well defined for $|s| \leq k$.

Finally, for $s < 0$ we define $H^s(\Gamma) := [H^{-s}(\Gamma)]^*$ by duality with respect to the $L_2(\Gamma)$ inner product,

$$||\lambda||_{H^s(\Gamma)} = \sup_{0 \neq v \in H^{-s}(\Gamma)} \frac{|\langle \lambda, v \rangle_{L_2(\Gamma)}|}{||v||_{H^{-s}(\Gamma)}} . \tag{1.10}$$

Theorem 1.1. [52, 53] *Let $\Omega \subset \mathbb{R}^n$ a bounded domain with Lipschitz boundary $\Gamma := \partial\Omega$. For any $u \in H^1(\Omega)$ there exists the trace $\gamma_0 u \in H^{1/2}(\Gamma)$ satisfying*

$$||\gamma_0 u||_{H^{1/2}(\Gamma)} \leq c_T \cdot ||u||_{H^1(\Omega)} . \tag{1.11}$$

Vice versa, for any $u \in H^{1/2}(\Gamma)$ there exists a bounded extension $\mathcal{E}u \in H^1(\Omega)$ satisfying $\gamma_0 \mathcal{E} u = u$ and

$$||\mathcal{E}u||_{H^1(\Omega)} \leq c_{IT} \cdot ||u||_{H^{1/2}(\Gamma)} . \tag{1.12}$$

Let $\Gamma_0 \subset \Gamma$ be an open subset of the closed boundary $\Gamma = \partial\Omega$. As in (1.4) we define two kinds of Sobolev spaces on Γ_0,

$$H^s(\Gamma_0) := \{ u : u = U_{|\Gamma_0} \text{ for some } U \in H^s(\Gamma) \}$$

$$\widetilde{H}^s(\Gamma_0) := \{ u \in H^s(\Gamma) : \operatorname{supp} u \subseteq \overline{\Gamma}_0 \}$$

with norms

$$||u||_{H^s(\Gamma_0)} := \inf_{u = U_{|\Gamma_0}} ||U||_{H^s(\Gamma)}, \quad ||u||_{\widetilde{H}^s(\Gamma_0)} := ||u||_{H^s(\Gamma)} .$$

These two families of spaces are related by duality with respect to $L_2(\Gamma_0)$,

$$[H^s(\Gamma_0)]^* = \widetilde{H}^{-s}(\Gamma_0), \quad [\widetilde{H}^s(\Gamma_0)]^* = H^{-s}(\Gamma_0) \quad \text{for } s \in \mathbb{R}.$$

Note that $\widetilde{H}^s(\Gamma_0)$ is often denoted by $H^s_{00}(\Gamma_0)$.

1.2 Saddle Point Problems

Let X and Π be some Hilbert spaces equipped with norms $||\cdot||_X$ and $||\cdot||_\Pi$, respectively. We assume that there are given some bounded bilinear forms

$$\begin{aligned} a(\cdot,\cdot) &: X \times X \to \mathbb{R}, \\ b(\cdot,\cdot) &: X \times \Pi \to \mathbb{R}. \end{aligned}$$

Note that by the Riesz representation theorem we can identify the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ with bounded operators $A : X \to X^*$ and $B : X \to \Pi^*$, respectively. In particular, for $u \in X$ we define $Au \in X^*$ such that

$$\langle Au, v\rangle = a(u,v) \quad \text{for all } v \in X$$

and $Bu \in \Pi^*$ satisfying

$$\langle Bu, \mu\rangle = b(u,\mu) \quad \text{for all } \mu \in \Pi.$$

For given $f \in X^*$ and $g \in \Pi^*$ we consider the saddle point problem to find $(u,\lambda) \in X \times \Pi$ such that

$$\begin{aligned} a(u,v) - b(v,\lambda) &= \langle f, v\rangle \\ b(u,\mu) &= \langle g, \mu\rangle \end{aligned} \tag{1.13}$$

for all $(v,\mu) \in X \times \Pi$.

