MECHANICS

by

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To my Father

PREFACE

This text is intended as the basis for an intermediate course in mechanics at the junior or senior level. Such a course, as essential preparation for advanced work in physics, has several major objectives. It must develop in the student a thorough understanding of the fundamental principles of mechanics. It should treat in detail certain specific problems of primary importance in physics; for example, the harmonic oscillator, and the motion of a particle under a central force. The problems suggested and those worked out in the text have been chosen with regard to their interest and importance in physics, as well as to their instructive value. This book contains sufficient material for a two-semester course, and is arranged in such a way that, with appropriate omissions, it can be used for a single three- or four-hour course for one semester. The author has used this material, with the omission of Chapters 8 and 9, and of a number of sections from earlier chapters, in a four-hour course in mechanics.

The choice of topics and their treatment throughout the book are intended to emphasize the modern point of view. Applications to atomic physics are made wherever possible, with an indication as to the extent of the validity of the results of classical mechanics. The inadequacies in classical mechanics are carefully pointed out, and the points of departure for quantum mechanics and for relativistic mechanics are indicated. The development, except for the last chapter, proceeds directly from Newton's laws of motion, which form a suitable basis from which to attack most mechanical problems. Some problems which are most easily treated by more advanced methods have been omitted; for example, the motion of a rigid body in space, which is most elegantly treated with the use of tensor algebra.

An important objective of a first course in mechanics is to train the student to think about physical phenomena in mathematical terms. Most students have a fairly good intuitive feeling for mechanical phenomena in a qualitative way. The study of mechanics should aim at developing an almost equally intuitive feeling for the precise mathematical formulation of physical problems, and for the physical interpretation of the mathematical solutions. The examples treated in the text have been worked out so as to integrate, as far as possible, the mathematical treatment with the physical interpretation. After working an assigned problem, the student should study it until he is sure he understands the physical interpretation of every feature of the mathematical treatment. He should decide whether the result agrees with his physical intuition about the problem. If the answer is fairly complicated, he should try to see whether it can be simplified in certain special or limiting cases. He should try to formulate and solve similar problems on his own.

Only a knowledge of differential and integral calculus has been presupposed. Mathematical concepts beyond those treated in the first year of calculus are introduced and explained as needed. A previous course in elementary differential equations or vector analysis may be helpful, but it is the author's experience that students with an adequate preparation in algebra and calculus are able to handle the elementary vector analysis and differential equations needed for this course with the explanations provided herein. A physics student is likely to get more out of his advanced courses in mathematics if he has previously encountered these concepts in physics.

The text has been written so as to afford maximum flexibility in the selection and arrangement of topics to be covered. With the exception of Chapter 1, the first five sections of Chapter 2, Sections 1, 3, 4, 5, 7, 8, 9, and 12 of Chapter 3, and Sections 1 through 3 of Chapter 4, almost any section or group of sections can be postponed or omitted without prejudice to the understanding of the remaining material.

In the first chapter, the basic concepts of mechanics are reviewed, and the laws of mechanics and of gravitation are formulated and applied to a few simple examples. The second chapter undertakes a fairly thorough study of the problem of one-dimensional motion. The chapter concludes with a study of the harmonic oscillator, as probably the most important example of one-dimensional motion. Use is made of complex numbers to represent oscillating quantities. The last section, on the principle of superposition, makes some use of Fourier series, and may be omitted or, better, passed over with a brief indication of the significance of the principle of superposition and the way in which Fourier series are used to treat the problem of an arbitrary applied force function.

Chapter 3 begins with a development of vector algebra and its use in describing motions in a plane or in space. Bold-face letters are used for vectors. Section 3–6 is a brief introduction to vector analysis, which is used very little in this book except in Chapter 8, and it may be omitted or skimmed rapidly if Chapter 8 and one or two proofs in Chapters 3 and 6 are omitted. The author feels there is some advantage in introducing the student to the concepts and notation of vector analysis at this stage, where the level of treatment is fairly easy; in later courses where the physical concepts and mathematical treatment become more difficult, it will be well if the notations are already familiar. The theorems stating the time rates of change of momentum, energy, and angular momentum are derived for a moving particle, and several problems are discussed, of which motion under central forces receives major attention. Examples are taken from astronomical and from atomic problems.

