Lie Groups, Lie Algebras, and Some of Their Applications

ROBERT GILMORE
University of South Florida

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Preface

Only a century has elapsed since 1873, when Marius Sophus Lie began his research on what has evolved into one of the most fruitful and beautiful branches of modern mathematics—the theory of Lie groups. These researches culminated twenty years later with the publication of landmark treatises by S. Lie and F. Engel [1–3] between 1888 and 1893, and by W. Killing [1–4] from 1888 to 1890. Matrices and matrix groups had been introduced by A. Cayley, Sir W. R. Hamilton, and J. J. Sylvester (1850–1859) about twenty years before the researches of Lie and Engel began. At that time mathematicians felt that they had finally invented something of no possible use to natural scientists. However, Lie groups have come to play an increasingly important role in modern physical theories. In fact, Lie groups enter physics primarily through their finite- and infinite-dimensional matrix representations.

Certain natural questions arise. For example, just how does it happen that Lie groups play such a fundamental role in physics? And how are they used?

Lie groups found their way into physics even before the development of the quantum theory. They were useful for the description of pseudo-Riemannian (locally) homogeneous symmetric spaces, being used in particular in geometric theories of gravitation. But Lie groups were virtually forced into physics by the development of the modern quantum theory in 1925–1926. In this theory, physical observables appear through their hermitian matrix representatives, whereas processes producing transformations are described by their unitary or antiunitary matrix representations. Operators that close under commutation belong to a finite-dimensional Lie algebra; transformation processes described by a finite number of continuous parameters belong to a Lie group.

The kinds of applications of Lie group theory in modern physics fall into three distinct stages:

1. As symmetry groups (1929–1960). Symmetry implies degeneracy. The greater the symmetry, the greater the degeneracy. Assume that a Lie group G with Lie algebra $\mathfrak{g}$ commutes with a Hamiltonian $\mathscr{H}$:

$$G\mathscr{H}G^{-1} = \mathscr{H} \Leftrightarrow [\mathscr{H}, \mathfrak{g}] = 0$$

Then by Wigner's theorem the basis vectors spanning a fixed energy eigenspace carry a representation of G. For example, the three-dimensional isotropic harmonic oscillator whose Hamiltonian is

$$\mathscr{H} = \hbar\omega(a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3 + 3/2)$$

where $$[a_i^\dagger, a_j] = -\delta_{ij}$$
$$[a_i, a_j] = [a_j^\dagger, a_i^\dagger] = 0$$

has spherical symmetry. Therefore, $\mathscr{H}$ commutes with the infinitesimal generators L_i of the rotation group $SO(3)$:

$$[\mathscr{H}, L_i] = 0 \qquad L_i \simeq a_j^\dagger a_k - a_k^\dagger a_j \qquad (i, j, k) = (1, 2, 3) \text{ cycl.}$$

The oscillator eigenfunctions therefore carry representations of the rotation group $SO(3)$.

However, the existence of an "accidental" degeneracy in this example gives a larger degeneracy than is demanded by the obvious geometric invariance group $SO(3)$. This suggests that a larger group, containing $SO(3)$ as a subgroup, may be a more useful symmetry group for this Hamiltonian. The group is $U(3)$, with Lie algebra U_{ij}:

$$[\mathscr{H}, U_{ij}] = 0 \qquad U_{ij} = a_i^\dagger a_j$$
$$\mathscr{H} = \hbar\omega \sum (a_i^\dagger a_i + 1/2); \qquad [U_{ij}, U_{rs}] = U_{is}\,\delta_{jr} - U_{rj}\,\delta_{si}$$

In fact, it is useful and even desirable from a calculational standpoint to label the oscillator eigenfunctions with $SU(3)$ representation labels (J. M. Jauch and E. L. Hill [1], J. P. Elliott [1]).