Denote

$$V := \ker B := \{v \in X \,:\, b(v,\tau) = 0 \quad \text{for all } \tau \in \Pi\}, \tag{1.14}$$

its orthogonal complement

$$V^\perp := \{w \in X \,:\, (w,v) = 0 \quad \text{for all } v \in V, \}. \tag{1.15}$$

and

$$V^0 := \{f \in X^* \,:\, \langle f, v\rangle = 0 \quad \text{for all } v \in V\}. \tag{1.16}$$

Theorem 1.2. [12, 21, 57] *Let the bounded bilinear form $a(\cdot,\cdot) : X \times X \to \mathbb{R}$ be elliptic on $V = \ker B$,*

$$a(v,v) \geq c_1^A \cdot ||v||_X^2 \quad \textit{for all } v \in V = \ker B. \tag{1.17}$$

If the bounded bilinear form $b(\cdot,\cdot) : X \times \Pi \to \mathbb{R}$ satisfies the inf–sup condition

$$\inf_{0 \neq \mu \in \Pi} \sup_{0 \neq v \in X} \frac{b(v,\mu)}{||v||_X ||\mu||_\Pi} \geq \gamma_S > 0, \tag{1.18}$$

and if $g \in \operatorname{Im} B$, then there exists a unique solution of (1.13) satisfying

$$||u||_X + ||\lambda||_\Pi \leq c \cdot \{||f||_{X^*} + ||g||_{\Pi^*}\}. \tag{1.19}$$

Moreover, $B : V^{\perp} \to \Pi^*$ is an isomorphism satisfying

$$\gamma_S \cdot ||v||_X \leq ||Bv||_{\Pi^*} \quad \text{for all } v \in V^{\perp}. \tag{1.20}$$

Finally, the operator $B^* : \Pi \to V^0$ is an isomorphism satisfying

$$\gamma_S \cdot ||\mu||_\Pi \leq ||B^*\mu||_{X^*} \quad \text{for all } \mu \in \Pi. \tag{1.21}$$

Let $X_h \subset X$ and $\Pi_h \subset \Pi$ be conforming finite dimensional trial spaces. The Galerkin variational formulation of the saddle point problem (1.13) is to find $(u_h, \lambda_h) \in X_h \times \Pi_h$ such that

$$\begin{aligned} a(u_h, v_h) - b(v_h, \lambda_h) &= \langle f, v_h \rangle \\ b(u_h, \mu_h) &= \langle g, \mu_h \rangle \end{aligned} \tag{1.22}$$

for all $(v_h, \mu_h) \in X_h \times \Pi_h$. As in the continuous case we define

$$V_h := \{v_h \in X_h : b(v_h, \tau_h) = 0 \quad \text{for all } \tau_h \in \Pi_h\} \tag{1.23}$$

and assume that the bilinear form $a(\cdot, \cdot)$ is V_h–elliptic,

$$a(v_h, v_h) \geq \widetilde{c}_1^A \cdot ||v_h||_X^2 \quad \text{for all } v_h \in V_h. \tag{1.24}$$

Theorem 1.3. [12, 21, 57] *Let the assumptions of Theorem 1.2 be satisfied and let the bilinear form $a(\cdot, \cdot)$ be V_h–elliptic. Let $V_h \subset V$. If the discrete inf–sup condition*

$$\inf_{0 \neq \mu_h \in \Pi_h} \sup_{0 \neq v_h \in X_h} \frac{b(v_h, \mu_h)}{||v_h||_X ||\mu_h||_\Pi} \geq \widetilde{\gamma}_S > 0 \tag{1.25}$$

is valid, then there exists a unique solution of (1.22) satisfying the error estimates

$$||u - u_h||_X \leq c_1 \cdot \inf_{v_h \in X_h} ||u - v_h||_X,$$

$$||\lambda - \lambda_h||_\Pi \leq c_2 \cdot \left\{ \inf_{v_h \in X_h} ||u - v_h||_X + \inf_{\mu_h \in \Pi_h} ||\lambda - \mu_h||_\Pi \right\}.$$

Hence we have convergence when assuming some approximation properties of X_h and Π_h. The crucial assumption of the preceding theorem is the discrete inf–sup condition (1.25). To characterize this condition we use a criteria due to Fortin [36]:

Theorem 1.4. *Assume that the continuous inf–sup condition (1.18) is satisfied. Let $P_h : X \to X_h$ be a projection operator satisfying the orthogonality*

$$b(v - P_h v, \mu_h) = 0 \qquad \text{for all } \mu_h \in \Pi_h$$

and the stability estimate

$$||P_h v||_X \leq c_S \cdot ||v||_X \qquad \text{for all } v \in X.$$

Then, (1.25) holds with $\widetilde{\gamma}_S = \gamma_S / c_S$.

