Geometry and Topology of Submanifolds

Proceedings of the Meeting at Luminy

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Preface

Two of the common main teachers of Jean-Marie Morvan and myself are Professor Radu Rosca, being retired now at Paris but still very active in mathematical research, and Professor Bang-Yen Chen, of the Michigan State University at East Lansing. From them we inherited a strong interest in differential geometry, and, in particular, in submanifold theory. When both of us were visiting Professor Chen once more at MSU, at the end of 1985 and the beginning of 1986, in his office, we discussed the organisation of international meetings on differential geometry. Actually, but on a rather small scale (with about 15 participants from 5 countries), such a meeting was organised by Jean-Marie already in 1980, at Limoges. Returning home in 1986, Jean-Marie obtained the support of the Société Mathématique de France and organised the meeting at Luminy (Marseille) in 1987, of which these are the proceedings. One of our purposes with this and subsequent meetings is to have some balanced mixture of some well established mathematicians in our field, on the one hand, and some younger mathematicians who are starting their research in differential geometry, on the other. This mixture is not merely expressed in attending the meeting, but we try to give also these younger people the chance to present their results during a lecture.

I want to take benefit of this occasion to thank Professor Jean-Marie Morvan, on behalf of all participants at Luminy, for his great efforts to make this meeting a success. Moreover, his kind and warm personality largely contributed to the friendly and pleasant atmosphere reigning at this meeting, and which is so important to the creation of new contacts and the smooth exchange of information and ideas; (we don't have to thank Jean-Marie for this, since that is just the way he is!)

Jean-Marie Morvan and I thank all participants for their contribution to our meeting. We thank the Société Mathématique de France for their generous support and also for allowing us to use their beautiful Centre International de Rencontres Mathématiques at Luminy. Finally, we thank Professor Bang-Yen Chen for his suggestion to present the lecture notes of this meeting to the World Scientific Publishing Co, and indeed we thank this Company for actually accepting to publish these proceedings, and both for the very friendly and efficient way in which this was done.

Leopold Verstraelen

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CONTENTS

NONNEGATIVELY CURVED HYPERSURFACES IN HYPERBOLIC SPACE

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§1. Introduction

This paper and [AC] concern the asymptotic behavior at infinity of complete, noncompact, nonnegatively curved hypersurfaces properly imbedded in hyperbolic space. These are the hypersurfaces which bound convex bodies in H^{n+1} and for which the product of any two eigenvalues of the second fundamental form is at least one. It is shown in [AC] that the asymptotic behavior of such a hypersurface M is related to the asymptotic behavior of slowly growing subharmonic functions on the plane. The consequence derived there is that M is weakly asymptotic, in a sense described below, to a nonnegatively curved rotation hypersurface. Thus we were led to ask for a simple characterization of the nonnegatively curved rotation hypersurfaces of hyperbolic space. Such a characterization is given in section 2, after a brief explanation of the background from [AC].

Theorem 1 [AC]. *Let M be a complete, noncompact, nonnegatively curved hypersurface which is properly imbedded in hyperbolic space and is not an equidistant hypersurface. Then M is the graph of a real-valued height function which is defined on an entire horosphere H. The restriction of the height function to any 2-plane in H is a subharmonic function of polynomial growth.*

Remark 1. The simplest examples of such nonnegatively curved hypersurfaces are the horospheres, which are isometric to Euclidean space, and the equidistant hypersurfaces (tubes of constant radius around geodesics), which are isometric to the product of a standard sphere and a line. For a horosphere, the height function over any concentric horosphere is constant. For an equidistant hypersurface about a geodesic β, the height function over a horosphere H orthogonal to β is $\log \rho + C$, where ρ is the distance in H to the point at which β strikes H.