In Chapter 4, the conservation laws of energy, momentum, and angular momentum are derived, with emphasis on their position as cornerstones of present-day physics. They are then applied to typical problems, particularly collision problems. The two-body problem is solved, and the motion of two coupled harmonic oscillators is worked out. The general theory of coupled oscillations is best treated by means of linear transformations in vector spaces, but the behavior of coupled oscillating systems is too important to be omitted altogether from an intermediate course. The rigid body is discussed in Chapter 5 as a special kind of system of particles. Only rotation about a fixed axis is treated; the more general study of the motion of a rigid body is left to a later course, where more advanced methods are used. The section on statics treats the problem of the reduction of a system of forces to an equivalent simpler system. Elementary treatments of the equilibrium of beams, flexible strings, and of fluids are given in Sections 5–9, 5–10, and 5–11.

The theory of gravitation is studied in some detail in Chapter 6. The last section, on the gravitational field equations, may be omitted without disturbing the continuity of the remaining material. The laws of motion in moving coordinate systems are worked out in Chapter 7, and applied to motion on the rotating earth and to the motion of a system of charged particles in a magnetic field. Particular attention is paid to the status in Newtonian mechanics of the "fictitious forces" which appear when moving coordinate systems are introduced, and to the role to be played by such forces in the general theory of relativity.

In Chapter 8, an introductory treatment of vibrating strings and of the motion of fluids is presented, with emphasis on the fundamental concepts and mathematical methods which are used in treating the mechanics of continuous media. The last chapter, on Lagrange's equations, is included as an introduction to the methods of advanced dynamics. In a shorter course, either or both of the last two chapters may be omitted without destroying the unity of the course.

The problems at the end of each chapter are arranged in the order in which the material is covered in the chapter, for convenience in assignment. They vary considerably in difficulty. Some are fairly easy; a few are probably too difficult for the average college junior or senior to solve without some assistance.

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K. R. S.

February 1953

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CHAPTER 1

ELEMENTS OF NEWTONIAN MECHANICS

1–1 Mechanics, an exact science. When we say that physics is an exact science, we mean that its laws are expressed in the form of mathematical equations which describe and predict the results of precise quantitative measurements. The advantage in a quantitative physical theory is not alone the practical one that it gives us the power accurately to predict and to control natural phenomena. By a comparison of the results of accurate measurements with the numerical predictions of the theory, we can gain considerable confidence that the theory is correct, and we can determine in what respects it needs to be modified. It is often possible to explain a given phenomenon in several rough qualitative ways, and if we are content with that, it may be impossible to decide which theory is correct. But if a theory can be given which predicts correctly the results of measurements to four or five (or even two or three) significant figures, the theory can hardly be very far wrong. Rough agreement might be a coincidence, but close agreement is unlikely to be. Furthermore, there have been many cases in the history of science when small but significant discrepancies between theory and accurate measurements have led to the development of new and more far-reaching theories. Such slight discrepancies would not even have been detected if we had been content with a merely qualitative explanation of the phenomena.

The symbols which are to appear in the equations that express the laws of a science must represent quantities which can be expressed in numerical terms. Hence the concepts in terms of which an exact science is to be developed must be given precise numerical meanings. If a definition of a quantity (mass, for example) is to be given, the definition must be such as to specify precisely how the value of the quantity is to be determined in any given case. A qualitative remark about its meaning may be helpful, but is not sufficient as a definition. As a matter of fact, it is probably not possible to give an ideally precise definition of every concept appearing in a physical theory. Nevertheless, when we write down a mathematical equation, the presumption is that the symbols appearing in it have precise meanings, and we should strive to make our ideas as clear and precise as possible, and to recognize at what points there is a lack of precision or clarity. Sometimes a new concept can be defined in terms of others whose meanings are known, in which case there is no problem. For example,

