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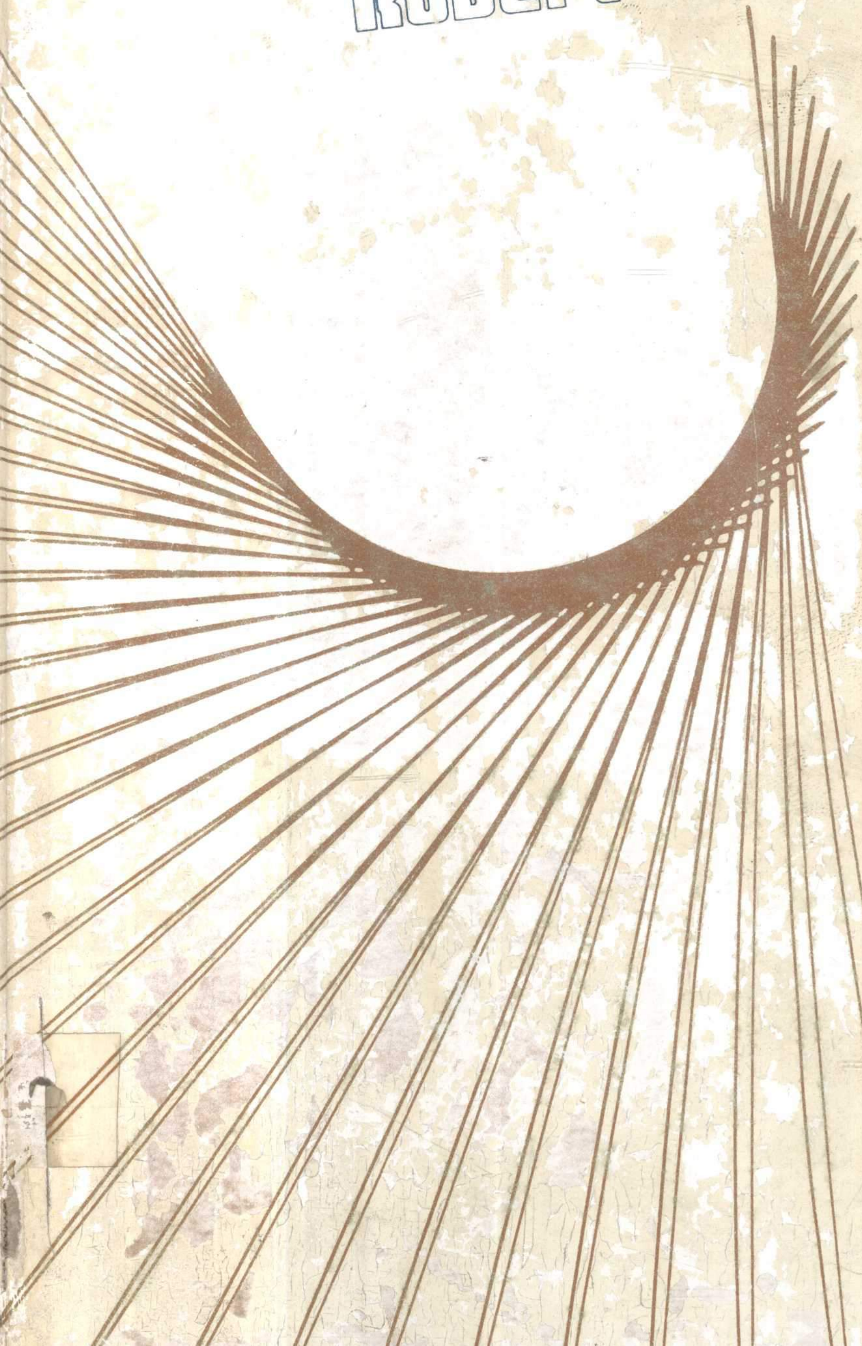

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Preface

This book is written for the one- or two-semester course in ordinary differential equations and was developed specifically with students of engineering and the physical sciences in mind. A knowledge of elementary calculus that is normally learned in a standard two- or three-semester university calculus sequence is assumed. The book has grown out of my experiences over the past several years in teaching various topics from the book in the ordinary differential equations course at North Carolina State University. During this time I have formed definite opinions as to the most effective instruction techniques which enable students to learn both the applications and the underlying theory of ordinary differential equations. It should be emphasized that without theory applications alone become a rote use of memorized techniques which are applied only with difficulty to nonstandard or previously unencountered problems.