There are three main points to Theorem 1. Given any geodesic ray β which lies in the convex set bounded by M, and any horosphere H orthogonal to β, it is easy to see that M may be expressed as the graph of a continuous height function with values in $\mathbb{R} \cup (-\infty)$ over a convex domain in H (where no graph points are assigned to the value $-\infty$). Here, height is hyperbolic distance to M along geodesics orthogonal to H and oriented in the direction of β. The first point is that the height function h is subharmonic on the intersection of its domain with any 2-plane in H. That is, when finite the restriction of h to a 2-plane satisfies $\Delta h \geq 0$ where Δ is the Euclidean Laplacian. Note that subharmonicity is not sufficient for nonnegative curvature, since by the theorem nonnegative curvature imposes growth conditions on h.

Secondly, if M is not an equidistant hypersurface then the geodesic rays lying in the convex set bounded by M are mutually asymptotic. In other words, the asymptotic boundary of M in the sphere at infinity consists of a single point, at which all the horospheres H of Theorem 1 are centered. Our proof uses analytic information about the nature of the set on which a subharmonic function can take the value $-\infty$. Epstein has recently given an independent proof, using conformal metrics and the uniformization theorem, that for a complete, nonnegatively curved, proper imbedding of the plane into H^3 the asymptotic boundary is a single point [E].

Finally, the *growth function* $G(\rho)$ of a subharmonic function h on the plane is defined by letting $G(\rho)$ be the maximum value of h on the circle of radius ρ about the origin. The smallest nonconstant subharmonic functions have *polynomial growth*, that is, $G(\rho)/\log \rho$ has a finite limit as $\rho \to \infty$. Theorem 1 says that the height function of M over any 2-plane is defined on the entire plane and has polynomial growth. By Remark 1, it suffices to show that the corresponding slice of M supports an equidistant surface at the asymptotic boundary point. (The statement *A supports B at p* will always mean that B lies on the inward side of A in a neighborhood of p.) One proof of this fact is given in section 2.

There are a number of theorems to the effect that if $G(\rho)$ grows sufficiently slowly, then h(z) is not much smaller than $G(\rho)$ for most values of $z = \rho e^{i\theta}$. For example, in the following theorem of Hayman an *$\mathcal{E}$-set* in the plane is a countable union of disks subtending angles at the origin whose sum is finite.

<u>Theorem 2</u> [H]. *If h is subharmonic on the plane and has polynomial growth then there is an $\mathcal{E}$-set with the property: for every $\epsilon > 0$ there is a $C > 0$ such that $0 \leq G(\rho) - h(\rho e^{i\theta}) \leq \epsilon$ if $\rho > C$ and $\rho e^{i\theta} \notin \mathcal{E}$-set.*

Using Hayman's theorem, the following theorem was proved about the asymptotic behavior of M:

Theorem 3 [AC]. *Let M be as in Theorem 1, and let p be the single point at infinity in the asymptotic boundary of M. Then M either is a horosphere; or is asymptotic to an equidistant hypersurface; or is weakly asymptotic to a nonnegatively curved rotation hypersurface M_0 which supports the equidistant hypersurfaces at p and is supported by the horospheres at p.*

Here M_0 is the *inner rotation hypersurface* of M about the geodesic orthogonal to H through an arbitrary point 0 in H. The height function $g(\rho)$ of M_0 is defined by maximizing the height function of M over the (n-1)- spheres in H of radius ρ about 0. Notice that M_0 is not necessarily smooth. (In this paper, objects are taken to be C^∞ unless otherwise identified.) By *asymptotic* we mean that the difference in the height functions is arbitrarily small off a sufficiently large ball in H. *Weakly asymptotic* means that there is a subset E of H which intersects every 2-plane through 0 in an $\mathcal{E}$-set and for which the difference in the height functions is arbitrarily small off the union of E with a sufficiently large ball in H. This implies, for example, that the height functions are asymptotic on all rays through 0, except for a rayset of measure zero.

