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Multiparameter Eigenvalue Problems and Expansion Theorems

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PREFACE

Since the publication of Atkinson's book "Multiparameter Eigenvalue Problems, Vol. I" in 1972, multiparameter spectral theory has become a subject of growing interest. Several authors have made important contributions to the theory; see the references at the end of this book. There are also two monographs of Sleeman (1978a) and McGhee and Picard (1988) on this subject.

The contents of the present book can be called "classical multiparameter theory" to distinguish it from more recent developments in multiparameter theory. The book contains the major results of Atkinson's book (most of them are proved differently), several of the contributions made in the last years and some new results. It is thought as a supplement to the excellent book of Atkinson which has the only drawback that we are waiting for volume II for quite a while.

We study two problems: the existence of eigenvalues and the expansion in series of eigenvectors. Each of these problems is treated for multiparameter eigenvalue problems involving (i) Hermitian matrices (ii) compact Hermitian operators, in particular, integral operators (iii) semibounded selfadjoint operators with compact resolvent, in particular, differential operators. Alltogether this leads to six chapters.

It is not assumed that the reader knows already some multiparameter spectral theory but it is supposed that the reader is familiar with the usual one-parameter eigenvalue problems for compact Hermitian operators and their inverses. Brouwer's degree of maps will be used in the first two chapters. Basic properties of the tensor product of linear spaces will be needed in Chapters 4,5,6.

Theorems in multiparameter spectral theory are usually proved under a so-called definiteness condition. It was my aim to prove these theorems under definiteness conditions as weak as possible. For instance, a bounded Hermitian operator A on a Hilbert space H with inner product $\langle \, , \, \rangle$ is positive definite if $\langle Au,u\rangle$ is positive for all unit vectors u. The operator A is strictly positive definite if there is a positive ε such that $\langle Au,u\rangle \geq \varepsilon$ for all unit vectors u. In such a situation I prefer to assume that A is positive definite even if the proofs would become simpler under strict definiteness. It is this weakening of definiteness conditions which leads to some new results. In particular, the abstract oscillation theorems of the Chapters 2 and 3 under local definiteness in the strong sense and the expansion theorems of the Chapters 5 and 6 under left definiteness are new. More detailed informations on the correspondence of the presented results with those of the literature are given in the Notes at the end of each chapter.

To avoid misunderstandings I have to say that the contents of this small book are far from being complete in any sense. There are many interesting fields in multiparameter theory which are not contained, for example, eigenvalue problems involving nonsymmetric operators (see Atkinson (1968) and Isaev (1980)), eigenvalue problems having a continuous spectrum (see Browne (1977a), (1977b), Volkmer (1982) and McGhee and Picard (1988)), nonlinear problems (see Binding (1980b) and Browne and Sleeman (1979b), (1980b), (1981)), indefinite problems (see Binding and Seddighi (1987a) and Faierman and Roach (1987), (1988a)) and .... Perhaps we should add "Vol. I" to the title of the book.

Finally, I wish to thank my colleagues working in multiparameter theory for their stimulation during the last years at various meetings. In particular, I thank Paul Binding and Patrick Browne for several fruitful discussions during my stay at the University of Calgary (March - July 1984). The results of the first three chapters are based on a joint paper with Binding (1986).

Essen, September 1988 Hans Volkmer

CONTENTS

INTRODUCTION

A multiparameter eigenvalue problem contains a finite number of spectral parameters whose number is denoted by k throughout the text. If k is equal to 1 then we obtain an ordinary one-parameter eigenvalue problem. Therefore the simplest "nontrivial" case is when k is equal to 2 . The following two-parameter examples illustrate some of the concepts and results of multiparameter spectral theory.