This experience has led me to adopt a philosophy of instruction which employs so-called qualitative methods as adjuncts to the more computational techniques for solving equation systems. An example of such a qualitative method is the procedure for sketching the graphs of solutions, introduced in Chapter 1 to complement the techniques for solving first order linear equations. Other qualitative methods are used throughout the book. The emphasis of these qualitative methods is to show how the methods actually apply to specific problems and models; they are not intended as substitutes for the standard methods of solution computation.

My experience in teaching applied science students has also influenced both the writing style and the general organization of this book. I have endeavored to keep my writing style simple, clear, and to the point but with no sacrifice of rigor. Students are introduced to proofs, but are not over-burdened with technicalities. Where a proof is beyond the mathematical knowledge of students at this level, either an intuitive proof or a simple explanation of the general direction a proof would take is substituted. In general, explanations of both theories and applications are very detailed, probably more so than in any other book at this level. For those who wish further to pursue a given topic, a short bibliography is provided at the end of the book, together with appropriate in-chapter references.

Examples are usually used to introduce a given topic. The approach to any given example is usually natural, and with a little thought would probably even occur to most students, although at this stage of their mathematical development many students would not likely be able to carry the mathematics through to a conclusion. In this respect the book is designed to be partially self-teaching, and by reducing the class time needed for basic explanations should free class time for analysis of topics especially pertinent to a given class's interests.

In keeping with my goal of making this book useful for as broad a range of students as practical, I have provided a wide variety of applications with clear, detailed explanations of the physical properties involved in the examples and problems. This variety allows the instructor to choose applications which most closely match the mathematical backgrounds and interests of the class. There is a generous number of problems. All problems are intended to be instructive, and not merely drill. Some problems are so structured as to require the student to bring together material from several different sections in order to develop a strategy for solving the problem. In effect, such problems are brief tests.

Finally, the inclusion of several not so standard but illuminating topics, such as the thorough discussions of critical points in Section 1.5 and limits of solutions in Section 2.5*b*, encourages the growth of the students' mathematical sophistication.

Chapters 1 and 2 emphasize basic methods of explicitly solving standard types of equations and also indicate some fundamental applications. Methods for first order linear and nonlinear equations are contained in Chapter 1 and methods for second order linear equations are contained in Chapter 2. A procedure for sketching the graph of solutions, the first of several qualitative methods, is introduced in these two chapters. A knowledge of the fundamental terminology and solution methods for first and second order equations in Chapters 1 and 2 is the basis for the entire book. The remaining chapters are mutually independent of one another and depend only on the ideas in the first two chapters. This allows the instructor to tailor the bulk of the course to the interests and abilities of the class.

Chapter 3 contains the basic properties of the Laplace transform and indicates the procedure for applying transform methods to solve nonhomogeneous second order equations with constant coefficients. The stress here is on the elementary properties of the Laplace transform and its application in determining solutions to second order linear equations that have discontinuous nonhomogeneous terms. The case of a jump discontinuity is considered, and some of the procedures and interpretations for nonhomogeneous terms involving the Dirac delta function are also indicated. Power

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series methods for second order linear equations with nonconstant coefficients are developed in Chapter 4. Equations having ordinary power series solutions and those having series solutions of the Frobenius type are both considered. Also, one section is devoted to analyzing some of the most important equations where power series methods are used (Bessel's equation, Legendre's equation, and the hypergeometric equation).