§2. Nonnegatively curved rotation hypersurfaces

In Euclidean space, a complete *rotation hypersurface* (by which we mean a hypersurface invariant under the isometries that leave a geodesic pointwise fixed) is nonnegatively curved precisely when its profile curve is convex, that is, when the hypersurface is globally supported at every point by a cone. We seek an analogous theorem for hyperbolic space. First, since a rotation hypersurface is intrinsically a warped product of an interval with a sphere, the metric is given by $g = ds^2 + [f(s)]^2\omega$, where s is arclength on the interval and ω is the standard sphere metric. For such a metric, a *radial sectional curvature*, K_{rad}, is the curvature of a section tangent to ∂s; a *tangential sectional curvature*, K_{tan}, is the curvature of a section tangent to a sphere. By a straightforward calculation,

$$K_{rad} = -f_{ss}/f \qquad\qquad K_{tan} = (1 - f_s^2)/f^2.$$

For rotation hypersurfaces, we have $f(s) = r(s)$ in Euclidean space, and $f(s) = \sinh r(s)$ in hyperbolic space. In both cases, $r(s)$ represents the distance of the point on the profile curve from the axial geodesic, and satisfies $|r_s| \leq 1$. In the Euclidean case, the equivalence of nonnegative curvature and convexity is immediate, since the tangential curvature is always nonnegative.

In either Euclidean or hyperbolic space, define a *cone* to be a rotation hypersurface for which the radial sectional curvatures vanish (equivalently, for which the slices by totally geodesic 3-planes through the axis are flat) and which, if incomplete, can be completed by the addition of one singular point. The cones that are complete without the addition of a singular point are the horospheres orthogonal to the axis and the hypersurfaces equidistant from the axis. In Euclidean space, these are hyperplanes and spherical cylinders. In hyperbolic space, but not in Euclidean space, there exist maximal rotation hypersurfaces with vanishing radial curvatures that are not cones. They can be completed by adding a disk, but not a point, and cannot be completed to have nonnegative radial sectional curvature (see Remark 2).

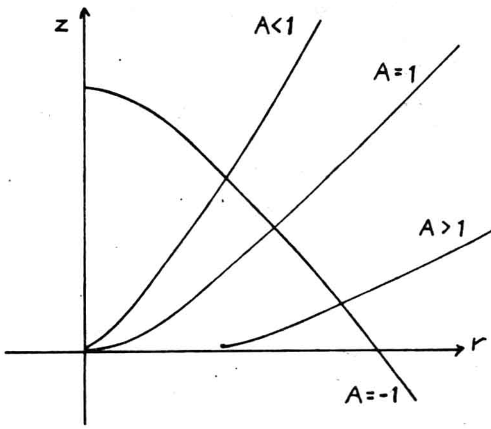
As above, in hyperbolic space the radial sectional curvatures of a rotation hypersurface are $-(\sinh r)_{ss}/\sinh r$. If these vanish then $\sinh r(s) = As + B$. For $A=0$, we have equidistant hypersurfaces. The other solutions are best described in Fermi coordinates, namely r (as above) for distance from the axis and z for arclength along the axis, extended to be constant on the totally geodesic hypersurfaces orthogonal to the axis. For a profile curve of the form $z = z(r)$, arclength is $s = \int_0^r [\cosh u]\,[z_r(u)^2 + \operatorname{sech}^2 u]^{1/2}\,du$. Thus the solutions satisfy

$$[z_r^2 + \operatorname{sech}^2 r] = A^{-2}. \qquad (1)$$

For $0 < |A| \leq 1$, we have

$$z = z_0 + A^{-1} \int_0^r [1 - A^2 \operatorname{sech}^2 u]^{1/2}\,du. \qquad (2)$$

Since $f(s) = \sinh r(s) = As + B$, these hypersurfaces have nonnegative tangential curvature. In particular, $A=1$ gives $z = z_0 + \log(\cosh r)$, namely, the horospheres. If $|A| > 1$, then $z = z_0 + A^{-1} \int_{\cosh^{-1}|A|}^r [1-A^2 \operatorname{sech}^2 u]^{1/2}\,du$. Here the tangential curvature is negative. These profile curves develop infinite curvature as r approaches its infimum, $\cosh^{-1}|A|$.



Thus the hyperbolic cones are the equidistant hypersurfaces and the surfaces given by equation (1) with $0<|A|\leq 1$, or equivalently by (2). Notice that changing the sign of A corresponds to reflection in the totally geodesic hypersurface $z=z_0$.