Let us consider the eigenvalue problem

$$\lambda_o A_{1o} x_1 + \lambda_1 A_{11} x_1 + \lambda_2 A_{12} x_1 = 0 \quad , \quad 0 \neq x_1 \in \mathbb{C}^3 \; , \tag{1}$$

$$\lambda_o A_{2o} x_2 + \lambda_1 A_{21} x_2 + \lambda_2 A_{22} x_2 = 0 \quad , \quad 0 \neq x_2 \in \mathbb{C}^2 \; , \tag{2}$$

where

$$A_{1o} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad A_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad A_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} ,$$

$$A_{2o} = \begin{pmatrix} 20 & 0 \\ 0 & 0 \end{pmatrix} , \qquad A_{21} = \begin{pmatrix} 0 & \sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix} , \qquad A_{22} = \begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix} .$$

The eigenvalues are nonzero tuples $\lambda = (\lambda_o,\lambda_1,\lambda_2)$ of complex numbers such that (1) and (2) can be solved simultaneously. It should be observed that an "eigenvalue" is a vector and not a scalar. The notion "eigenvector" is reserved for the tensor products of the solutions x_1 and x_2 of (1) and (2), respectively. There are three spectral parameters $\lambda_o,\lambda_1,\lambda_2$ but they only count for two because (1),(2) is written in a homogeneous formulation i.e. if λ is an eigenvalue then also $\alpha\lambda$ is one for each nonzero complex number α . Our example is so simple that we can calculate the eigenvalues explicitely. Of course, this will be possible only in a very limited number of examples. We first determine the values of $\lambda_o,\lambda_1,\lambda_2$ such that the matrix

$$\lambda_o A_{1o} + \lambda_1 A_{11} + \lambda_2 A_{12} \tag{3}$$

becomes singular. These values are given by

$$(4\lambda_o + 2\lambda_1)\, \lambda_2^2 = 6(4\lambda_o + \lambda_1)\, \lambda_1^2 \ . \tag{4}$$