Elementary concepts and techniques for first order systems of two equations (linear and nonlinear) are developed in Chapter 5. This chapter divides naturally into two parts: the explicit computation of solutions to linear equations with constant coefficients, and the sketching of solution curves in the plane for time-independent nonlinear equations. Several interesting and important models involving planar systems are included in the chapter: mixing in interconnected tanks, epidemics, interacting populations, and nonlinear oscillations.

Chapter 6 introduces some of the basic concepts and methods associated with the numerical approximation of solutions to differential equations. These methods include Taylor series expansions, one-step (Runge-Kutta) methods, and multistep (predictor-corrector) methods. The techniques are mainly discussed relative to single first order equations; however, some consideration is given to approximation of systems of two first order equations. Linear differential equations of arbitrary order are discussed in Chapter 7. The main emphasis is on equations with constant coefficients, and it is shown that many of the properties and techniques for second order equations developed in Chapter 2 have natural extensions to higher-order equations.

Chapter 8 introduces the basic methods involving the concepts and techniques for matrix and vector algebra. The type of differential equation studied is a first order linear system with constant coefficients. A rather detailed development of the techniques needed from matrix theory is given, and the student should be able to grasp these ideas without any previous study in matrix algebra. Solution computations using eigenvalue-eigenvector techniques are stressed, but other matrix methods are also indicated.

In the one-semester course in introductory differential equations at North Carolina State, I usually cover the basic solution methods in Chapter 1 (Sections 1.2, 1.4, 1.6, and 1.7), curve sketching of solutions (Section 1.5), and some applications (e.g., radioactive decay, mixing problems, and population models). The first five sections in Chapter 2 on second order equations are then covered, with the emphasis on Sections 2.3 to 2.5. This chapter is concluded with an application (usually the analysis of vibrations in linear springs—Section 2.6) and an analysis of Euler's equation in Section 2.7.

After the first two chapters are covered, the rest of the course can be taken from any of the remaining chapters and in any order. In the previously mentioned course I usually go from Chapter 2 directly into Chapter 5, emphasizing solving linear equations (Section 5.3) and sketching solution curves for nonlinear equations (Section 5.4). In the remaining time (which varies considerably, depending on how much detail is covered in Chapter 5)

I try to cover as much as possible from either Laplace transform methods or power series methods.

Of course, once Chapters 1 and 2 are covered there is great flexibility in the choice of additional material. For example, one could cover Laplace transform methods (Chapter 3), power series methods (Chapter 4), and, depending on the remaining time, systems of two linear equations (Sections 5.2 and 5.3) or higher-order linear equations (Chapter 7). Another possibility is to go directly from Chapter 2 to higher-order linear equations (Chapter 7) and conclude with a detailed development of matrix methods (Chapter 8). Most of the topics in this text can be covered in a two-semester course.

The chapters in this text are divided into sections, and some of the sections are further divided into subsections. An asterisk preceding the section or subsection number indicates that the material in this section is perhaps more difficult to understand than what might be expected of an average student at the level of this book. Therefore, some care should be taken in covering those topics. Each section concludes with a set of problems that pertains to the topics of that section. Generally speaking, the first few problems are more basic and the latter ones more difficult. Problems requiring a good deal more than basic procedures are indicated by an asterisk. Answers to selected problems are included at the end of the book, and almost all of the computational exercises have answers in this section.

No book emerges fully formed from an author's forehead. I would like to acknowledge the inspiration and encouragement I received from my colleagues at North Carolina State and the help of my students, who class-tested early versions of the book. I am especially grateful to Professor James Selgrade who carefully read the entire manuscript and provided answers to most of the problems. Special thanks also are due to Margaret Memory and Dale Boger for checking the examples and answers for accuracy in several of the chapters.