$$\text{momentum} = \text{mass} \times \text{velocity}$$

gives a perfectly precise definition of "momentum" provided "mass" and "velocity" are assumed to be precisely defined already. But this kind of definition will not do for all terms in a theory, since we must start somewhere with a set of basic concepts or "primitive" terms whose meanings are assumed known. The first concepts to be introduced in a theory cannot be defined in the above way, since at first we have nothing to put on the right side of the equation. The meanings of these primitive terms must be made clear by some means that lies outside of the physical theories being set up. We might, for example, simply use the terms over and over until their meanings become clear. This is the way babies learn a language and probably, to some extent, freshman physics students learn the same way. We might define all primitive terms by stating their meaning in terms of observation and experiment. In particular, nouns designating measurable quantities, like force, mass, etc., may be defined by specifying the operational process for measuring them. One school of thought holds that all physical terms should be defined in this way. Or we might simply state what the primitive terms are, with a rough indication of their physical meaning, and then let the meaning be determined more precisely by the laws and postulates we lay down and the rules that we give for interpreting theoretical results in terms of experimental situations. This is the most convenient and flexible way, and is the way physical theories are usually set up. It has the disadvantage that we are never sure that our concepts have been given a precise meaning. It is left to experience to decide not only whether our laws are correct, but even whether the concepts we use have a precise meaning. The modern theories of relativity and quanta arise as much from fuzziness in classical concepts as from inaccuracies in classical laws.

Historically, mechanics is the earliest branch of physics to be developed as an exact science. The laws of levers and of fluids in static equilibrium were known to Greek scientists in the third century B.C. The tremendous development of physics in the last three centuries began with the discovery of the laws of mechanics by Galileo and Newton. The laws of mechanics as formulated by Isaac Newton in the middle of the seventeenth century and the laws of electricity and magnetism as formulated by James Clerk Maxwell about two hundred years later are the two basic theories of classical physics. Relativistic physics, which began with the work of Einstein in 1905, and quantum physics, as based upon the work of Heisenberg and Schroedinger in 1925–1926, require a modification and reformulation of mechanics and electrodynamics in terms of new physical concepts. Nevertheless, modern physics builds on the foundations laid by classical physics, and a clear understanding of the principles of classical mechanics and electrodynamics is still essential in the study of relativistic and quantum physics. Furthermore, in the vast majority of practical applications of mechanics to the various branches of engineering and to astronomy, the laws of classical mechanics can still be applied. Except when bodies travel at speeds approaching the speed of light, or when enormous masses or enormous distances are involved, relativistic mechanics gives the same results as classical mechanics; indeed, it must, since we know from experience that classical mechanics gives correct results in ordinary applications. Similarly, quantum mechanics should and does agree with classical mechanics except when applied to physical systems of molecular size or smaller. Indeed, one of the chief guiding principles in formulating new physical theories is the requirement that they must agree with the older theories when applied to those phenomena where the older theories are known to be correct.

Mechanics is the study of the motions of material bodies. Mechanics may be divided into three subdisciplines, *kinematics*, *dynamics*, and *statics*. Kinematics is the study and description of the possible motions of material bodies. Dynamics is the study of the laws which determine, among all possible motions, which motion will actually take place in any given case. In dynamics we introduce the concept of force. The central problem of dynamics is to determine for any physical system the motions which will take place under the action of given forces. Statics is the study of forces and systems of forces, with particular reference to systems of forces which act on bodies at rest.

We may also subdivide the study of mechanics according to the kind of physical system to be studied. This is, in general, the basis for the outline of the present book. The simplest physical system, and the one we shall study first, is a single particle. Next we shall study the motion of a system of particles. A rigid body may be treated as a special kind of system of particles. Finally, we shall study the motions of continuous media, elastic and plastic substances, solids, liquids, and gases.