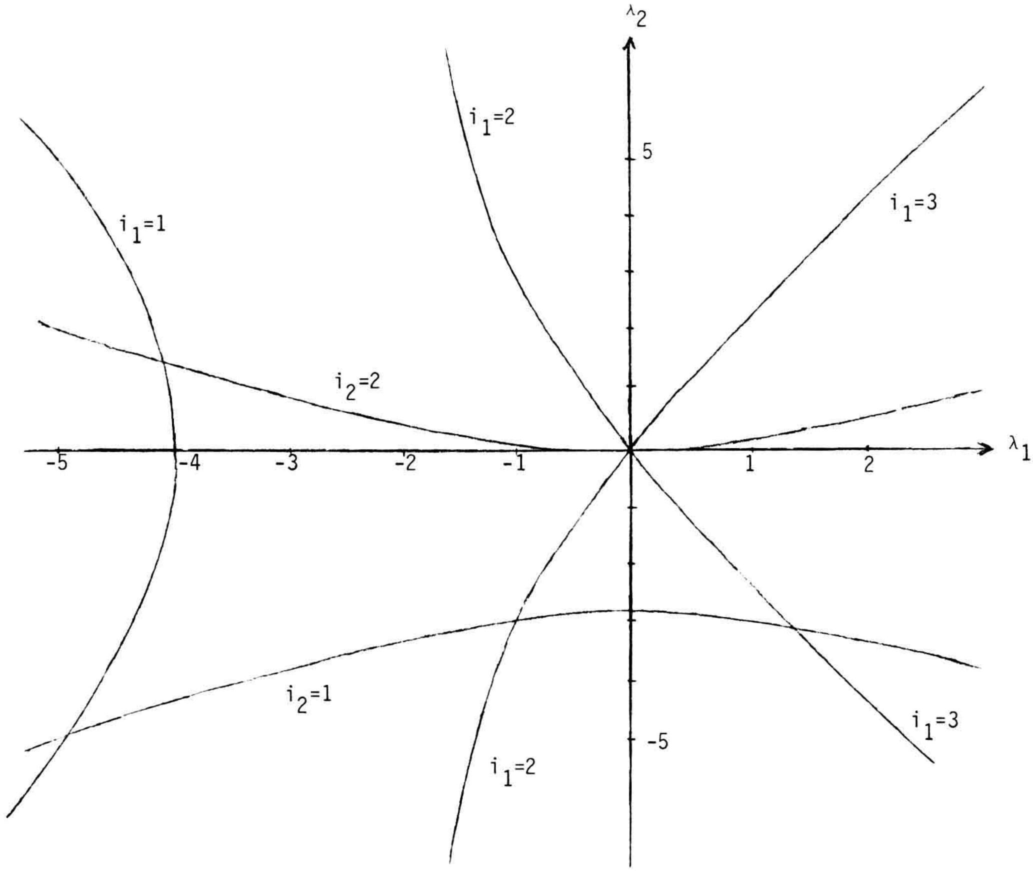
Similarly, the matrix

$$\lambda_o A_{2o} + \lambda_1 A_{21} + \lambda_2 A_{22} \tag{5}$$

is singular if

$$3\lambda_1^2 = 20\, \lambda_o \lambda_2 + 7\lambda_2^2 \ . \tag{6}$$

The eigenvalues $\lambda = (\lambda_o, \lambda_1, \lambda_2)$ are the simultaneous solutions of (4) and (6). It is easily seen that λ_o is nonzero for eigenvalues $(\lambda_o, \lambda_1, \lambda_2)$. Hence we can go over from the homogeneous formulation of the eigenvalue problem (1),(2) to an inhomogeneous one by setting $\lambda_o = 1$. Now the eigenvalues $(1, \lambda_1, \lambda_2)$ have real components which can be determined by a picture.



The curves indicated by $i_1 = 1$, $i_1 = 2$, $i_1 = 3$ consist of those pairs (λ_1,λ_2) for which the i_1th greatest eigenvalue of (3), counted according to multiplicity, is equal to 0 . Similarly, the curves indicated by $i_2 = 1$, $i_2 = 2$ consist of the pairs (λ_1,λ_2) for which the i_2th greatest eigenvalue of (5) is 0 . The eigenvalues of problem (1),(2) correspond to the points of intersection of these eigencurves. For example, (-5,-5) lies on the curves $i_1 = 1$ and $i_2 = 1$. We say that the eigenvalue (1,-5,-5) has index $(i_1,i_2) = (1,1)$. The eigenvalue (1,0,0) has two indices, namely $(i_1,i_2) = (2,2)$ and $(i_1,i_2) = (3,2)$. We say that the eigenvalue (1,0,0) has multiplicity 2 .

We see that, for every given index (i_1,i_2) , $i_1 = 1,2,3$, $i_2 = 1,2$, there is exactly one eigenvalue $(1,\lambda_1,\lambda_2)$ which has index (i_1,i_2) . This is a special case of the statement of Theorem 1.4.1 because our eigenvalue problem (1),(2) is definite with respect to (1,0,0) i.e. we have

$$\det \begin{pmatrix} 1 & 0 & 0 \\ \langle A_{10}u_1,u_1\rangle_1 & \langle A_{11}u_1,u_1\rangle_1 & \langle A_{12}u_1,u_1\rangle_1 \\ \langle A_{20}u_2,u_2\rangle_2 & \langle A_{21}u_2,u_2\rangle_2 & \langle A_{22}u_2,u_2\rangle_2 \end{pmatrix} > 0 ,$$

for all unit vectors $u_1 \in H_1$ and $u_2 \in H_2$, where $\langle\, ,\, \rangle_1$ and $\langle\, ,\, \rangle_2$ are the usual inner products in $\mathbb{C}^3$ and $\mathbb{C}^2$, respectively.

There is also an expansion theorem for the eigenvalue problem (1),(2). For every eigenvalue $(1,\lambda_1,\lambda_2)$, we choose solutions x_1,x_2 of (1),(2), respectively. For the eigenvalue (1,0,0) of multiplicity 2 , we take two linear independent solutions x_1 of (1). Then we obtain 6 decomposable tensors $x_1 \otimes x_2$ which form a basis of the tensor product $\mathbb{C}^3 \otimes \mathbb{C}^2$. This is a special case of the results of Section 4.5.