Additionally, I would like to thank Vasilios Alexiades, University of Tennessee; Prem N. Bajaj, Witchita State University; Peter W. Bates, Texas A & M University; John Bradley, University of Tennessee; Hsim Chu, University of Maryland; Maurice Eggen, Trinity University; Donald L. Goldsmith, Western Michigan University; Herman Gollwitzer, Drexel University; Ronald B. Guenther, Oregon State University; Terry Herdman, Virginia Polytechnic Institute; Allan M. Krall, Pennsylvania State University; Kenneth R. Meyer, University of Cincinnati; James A. Morrow, University of Washington; David A. Sánchez, University of New Mexico; Thomas J. Smith, Manhattan College; Ralph E. Showalter, University of Texas at Austin; and Jacob Towber, DePaul University; who read all or parts of the draft manuscript. Their comments and occasional criticisms were always to the point and contributed to what I hope is an excellent introduction to the study of ordinary differential equations. Any remaining errors of omission or commission are solely my responsibility. I would be pleased if readers would write to me, to deliver either compliments or criticisms. I would also like to thank Sharon Jones for her help in turning rough drafts into beautifully prepared final manuscript. Thanks also to John Corrigan and James Amar of McGraw-Hill for their help throughout the project.

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Robert H. Martin, Jr.

Contents

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1 First Order Equations

The purpose of this chapter is to develop elementary methods for determining solutions to simple first order ordinary differential equations and to indicate some basic applications of such equations. The principal methods are integration of linear equations in Section 1.2, separation of variables in Section 1.4, change of variables in Section 1.6, and exact equations in Section 1.7. In Section 1.5 elementary techniques for sketching the graphs of solutions for autonomous equations are developed and these ideas are used in several applications and models (see, for example, the population models analyzed in Section 1.8). The applications considered in this chapter come not only from physics and engineering but also from the social sciences, chemistry, and economics.

1.1 INTRODUCTORY CONCEPTS AND EXAMPLES

A *differential equation* is an equation that involves an unknown function and its derivatives. An *ordinary differential equation* is a differential equation whose unknown is a function of a single independent variable. In this chapter we consider only ordinary differential equations that are *real* and *first order*: the unknown is a real-valued function of a single real variable, and the only derivative appearing in the equation is the first. When an ordinary differential equation arises as a model or description of a scientific phenomenon, the independent variable is often time. Therefore the independent variable in this text is usually denoted by t. Also, if y is a real-valued function of the real variable t, then y' or dy/dt denotes the first derivative of y. The second derivative of y is denoted by y'' or d^2y/dt^2 and the third derivative by y''' or d^3y/dt^3. In general, for each positive integer n, $y^{(n)}$ or d^ny/dt^n is used to denote the nth derivative of the function y. A differential equation is said to be of *order n* if the nth derivative $y^{(n)}$ appears in the equation and no derivatives of y larger than n appear in the equation. For example, the equation

$$y'' + 2y^4 - e^{-y'} \sin t = 5$$

is a differential equation of order 2 since y'' occurs in the equation and no higher derivatives of y appear. As a further example,

$$\cos ty + e^t \frac{dy}{dt} = \frac{t}{1 + y^2}$$

is a first order differential equation.

In the general case, a *real first order ordinary differential equation* has the form

$$\Phi(t, y, y') = 0$$

where the function Φ defines the relationship of the derivative $y' = dy/dt$ with the dependent variable y and the independent variable t. In this chapter, however, only those equations where the derivative y' can be explicitly written in terms of t and y are considered. Therefore, it is assumed that f is a continuous function of two variables and the differential equation

(1) $$y' = f(t, y)$$

is considered. A differentiable function y on an interval I is said to be a solution to (1) on I if $y'(t) = f(t, y(t))$ for each t in I. As a simple example, a solution to the differential equation

$$y' = 3y$$

is the function $y(t) = e^{3t}$ for all t. For if $y(t) = e^{3t}$, then

$$y'(t) = 3e^{3t} = 3y(t)$$

and $y = e^{3t}$ is a solution to $y' = 3y$ by definition. In fact, the student should verify that $y(t) = ce^{3t}$ is a solution to $y' = 3y$ for any constant c. One of the fundamental problems associated with equation (1) is developing methods for special types of functions f that lead to the determination of the solutions to (1) on a given interval I. In general, there are an infinite number of solutions to (1) on any given interval. For example, if $f(t, y) \equiv 0$, then $y(t) \equiv c$ on I is a solution to (1) for any constant c and any interval I.