Remark 2. A flat rotation surface in H^3 with $A>1$ supports, at its asymptotic boundary point, every horosphere centered at that point. By piecing and rounding at the joins, one can also obtain nonflat examples of incomplete, nonnegatively curved surfaces which support a horosphere at infinity. In contrast, the only complete, nonnegatively curved, imbedded hypersurfaces of hyperbolic space which support a horosphere at infinity are the concentric horospheres. This is a consequence of the fact that a bounded subharmonic function defined on the entire plane is constant.

Theorem 4. *A complete, noncompact rotation hypersurface in hyperbolic space has nonnegative curvature if and only if it is globally supported at every point by a hyperbolic cone.*

Proof: For a rotation hypersurface given in Fermi coordinates by $z = z(r)$, the radial curvature is nonnegative if and only if

$$[\, z_r{}^2 + \operatorname{sech}^2 r\,]_r \geq 0. \tag{3}$$

This follows by a simple computation from the equations
$K_{rad} = -(\sinh r)_{ss}/\sinh r$ and $s_r = \cosh r\,[z_r^2 + \operatorname{sech}^2 r]^{1/2}$.

Let M denote a complete, noncompact rotation hypersurface which is not an equidistant hypersurface, and supppose first that M has nonnegative radial curvature. Then z_r is not always infinite, and without loss of generality is positive somewhere. It is a consequence of (3) that $z_r \neq 0$ for $r>0$, and $\dot{z}_r$ and z_{rr} always have the same sign. By completeness and noncompactness, it follows that M has the form $z = z(r)$, $z_r > 0$, on a maximal interval $(0,a)$, where if $a < \infty$ then the remainder of M coincides with the equidistant hypersurface $r = a$. Now (3) implies that the inequality $[z_r^2 + \operatorname{sech}^2 r] \geq 1$ holds everywhere because it holds at $r = 0$. Thus M is tangent at each point to a hyperbolic cone (and so has nonnegative tangential curvature). If $z = z_C(r)$ is the equation of such a cone, then the expression $g(r) = [z_r^2 + \operatorname{sech}^2 r]$ is increasing while the same expression in z_C, say g_C, remains constant. Since $g(r_0) = g_C(r_0)$, then the sign of $z_r - (z_C)_r$ agrees with that of $r - r_0$; and since $z(r_0) = z_C(r_0)$, then $z(r) - z_C(r)$ is nonnegative. It follows that the hyperbolic cone supports M globally.

Conversely, suppose M is globally supported by hyperbolic cones. Then $z_r \neq 0$ for $r>0$, and it follows from noncompactness that z_r cannot take opposite signs on M. Therefore we may assume that M takes the form $z = z(r)$ with $z_r>0$ and $r \in (0,a)$, as described above. Furthermore, the tangential curvatures of M are nonnegative since they coincide with those of cones. Local support by the cone $z = z_C(r)$ at $r = r_0$ implies

$$z_{rr}(r_0) \geq (z_C)_{rr}(r_0) = (z_C)_r^{-1}(r_0)\,(\operatorname{sech}^2 r_0)\,(\tanh r_0).$$

Since $(z_C)_r$ and z_r agree at r_0, M has nonnegative radial curvature. Since the extreme values of curvature at a point are the radial and tangential curvatures, this completes the proof.

Now let M be a nonnegatively curved hypersurface as described in Theorem 1. Note that the proof of Theorem 4 does not apply directly to the inner rotation hypersurfaces M_0 of M because they are not smooth. However, the proof is retrievable:

Corollary. *The inner rotation hypersurfaces M_0 of M are globally supported at every point by hyperbolic cones.*