Multiparameter eigenvalue problems arise in mathematical physics if the method of separation of variables can be used to solve boundary eigenvalue problems. Usually, we have a partial differential equation in k independent variables which contains one spectral parameter. If it is possible to apply the method of separation of variables to solve the equation then we obtain k ordinary differential equations linked by k-1 separation constants and the original spectral parameter. Together we have k parameters which are the spectral parameters of a multiparameter eigenvalue problem. In many cases the original equation must first be transformed in a suitable coordinate system before the separation process is possible. It is not part of multiparameter theory to find such coordinate systems. Concerning this problem we refer to Miller (1968) and Kalnins (1986). In multiparameter spectral theory one starts with a given system of k equations linked by k spectral parameters. Then, if necessary, the separation process is reversed to find the associated partial differential operators. The above remarks show that multiparameter spectral theory is closely related to the spectral theory of partial differential operators. There is also a relationship to the theory of special functions because these functions are the solutions of ordinary differential equations which arise from the separation of the wave equation.

As an example, let us consider the problem of the vibrating elliptic membrane with fixed boundary; see [Meixner and Schäfke (1954), Section 4.31]. Let the membrane be given by

$$\Omega := \{(\eta_1,\eta_2) \in \mathbb{R}^2 \mid (\eta_1/c_1)^2 + (\eta_2/c_2)^2 \leq 1\} ,$$

where it is assumed that $c_1 > c_2 > 0$. Then our boundary eigenvalue problem consists in finding values of ν such that the two-dimensional wave equation

$$\frac{\partial^2 y}{\partial \eta_1^2} + \frac{\partial^2 y}{\partial \eta_2^2} + \nu y = 0 \tag{7}$$

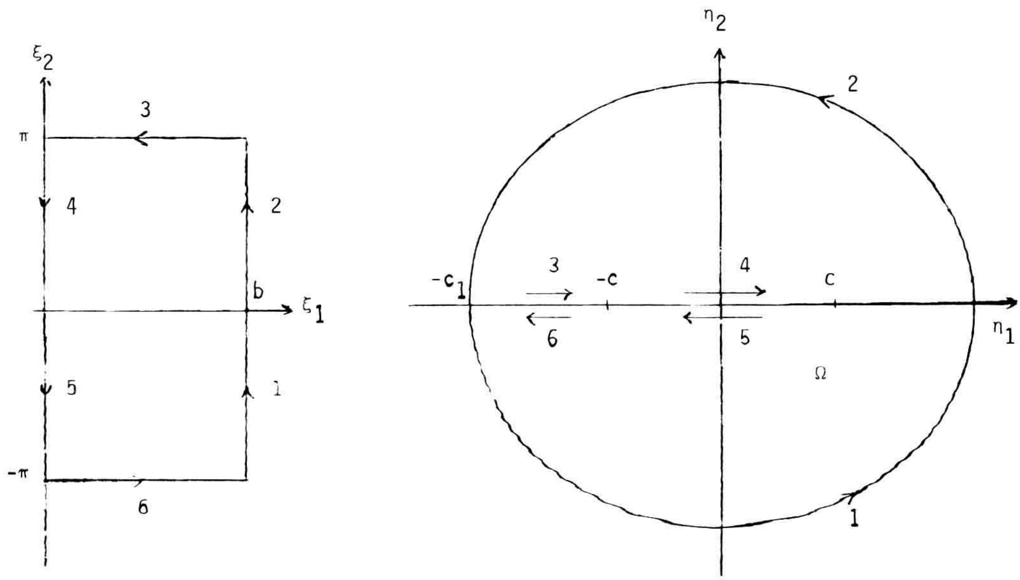
has a nontrivial solution y defined on Ω satisfying the boundary condition

$$y(\eta_1,\eta_2) = 0 \quad \text{for} \quad (\eta_1,\eta_2) \in \partial\Omega \; . \tag{8}$$