A great many of the applications of classical mechanics may be based directly on Newton's laws of motion. All of the problems studied in this book, except in the last chapter, are treated in this way. There are, however, a number of other ways of formulating the principles of classical mechanics. The equations of Lagrange and of Hamilton are examples. They are not new physical theories, for they may be derived from Newton's laws, but they are different ways of expressing the same physical theory. They use more advanced mathematical concepts, they are in some respects more elegant than Newton's formulation, and they are in some cases more powerful in that they allow the solutions of some problems whose solution based directly on Newton's laws would be very difficult. The more different ways we know to formulate a physical theory, the better chance we have of learning how to modify it to fit new kinds of phenomena as they are discovered. This is one of the main reasons for the importance of the more advanced formulations of mechanics. They are a starting point for the newer theories of relativity and quanta.

1–2 Kinematics, the description of motion. Mechanics is the science which studies the motions of physical bodies. We must first describe motions. Easiest to describe are the motions of a particle, which is an object whose size and internal structure are negligible for the problem with which we are concerned. The earth, for example, could be regarded as a particle for most problems in planetary motion, but certainly not for terrestrial problems. We can describe the position of a particle by specifying a point in space. This may be done by giving three coordinates. Usually, rectangular coordinates are used. For a particle moving along a straight line (Chapter 2) only one coordinate need be given. To describe the motion of a particle, we specify the coordinates as functions of time:

$$\begin{aligned}\text{one dimension:}&\quad x(t),\\ \text{three dimensions:}&\quad x(t),\ y(t),\ z(t).\end{aligned} \tag{1–1}$$

The basic problem of classical mechanics is to find ways to determine functions like these which specify the positions of objects as functions of time, for any mechanical situation. The physical meaning of the function $x(t)$ is contained in the rules which tell us how to measure the coordinate x of a particle at a time t. Assuming we know the meaning of $x(t)$, or at least that it has a meaning (this assumption, which we make in classical mechanics, is not quite correct according to quantum mechanics), we can define the x-component of velocity v_x at time t as*

$$v_x = \dot{x} = \frac{dx}{dt}, \tag{1–2}$$

and, similarly,

$$v_y = \dot{y} = \frac{dy}{dt}, \quad v_z = \dot{z} = \frac{dz}{dt}.$$

We now define the components of acceleration a_x, a_y, a_z as the derivatives of the velocity components with respect to time (we list several equivalent notations which may be used):

$$\begin{aligned}a_x &= \dot{v}_x = \frac{dv_x}{dt} = \ddot{x} = \frac{d^2x}{dt^2},\\ a_y &= \dot{v}_y = \frac{dv_y}{dt} = \ddot{y} = \frac{d^2y}{dt^2},\\ a_z &= \dot{v}_z = \frac{dv_z}{dt} = \ddot{z} = \frac{d^2z}{dt^2}.\end{aligned} \tag{1–3}$$

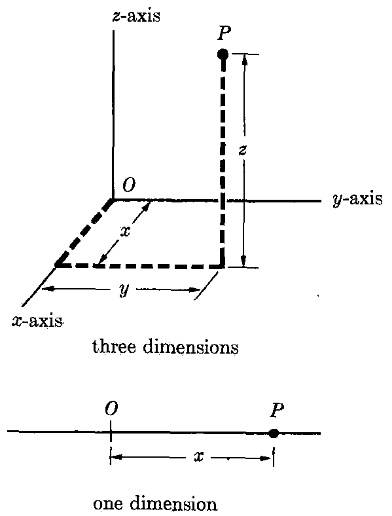

FIG. 1–1. Rectangular coordinates specifying the position of a particle P relative to an origin O.

* We shall denote a time derivative either by d/dt or by a dot. Both notations are given in Eqs. (1–2).

For many purposes some other system of coordinates may be more convenient for specifying the position of a particle. When other coordinate systems are used, appropriate formulas for components of velocity and acceleration must be worked out. Spherical, cylindrical, and plane polar coordinates will be discussed in Chapter 3. For problems in two and three dimensions, the concept of a vector is very useful as a means of representing positions, velocities, and accelerations. A systematic development of vector algebra will be given in Section 3–1.