Instead of the saddle point problem (1.13) we now consider a two–fold saddle point problem. Let X, Π_1 and Π_2 be some Hilbert spaces with norms $||\cdot||_X$, $||\cdot||_{\Pi_1}$ and $||\cdot||_{\Pi_2}$. Then we want to find $(u, \lambda, w) \in X \times \Pi_1 \times \Pi_2$ such that

$$\begin{aligned} b_1(u,\mu) - b_2(\mu,w) &= \langle f_1, \mu\rangle \\ -b_1(v,\lambda) + a(u,v) &= \langle f_2, v\rangle \\ b_2(\lambda,z) &= \langle g, z\rangle \end{aligned} \tag{1.26}$$

for all $(v, \mu, z) \in X \times \Pi_1 \times \Pi_2$. Here,

$$\begin{aligned} a(\cdot,\cdot) &: X \times X \to \mathbb{R}, \\ b_1(\cdot,\cdot) &: X \times \Pi_1 \to \mathbb{R}, \\ b_2(\cdot,\cdot) &: \Pi_1 \times \Pi_2 \to \mathbb{R} \end{aligned}$$

are bounded bilinear forms implying, by the Riesz representation theorem, bounded operators $A : X \to X^*$, $B_1 : X \to \Pi_1^*$ and $B_2 : \Pi_1 \to \Pi_2^*$, respectively.

Two–fold saddle point problems (1.26) appear in many applications: the coupling of mixed finite elements and symmetric boundary element methods [37]; hybrid boundary element methods [64] or three–field domain decomposition methods, see [24].

To prove unique solvability of the two–fold saddle point problem (1.26) we apply Theorem 1.2 twice. As in (1.14) we define

$$\ker B_2 := \{\mu \in \Pi_1 \; : \; b_2(\mu, z) = 0 \quad \text{for all } z \in \Pi_2\}. \tag{1.27}$$

Moreover,

$$\ker_{B_2} B_1 := \{v \in X \; : \; b_1(v,\mu) = 0 \quad \text{for all } \mu \in \ker B_2\}. \tag{1.28}$$

Theorem 1.5. *Let the bounded bilinear form $a(\cdot,\cdot) : X \times X \to \mathbb{R}$ be elliptic on $\ker_{B_2} B_1$,*

$$a(v,v) \ge c \cdot ||v||_X^2 \quad \text{for all } v \in \ker_{B_2} B_1. \tag{1.29}$$

Let the bounded bilinear forms $b_1(\cdot,\cdot) : X \times \Pi_1$ and $b_2(\cdot,\cdot) : \Pi_1 \times \Pi_2$ satisfy the inf–sup conditions

$$\inf_{0\neq\mu\in\Pi_1} \sup_{0\neq v\in X} \frac{b_1(v,\mu)}{||v||_X ||\mu||_{\Pi_1}} \ge \gamma_S > 0, \tag{1.30}$$

$$\inf_{0\neq z\in\Pi_2} \sup_{0\neq \mu\in\Pi_1} \frac{b_2(\mu,z)}{||\mu||_{\Pi_1} ||z||_{\Pi_2}} \ge \gamma_S > 0. \tag{1.31}$$

If $g \in Im B_2$ and $f_1 \in Im B_1$ is satisfied, then there exists a unique solution $(u, \lambda, w) \in X \times \Pi_1 \times \Pi_2$ of the two–fold saddle point problem (1.26) satisfying the stability estimate

$$||u||_X + ||\lambda||_{\Pi_1} + ||w||_{\Pi_2} \le c \cdot \left\{ ||f_1||_{\Pi_1^*} + ||f_2||_{X^*} + ||g||_{\Pi_2^*} \right\}. \tag{1.32}$$