We introduce ellipsoidal coordinates ξ_1,ξ_2 defined by

$$\eta_1 = c \cosh \xi_1 \cos \xi_2 \; , \; \eta_2 = c \sinh \xi_1 \sin \xi_2 \quad ,$$

where $(\pm c,0)$, $c > 0$, denote the foci of the ellipse $\partial\Omega$. Let b be the positive number satisfying $c_1 = c \cosh b$, $c_2 = c \sinh b$. Then the map $(\xi_1,\xi_2) \mapsto (\eta_1,\eta_2)$ is one-to-one from the open rectangle $]0,b[\times]-\pi,\pi[$ onto the interior of Ω minus the cut from $(-c_1,0)$ to $(c,0)$. The behaviour on the boundary is shown by the following picture.



Setting

$$y(\eta_1,\eta_2) = x(\xi_1,\xi_2) \quad ,$$

the equation (7) transforms into

$$\frac{\partial^2 x}{\partial \xi_1^2} + \frac{\partial^2 x}{\partial \xi_2^2} + \frac{c^2 \nu}{2} (\cosh 2\xi_1 - \cos 2\xi_2)\, x = 0 \quad .$$

If we assume a solution of this equation of the form

$$x(\xi_1,\xi_2) = x_1(\xi_1)\, x_2(\xi_2) \quad ,$$

we obtain for x_1 and x_2 the ordinary differential equations

$$x_1'' + (2\lambda_1 \cosh 2\xi_1 - \lambda_2)\, x_1 = 0 \quad , \tag{9}$$

$$x_2'' - (2\lambda_1 \cos 2\xi_2 - \lambda_2)\, x_2 = 0 \quad , \tag{10}$$

where λ_2 is the separation constant and $4\lambda_1 = c^2\nu$. Equation (10) is Mathieu's differential equation and (9) is its modified form. The function x_2 has to be a periodic function with period 2π . The solutions of (10) with period 2π are usually divided into the following four sets.

$$\left.\begin{array}{lll} \text{I:} & x_2'(0) = x_2'(\pi/2) = 0 & \text{i. e. } x_2 \text{ is even and of period } \pi , \\ \text{II:} & x_2'(0) = x_2(\pi/2) = 0 & \text{i. e. } x_2 \text{ is even and of half-period } \pi , \\ \text{III:} & x_2(0) = x_2'(\pi/2) = 0 & \text{i. e. } x_2 \text{ is odd and of half-period } \pi , \\ \text{IV:} & x_2(0) = x_2(\pi/2) = 0 & \text{i. e. } x_2 \text{ is odd and of period } \pi ; \end{array}\right\} \quad (11)$$

see [Meixner and Schäfke (1954), pages 108, 149].

Further, the boundary condition (8) yields

$$x_1(b) = 0 . \qquad (12)$$

Since y is continuously differentiable in the neighborhood of the focal line $(\eta_1,0)$, $-c \le \eta_1 \le c$, the corresponding function x satisfies

$$x(0,\xi_2) = x(0,-\xi_2) , \quad \frac{\partial x}{\partial \xi_1}(0,\xi_2) = - \frac{\partial x}{\partial \xi_1}(0,-\xi_2) .$$

Hence we obtain the boundary conditions

$$\left.\begin{array}{lll} x_1'(0) = 0 & \text{in the cases} & \text{I , II ,} \\ x_1(0) = 0 & \text{in the cases} & \text{III , IV .} \end{array}\right\} \quad (13)$$