▶ **EXAMPLE 1.1-1**

Consider the equation $y' = y - t$ and let I be any interval. For each real constant c the function $y = ce^t + t + 1$ for t in I is a solution to this equation, since

$$y'(t) = ce^t + 1 = (ce^t + t + 1) - t = y(t) - t$$

for all t in I. At this time the reader is not expected to produce a solution to equation (1). However, one should be able to determine if a *given* function is a solution to (1) (see Problem 1.1-1).

In actually trying to explicitly determine a solution to equation (1), the simplest case is when the function f does not depend on y. Therefore, assume that g is a continuous real-valued function on an interval I and consider the equation

(2) $$y' = g(t)$$

In order to solve (2) on I, one needs to determine all differentiable functions y on I such that $y'(t) = g(t)$ for all $t \in I$: that is, the solutions to (2) on I are precisely the antiderivatives of g. The solutions to the equation $y' = t^2$ are all functions y of the form $y = c + t^3/3$, where c is any constant. In general, the solutions to equation (2) on I are precisely the functions y on I having the form

(3) $$y(t) = c + \int_{t_0}^{t} g(s)\,ds \qquad \text{for all } t \text{ in } I$$

where c is any constant.

Therefore, if y is a solution to (2) on I, then y has the form indicated in (3) for some constant c [and, in fact, $c = y(t_0)$]; and conversely, if y has the form in (3), then y is a solution to (2). It should be noted that if G is *any* antiderivative of g on I [that is, $G'(t) = g(t)$ for all t in I] then the family

(3′) $$y(t) = \bar{c} + G(t) \qquad \text{for } t \text{ in } I$$

where $\bar{c}$ is any constant, describes precisely the same family of functions as (3). The family (3) of all solutions to equation (2) on I is called the *general solution to* (2) *on* I.

▶ **EXAMPLE 1.1-2**

Consider the equation $y' = 4 - 3t^2$ on $(-\infty, \infty)$. Since $4t - t^3$ is an antiderivative of $4 - 3t^2$ on $(-\infty, \infty)$, the family of functions

$$y(t) = c + 4t - t^3 \qquad \text{for } t \text{ in } (-\infty, \infty),\ c \text{ a constant}$$

is the general solution to this equation on $(-\infty, \infty)$.

The solution set to equation (2) indicates that there may be many different solutions to a first order ordinary differential equation. Usually one is interested in determining a function y on I that is not only a solution to (1) but also satisfies some additional property or condition. [For example, does (1) have a solution y on $(-\infty, \infty)$ that is periodic in t: $y(t + T) = y(t)$ for some $T > 0$ and all t in $(-\infty, \infty)$?] The most important case is determining a solution to (1) that assumes a specified value y_0 at some given time t_0 in I. Such a problem is called an *initial value problem* and is denoted in the following manner:

(4) $$y' = f(t, y) \qquad y(t_0) = y_0$$

where (t_0, y_0) is some given pair in the domain of f. If I is an interval and t_0 is in I, then a solution y to (4) on I is defined to be a solution to (1) on I that also has the value y_0 at time t_0: $y'(t) = f(t, y(t))$ for t in I and $y(t_0) = y_0$.

EXAMPLE 1.1-3

Consider the initial value problem $y' = y - t$, $y(0) = 4$. According to Example 1.1-1, the function $y(t) = ce^t + t + 1$ is a solution to the corresponding differential equation for each constant c. Selecting c so that $y(0) = c + 1 = 4$ it follows that $y(t) = 3e^t + t + 1$ is a solution to the given initial value problem.

EXAMPLE 1.1-4

Suppose that the position of an object on the x axis is denoted by $x(t)$ for all $t \geq 0$, and that the instantaneous velocity of this object is $\sin t$ for all $t \geq 0$. If initially (at time $t = 0$) the object is 2 units to the right of the origin, determine the position $x(t)$ for all times $t \geq 0$. Since the instantaneous velocity is $x'(t)$, the function x should be a solution to the initial value problem

$$x' = \sin t \qquad x(0) = 2$$

By (3′) x belongs to the family of functions $c - \cos t$ on $[0, \infty)$ where c is a constant. Therefore, since $c - \cos 0 = 2$ implies that $c = 3$, the position $x(t)$ of this object is given by $x(t) = 3 - \cos t$ for all $t \geq 0$.