2. As nonsymmetry groups (1960–). Around 1960 physicists were gradually forced to realize that groups that do not commute with $\mathscr{H}$ can be even more useful than symmetry groups from a computational viewpoint. As an example, it is possible to find a 16-dimensional nonsymmetry group with generators $a_i^\dagger a_j$, $a_i^\dagger$, a_j, I

$$[\mathscr{H}, a_i^\dagger a_j] = 0$$
$$[\mathscr{H}, a_i^\dagger] = +\hbar\omega a_i^\dagger$$
$$[\mathscr{H}, a_j] = -\hbar\omega a_j$$
$$[\mathscr{H}, I] = 0$$

This nonsymmetry group is contracted from the noncompact group $U(3, 1)$. Using this noncompact algebra, any eigenstate can be obtained from any other by applying a sequence of elements in the Lie algebra. In particular, all excited states can be computed from the ground state, which, in turn, can be computed either by algebraic or by analytic (variational) methods. The hydrogen atom, superfluid and superconductor models, laser systems, and charged particles in external fields are some of the problems amenable to such treatment.

3. ? (1970–). Strictly speaking, the third class of applications is not yet known, although its appearance is probably around the corner. It now seems possible that Lie group theory, together with differential geometry, harmonic analysis, and some devious arguments, might be able to predict some of Nature's dimensionless numbers (α, m_p/m_e, m_μ/m_e, G^2/hc, ...). In retrospect, it seems clear that the application of group theory to physical problems represents the dividing line between kinematics and dynamics. The group theory gives the overall structure of the spectrum; the dynamics serves to define only the scale. We are looking forward to the day when Lie groups can be pushed to give also the dynamics, or scale, of a physical process. In terms of our model harmonic osciliator Hamiltonian, this means that we hope some day to be able to derive the scaling factor $\hbar\omega$ from fundamental group theoretical arguments.

The work presented here has evolved from a course on Lie groups and their physical applications which I taught several times at M.I.T. and at the University of South Florida. The course covered Lie groups and algebras, representation theory, realizations and special functions, and physical applications. Using the theory of Lie groups as a unifying vehicle, many different aspects of many fields of physics can be presented in an extremely economical way. A great number of calculations remain fundamentally unchanged from one field of physics to another; it is only the interpretation of the symbols and the language used which changes. Thus the Jahn-Teller effect and the Nilsson nucleus are but two aspects of the same phenomenology.

During the development of the course, I realized that a relatively small number of physicists have mastered the theory of Lie groups and are able to use it actively as a tool in their researches. These physicists spend their time primarily writing beautiful papers for one another. On the outside looking in are the relatively large number of physicists who would like to learn the material, who appreciate its power and usefulness, but who are hampered by the lack of an adequate text.

In this context, two established books deserve special mention and praise. These books may profitably be consulted by readers interested in alternative treatments of overlapping material. M. Hamermesh's book [1] has done yeoman service for the physics community during the last decade.

Unfortunately, it stops short of a thorough discussion of Lie group theory. S. Helgason's book [1], which has been equally important in the mathematics community, provides an excellent discussion of Lie group theory but is unfortunately beyond the grasp of most working physicists.

The purpose of this book is to bridge the gap between those who do not know Lie group theory and those who do know. In this sense, this work "fits between" the books of Hamermesh and Helgason.

It has been my intention throughout to present the material in such a way that it is accessible to physicists. I have tried to be as rigorous as possible. But when rigor and clarity have clashed, clarity has won out. There are a sufficient number of treatises on Lie groups by and for mathematicians, and the reader interested in complete rigor will have no trouble filling the gaps I have left.

This work has been aimed at the level of the graduate student. Problems of an illustrative nature have been worked out and included throughout the text. For a physicist it is not only desirable to understand the material, but necessary to be able to make calculations. It is hoped that the solved problems will lead more swiftly to this facility. Exercises have been included at the end of each chapter. Many of them are designed to bring on an awareness of how and where the mathematics presented finds its way into physics. Numerous figures—perhaps too many—have been included, in an attempt to foster easier understanding of the arguments presented in the text. This vice dates from many encounters with Professor I. M. Singer, who always managed to make an argument clearer with one or two telling sketches. The references within each chapter (superscript numbers) refer to the Notes and References section at the end of that chapter. The references in the closing section of each chapter refer to entries in the master bibliography at the end of the book.