In each of the four cases I, II, III, IV we have thus obtained a two-parameter Sturm-Liouville eigenvalue problem (9) , (10), (11), (12), (13). For example, in case I our eigenvalue problem consists in finding values of λ_1 and λ_2 such that equation (9) has a nontrivial solution satisfying the boundary conditions $x_1'(0) = 0$, $x_1(b) = 0$ and, simultaneously, (10) has a nontrivial solution satisfying $x_2'(0) = 0$, $x_2'(\pi/2) = 0$. If (λ_1,λ_2) is an eigenvalue then $\nu = 4\lambda_1/c^2$ is an eigenvalue of the original eigenvalue problem (7), (8). Using expansion theorems for the two-parameter eigenvalue problems, we can also show that the converse statement is true: if ν is an eigenvalue of (7), (8) then there exists an eigenvalue (λ_1,λ_2) of one of the four two-parameter problems such that $\nu = 4\lambda_1/c^2$.

There holds the following result concerning the existence of eigenvalues in each of the four cases. For each pair n_1 , n_2 of nonnegative integers, there are uniquely determined real numbers λ_1 , λ_2 such that (9) admits a solution satisfying the boundary conditions (12), (13) and having exactly n_1 zeros in the open interval $]0,b[$ and such that (10) admits a solution satisfying the boundary conditions (11) and having exactly n_2 zeros in $]0,\pi/2[$. This result is a particular case of the Theorems 3.5.1, 3.5.2 because all four eigenvalue problems are right definite according to Theorem 3.6.2 (ii) :

$$d_o(\xi_1,\xi_2) := \det \begin{pmatrix} 2\cosh\xi_1 & -1 \\ -2\cos\xi_2 & 1 \end{pmatrix} > 0 \quad \text{on} \quad]0,b[\times]0,\pi/2[\quad .$$

In Section 6.8 we shall prove that every function in $L^2([0,b] \times [0,\pi/2])$ can be expanded in a series of products $x_1(\xi_1)x_2(\xi_2)$ of solutions of (9),(10) associated with the countable number of eigenvalues. The series converges with respect to the inner product

$$\varphi_o(x,y) := \int_0^b \int_0^{\pi/2} d_o(\xi_1,\xi_2)\, x(\xi_1,\xi_2)\, \overline{y(\xi_1,\xi_2)}\, d\xi_2\, d\xi_1 \quad .$$

We can also apply the results of Section 6.7 to obtain expansions which converge with respect to the norm of suitably chosen Sobolev spaces.

One of the most interesting two-parameter eigenvalue problems is that which leads to the definition of Lamé polynomials. We refer to Sections 3.8 and 6.10.

CHAPTER 1

MULTIPARAMETER EIGENVALUE PROBLEMS FOR HERMITIAN MATRICES

1.1 Introduction

We suppose given k linear spaces H_r, $r = 1,\dots,k$, all over the complex field, nonzero, and of finite dimension. In each H_r there is an inner product $< , >_r$ with associated unit sphere U_r . For each r , let A_{rs}, $s = 0,\dots,k$, be a set of $k+1$ Hermitian operators on H_r .

We shall study the multiparameter eigenvalue problem

$$\sum_{s=0}^{k} \lambda_s A_{rs} u_r = 0, \quad u_r \in U_r, \; r = 1,\dots,k \;. \tag{1.1.1}$$

We shall use the term *eigenvalue* to denote a nonzero (k+1)-tuple of scalars $\lambda = (\lambda_0,\dots,\lambda_k)$ such that there exist vectors $u_r \in U_r$, $r = 1,\dots,k$, satisfying the k equations (1.1.1).