Proof. Suppose first that M is 2-dimensional. As in Theorem 1, M is the graph of a subharmonic height function h defined on a horosphere H. M_0 is the graph of the growth function $G(\rho)$ defined by maximizing h on the circles of radius ρ about the point 0 in H. It is a consequence of subharmonicity that the growth function is nondecreasing and is convex in $\log \rho$ [HK, p. 66]. Thus the profile curve of M_0 possesses regularity properties inherited from those of convex functions. In particular, it has one-sided tangents everywhere, is differentiable except at countably many points at which the tangent undergoes a positively oriented rotation, and its tangents are continuous from each side. Here the orientation of the totally geodesic halfplane in which the profile curve lies coincides with that of Fermi coordinates (r, z) where z increases toward the inward side of M_0. The profile curve is oriented in the direction of increasing ρ. We also need the fact that, since h is smooth, the left and right derivatives of $G(\rho)$ at any ρ_0 coincide respectively with the minimum and maximum values of $\{h_\rho(\rho_0 e^{i\theta}): h(\rho_0 e^{i\theta}) = G(\rho_0)\}$.

Fix a point q on the profile curve of M_0, and assume that the totally geodesic halfplane in which it lies has been chosen so that q falls on M. Now slice a neighborhood of q in M by that same halfplane, and let M_S be the smooth rotation hypersurface generated by the slice. Then M and M_S have the same normal at q. The tangential and radial principal curvatures of M_S at q are no less in magnitude than the corresponding normal curvatures of M. It follows from the Gauss equation that the curvature of M_S at q is nonnegative. Thus at each of its points and for each of the two extreme positions of its tangent plane at that point, M_0 is locally supported by a smooth rotation surface which has nonnegative curvature at that point and is tangent to that plane.

In the coordinates (r, z), the tangents to the profile curve of M_0 point strictly upward when $r>0$. Any point of the profile curve which has a nonvertical one-sided tangent lies on a nontrivial closed subsegment P whose tangents never pass through the vertical and which therefore has the form $z = z(r)$. By the preceding paragraph and equation (3), at each of its points and for each of its extreme tangents at that point, P is locally supported on its negative side by a smooth curve $z = z_S(r)$ which has the same tangent and satisfies $[(z_S)_r{}^2 + \operatorname{sech}^2 r]_r \geq -\varepsilon$, where ε can be taken arbitrarily small. Now suppose that all the one-sided derivatives z_r are positive on P. We apply an argument from the previous proof to the differential inequality $[z_r{}^2 + \operatorname{sech}^2 r + \varepsilon r]_r \geq 0$. It follows that at each of its points and for each of its tangents (not necessarily extreme) at that point, P is locally supported below by a curve $z = z_C(r)$ which has the same tangent and satisfies

$$(z_C)_r{}^2 + \operatorname{sech}^2 r + \varepsilon r = C. \qquad (4)$$

Furthermore, this local support property holds on intervals of uniform width around any point. But then it is straightforward to show that $[z_r{}^2 + \operatorname{sech}^2 r + \varepsilon r]$'s nondecreasing in r on any given bounded subsegment of P, if ε is sufficiently small. Here "nondecreasing" means that the largest value at r_0 is no larger than the smallest value at $r > r_0$. Therefore $[z_r{}^2 + \operatorname{sech}^2 r]$ is nondecreasing in r, and in particular $z(r)$ is convex.

Similarly, a subsegment P of the profile curve for which z_r is negative is supported above by curves of the form (4). Moreover, $[z_r{}^2 + \operatorname{sech}^2 r]$ is nondecreasing in r, and hence $z(r)$ is concave. Now we have the information necessary to carry out the argument in the second paragraph of the proof of Theorem 4.

In particular, M_0 has the form $z = z(r)$, $z_r(\pm) > 0$, on a maximal interval $(0,a)$, where if $a < \infty$ then the remainder of M_0 coincides with the equidistant hypersurface $r = a$. (This proves that M itself supports some equidistant hypersurface at infinity, as claimed in section 1.) Furthermore, the corollary holds if M is 2-dimensional. Suppose M is of higher dimension. Then the corollary follows from the fact that the height function $g(\rho)$ of the inner rotation hypersurface of M is globally supported at each point by the height function $G(\rho)$ of the inner rotation surface of some 2-dimensional slice of M. This completes the proof.

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