It is sometimes convenient in studying the behavior of solutions to look at equation (1) from a geometric point of view. At each point (t, y) in the plane the value $f(t, y)$ is the slope of a solution at this point. Therefore, in order to estimate the graphs of solutions to (1), it is helpful to select "appropriate" points in the ty plane and indicate the slope of the solutions at these points by drawing a short line with slope $f(t, y)$. The graph of these slopes is called the *direction field* for the equation (1).

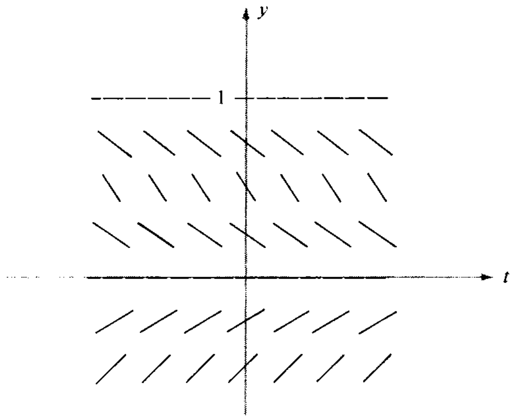

Figure 1.1 Sketch of direction field.

▶ **EXAMPLE 1.1-5**

Consider the equation $y' = 5y(y-1)$ where $I = (-\infty, \infty)$. Since $f(t, y)$ in this case is independent of t, for any given y_0 the slopes of the solutions at (t, y_0) are the same for all t. Noting that the right-hand side of this equation is zero when y is 0 or 1, positive when y is in $(-\infty, 0) \cup (1, \infty)$, and negative when y is on $(0, 1)$, one can readily verify that the sketch of the slopes given in Figure 1.1 gives a reasonable indication of the direction field. For example, if $y = \frac{1}{2}$ then $y' = 5(\frac{1}{2})(-\frac{1}{2}) = -\frac{5}{4}$, so the solutions have slope $-\frac{5}{4}$ when they cross the line $y = \frac{1}{2}$. Also, if $y = \frac{1}{4}$ then $y' = -\frac{15}{16}$ and if $y = -\frac{1}{4}$ then $y' = \frac{25}{16}$ (these values are indicated in Figure 1.1).

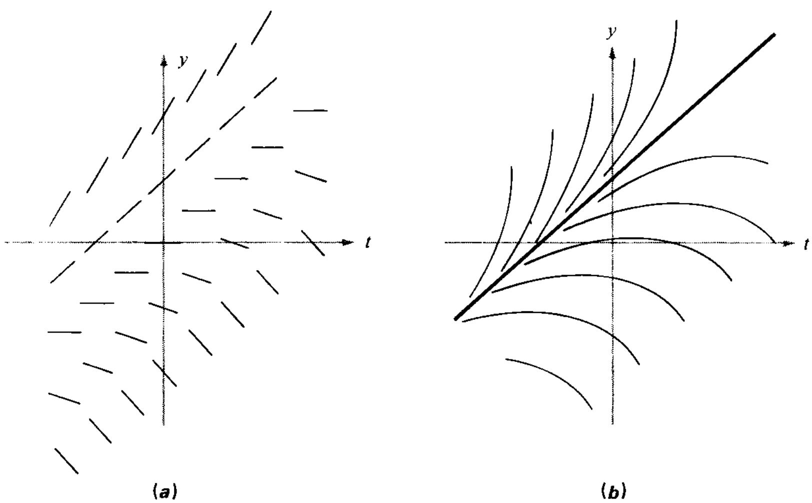

Figure 1.2 *a* Sketch of direction field. *b* Sketch of solution curves.

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