In most cases the eigenvalue problem (1.1.1) will be treated under the hypothesis of local definiteness. We call (1.1.1) *locally definite* if the real k by $k+1$ matrix

$$W(u) := \begin{pmatrix} <A_{10}u_1,u_1>_1 & \cdots & <A_{1k}u_1,u_1>_1 \\ \vdots & & \vdots \\ <A_{k0}u_k,u_k>_k & \cdots & <A_{kk}u_k,u_k>_k \end{pmatrix} \tag{1.1.2}$$

is of maximal rank for all $u = (u_1,\dots,u_k) \in U := U_1 \times \dots \times U_k$, i. e.

$$\text{rank } W(u) = k \quad \text{for all} \quad u \in U \;. \tag{1.1.3}$$

In the next section we show that the eigenvalues of a locally definite problem (1.1.1) can be indexed in a natural manner, and in Theorem 1.2.3 we prove that the eigenvalues are uniquely determined by a signed index. In Section 1.3 we provide the basic properties of Brouwer's degree of maps which we need in order to prove Theorem 1.4.1 on the existence of eigenvalues of given signed index. Subsequently, we note two corollaries of the existence theorem, a perturbation theorem and a criterion for the problem (1.1.1) to be locally definite. In Section 1.5 we consider definite problems. In Section 1.6 we refer to the literature.

1.2 Indexed eigenvalues

We consider the eigenvalue problem (1.1.1). For given $\lambda = (\lambda_0,\ldots,\lambda_k) \in \mathbb{R}^{k+1}$ and $r = 1,\ldots,k$, we list the eigenvalues of the Hermitian operator

$$\sum_{s=0}^{k} \lambda_s A_{rs} \tag{1.2.1}$$

in decreasing order, according to multiplicity, as

$$\rho_r(\lambda,1) \geq \rho_r(\lambda,2) \geq \ldots \geq \rho_r(\lambda, \dim H_r) \quad . \tag{1.2.2}$$

Then a nonzero $\lambda \in \mathbb{R}^{k+1}$ is an eigenvalue of (1.1.1) if and only if there exists a multiindex

$$i = (i_1,\ldots,i_k) \quad , \quad i_r = 1,\ldots,\dim H_r \quad , \quad r = 1,\ldots,k \quad , \tag{1.2.3}$$

such that

$$\rho_r(\lambda,i_r) = 0 \quad \text{for every} \quad r = 1,\ldots,k \quad . \tag{1.2.4}$$

If the equations (1.2.4) are satisfied then we say that the eigenvalue λ has index i. In general, an eigenvalue can have several indices. The number of these indices is called the *multiplicity* of the eigenvalue.

The maximum-minimum-principle for the eigenvalues of the Hermitian operators (1.2.1) yields the useful representation

$$\rho_r(\lambda,i_r) = \max\{\min\{w_r(u_r)\lambda \mid u_r \in F_r \cap U_r\} \mid F_r \subset H_r, \dim F_r = i_r\} \quad , \tag{1.2.5}$$

where $w_r(u_r)$ denotes the r^{th} row of the matrix (1.1.2) and $w_r(u_r)\lambda$ is the usual product of a row and column vector. Here and in the sequel we consider λ as a column vector. For the maximum-minimum-principle we refer to [Weinstein and Stenger (1972),Chapter 2]. There it is called minimum-maximum-principle because the eigenvalues are listed in increasing order.

We now turn to consequences of local definiteness of the eigenvalue problem (1.1.1). Let us first express the rank condition (1.1.3) by determinants as usual. For $u \in U$ and $s = 0,\ldots,k$, we denote by $(-1)^s \, \delta_s(u)$ the determinant of the matrix $W(u)$ with s^{th} column deleted. Then the eigenvalue problem is locally definite if and only if the column vector

$$\delta(u) := (\delta_0(u),\ldots,\delta_k(u)) \text{ is nonzero for all } u \in U \quad . \tag{1.2.6}$$

We also know from the elementary theory of matrices that local definiteness is equivalent to the condition that

$$\text{Ker } W(u) = \{\alpha\ \delta(u) \mid \alpha \text{ complex}\} \quad \text{for all} \quad u \in U \quad , \tag{1.2.7}$$

where Ker $W(u)$ denotes the kernel of the matrix $W(u)$. In the rest of this section we shall assume that (1.1.1) is locally definite.