To describe a system of particles, we may specify the coordinates of each particle in any convenient coordinate system. Or we may introduce other kinds of coordinates, for example, the coordinates of the center of mass, or the distance between two particles. If the particles form a rigid body, the three coordinates of its center of mass and three angular coordinates specifying its orientation in space are sufficient to specify its position. To describe the motion of continuous matter, for example a fluid, we would need to specify the density $\rho(x,y,z,t)$ at any point (x,y,z) in space at each instant t in time, and the velocity vector $v(x,y,z,t)$ with which the matter at the point (x,y,z) is moving at time t. Appropriate devices for describing the motion of physical systems will be introduced as needed.

1–3 Dynamics. Mass and force. Experience leads us to believe that the motions of physical bodies are controlled by interactions between them and their surroundings. Observations of the behavior of projectiles and of objects sliding across smooth, well-lubricated surfaces suggest the idea that changes in the velocity of a body are produced by interaction with its surroundings. A body isolated from all interactions would have a constant velocity. Hence, in formulating the laws of dynamics, we focus our attention on accelerations.

Let us imagine two bodies interacting with each other and otherwise isolated from interaction with their surroundings. As a rough approximation to this situation, imagine two boys, not necessarily of equal size, engaged in a tug of war over a rigid pole on smooth ice. Although no two actual bodies can ever be isolated completely from interactions with all other bodies, this is the simplest kind of situation to think about and one for which we expect the simplest mathematical laws. Careful experiments with actual bodies lead us to conclusions as to what we should observe if we could achieve ideal isolation of two bodies. We should observe that the two bodies are always accelerated in opposite directions, and that the ratio of their accelerations is constant for any particular pair of bodies no matter how strongly they may be pushing or pulling each other. If we measure the coordinates x_1 and x_2 of the two bodies along the line of their accelerations, then

$$\ddot{x}_1/\ddot{x}_2 = -k_{12}, \tag{1–4}$$

where k_{12} is a positive constant characteristic of the two bodies concerned. The negative sign expresses the fact that the accelerations are in opposite directions.

Furthermore, we find that in general the larger or heavier or more massive body is accelerated the least. We find, in fact, that the ratio k_{12} is proportional to the ratio of the weight of body 2 to that of body 1. The accelerations of two interacting bodies are inversely proportional to their weights. This suggests the possibility of a dynamical definition of what we shall call the *masses* of bodies in terms of their mutual accelerations. We choose a standard body as a unit mass. The mass of any other body is defined as the ratio of the acceleration of the unit mass to the acceleration of the other body when the two are in interaction:

$$m_i = k_{1i} = -\ddot{x}_1/\ddot{x}_i, \tag{1-5}$$

where m_i is the mass of body i, and body 1 is the standard unit mass.

In order that Eq. (1–5) may be a useful definition, the ratio k_{12} of the mutual accelerations of two bodies must satisfy certain requirements. If the mass defined by Eq. (1–5) is to be a measure of what we vaguely call the amount of matter in a body, then the mass of a body should be the sum of the masses of its parts, and this turns out to be the case to a very high degree of precision. It is not essential, in order to be useful in scientific theories, that physical concepts for which we give precise definitions should correspond closely to any previously held common-sense ideas. However, most precise physical concepts have originated from more or less vague common-sense ideas, and mass is a good example. Later, in the theory of relativity, the concept of mass is somewhat modified, and it is no longer exactly true that the mass of a body is the sum of the masses of its parts.

One requirement which is certainly essential is that the concept of mass be independent of the particular body which happens to be chosen as having unit mass, in the sense that the ratio of two masses will be the same no matter what unit of mass may be chosen. This will be true because of the following relation, which is found experimentally, between the mutual acceleration ratios defined by Eq. (1–4) of any three bodies:

$$k_{12}k_{23}k_{31} = 1. \tag{1-6}$$

Suppose that body 1 is the unit mass. Then if bodies 2 and 3 interact with each other, we find, using Eqs. (1–4), (1–6), and (1–5),

$$\begin{aligned} \ddot{x}_2/\ddot{x}_3 &= -k_{23} \\ &= -\frac{1}{k_{12}k_{31}} \\ &= -k_{13}/k_{12} \\ &= -m_3/m_2. \end{aligned} \tag{1-7}$$