If λ is an eigenvalue then the equations (1.1.1) hold for some $u = (u_1,\ldots,u_k)$. It follows immediately that $W(u)\lambda = 0$. Hence, by (1.2.7), λ is a complex multiple of $\delta(u)$, a vector which has real components. Therefore, and since the eigenvalue problem (1.1.1) is homogeneous in λ , it will be sufficient to search eigenvalues in the unit sphere S^k of $\mathbb{R}^{k+1}$.

We now define three subsets of S^k which will play the crucial role in this chapter:

$$P \ := \{\lambda \in S^k \mid W(u)\lambda = 0 \ \text{ for some } \ u \in U\} \quad ,$$

$$P^+ := \{\delta(u) \ / \ \|\delta(u)\| \ \mid u \in U\} \quad ,$$

$$P^- := \{-\delta(u) \ / \ \|\delta(u)\| \ \mid u \in U\} = -P^+ \quad ,$$

where $\| \ \|$ denotes the Euclidean norm in $\mathbb{R}^{k+1}$. We note that P contains all eigenvalues of (1.1.1) lying in S^k .

LEMMA 1.2.1. *Assume that the eigenvalue problem (1.1.1) is locally definite. Then* P *is the disjoint union of the compact and arcwise connected sets* P^+ *and* P^- .

Proof. It follows from (1.2.7) that P is the union of P^+ and P^- . In order to prove that P^+ and P^- are disjoint, let $\lambda \in P$ and let Y be the set of all $u \in U$ such that $W(u)\lambda = 0$. This set is a product $Y = Y_1 \times \ldots \times Y_k$, where

$$Y_r = \{u_r \in U_r \mid w_r(u_r)\lambda = 0\} \quad .$$

The following general Lemma 1.2.2 shows that Y_r is arcwise connected for every r. Hence Y is arcwise connected, too. Now there is a continuous real-valued function α on Y such that $\lambda = \alpha(u)\delta(u)$ for all $u \in Y$. Since Y is connected and α has no zeros, α has constant sign on Y. This proves that λ cannot belong to both P^+ and P^-.

The sets P^+ and P^- are continuous images of the compact and arcwise connected set U. Hence P^+ and P^- are compact and arcwise connected, too. We note that U is compact because the spaces H_r are finite dimensional, and that U is arcwise connected because of Lemma 1.2.2.□

LEMMA 1.2.2. *Let* H *be a complex linear space with inner product* $<\,,\,>$ *and unit sphere* U. *Let* ψ *be a Hermitian sesquilinear form on* H. *Then the set*

$$Y = \{u \in U \mid \psi(u,u) = 0\}$$

is arcwise connected. In particular, U *is arcwise connected.*

Proof. Let $x,y \in Y$. If x,y are linearly dependent then there is a real number θ such that $y = \exp(i\theta)x$. Then the continuous path $\exp(it\theta)x,\ 0 \leq t \leq 1$, connects x and y within Y. Now let x,y be linearly independent. Then we choose the real number θ such that the real part of $\exp(i\theta)\psi(x,y)$ vanishes. The segment

$$z(t) = t\exp(i\theta)x + (1-t)y, \quad 0 \leq t \leq 1 ,$$

connects y and $\exp(i\theta)x$, does not cross 0 and satisfies $\psi(z(t), z(t)) = 0$ for all $0 \leq t \leq 1$. Hence the continuous path $z(t) / <z(t), z(t)>^{1/2}$ connects y and $\exp(i\theta)x$ within Y. By what we have shown in the first part of the proof, we can also connect x and y by a continuous path within Y. This shows that Y is arcwise connected. Choosing $\psi = 0$, we see that U is arcwise connected. □

If λ is in P^+ or P^- then we say that λ has *signum* $+1$ or -1, respectively. We now are in a position to prove the uniqueness of an eigenvalue of given signed index (i,σ), i. e. an eigenvalue which has index i and signum σ.

THEOREM 1.2.3. *Let the eigenvalue problem (1.1.1) be locally definite. Then there is at most one eigenvalue in* S^k *of given signed index.*