The structure of this book resembles that of a concerto. The study develops (*allegro*) in Chapters 1 to 4, where the general properties of Lie groups and algebras are discussed. It continues and concludes (more *allegro*) in Chapters 7 to 10, which are principally devoted to the properties of the semisimple Lie groups. Chapters 5 and 6 provide a relief (*moderato*) from the development. In these chapters specific examples are used both to illustrate concepts developed earlier and to presage concepts to be dealt with subsequently.

Chapter 1, which is devoted to fundamental working definitions and notations, has been included to make the book as self-contained as possible. A cursory familiarity with modern algebra will allow the reader to bypass this chapter. I have tried to present here some of the basic concepts of modern mathematics in such a way that they are less mysterious to a student of physics.

Chapter 2 describes examples of Lie groups. In particular, the classical Lie groups are described following, to some extent, the treatment given by F. D. Murnaghan [1, 2]. This is a not altogether satisfying approach, and we return to the problem of enumerating all the real forms of the simple classical Lie algebras in Chapter 6, where a complete and elegant summary is presented.

In Chapter 3 we define, describe, and work with continuous groups and some of their properties. This treatment culminates in a definition of a Lie group, described more thoroughly in Chapter 4. In this chapter we display the relationship between Lie groups and Lie algebras; we also prove the three theorems of Lie. These theorems relate a Lie algebra to a Lie group by the linearization process. The converses to these three theorems—stated but not proved—relate Lie groups to Lie algebras by the inverse process, exponentiation.

Chapters 5 and 6 represent a watershed in our formal discussion of Lie groups and their algebras. Chapter 5, an elaboration of the concepts developed in the preceding chapters, takes the form of applications of the formal machinery to some of the classical groups—chiefly $SU(2)$. We indicate here also how this machinery can be applied to some useful physical problems. In Chapter 6 we describe more thoroughly the simple classical matrix groups and their algebras. The focal point of this chapter is the summary of all the real forms of the simple classical Lie algebras, and the coset spaces related to these real forms.

In Chapter 7 we resume our formal study of Lie groups and their algebras. All the major tools used in the classification theory of Lie algebras are trotted out one by one, dusted off, and applied to this classification problem. At the end of this chapter we present the commutation relations for all the classical complex simple Lie algebras in canonical form, using the concept of a root space diagram.

The canonical commutation relations are presented again at the beginning of Chapter 8 and are used in making a complete classification of all the root space diagrams. The completeness classification of B. L. van der Waerden [3] is used to construct the complete set of roots in any root space. E. B. Dynkin's approach [1], using Coxeter-Dynkin diagrams, then serves to furnish a convincing proof of the completeness of the classification.

Once all the root space diagrams have been classified, there remains only the problem of classifying the real forms which the complex simple algebras can have. This problem is treated in Chapter 9. The approach in itself leads to nothing surprising: all such real forms have already been encountered, using different arguments, in Chapter 6. This approach merely shows the completeness of the list in Chapter 6. The classification of the real forms used here involves a listing of the irreducible Riemannian symmetric spaces, which are cosets of a simple Lie group by a maximal compact subgroup. These spaces are interesting objects in their own right, and are of course intimately related to Lie groups. Moreover, the concepts and methods developed in Chapters 7 and 8 are applicable to the study of the irreducible Riemannian symmetric spaces. Application of these methods leads immediately to a complete listing of all the globally symmetric pseudo-Riemannian symmetric spaces.

The closing chapter is devoted to a study of how Lie groups and their algebras can be altered. We begin by studying the process of contraction, in which nonsemisimple Lie groups can be constructed from semisimple Lie groups by a limiting procedure. Chapter 10 closes with an indication of how the reverse process can take place; this is called group expansion.

Some terms appearing in this work are used in an unusual way. For example, the term "basis" is usually applied to an element in a linear vector space (basis vector); however, since I apply this term to analogous elements in a group, field, and algebra, the shortened term "basis" is much more appropriate. Such usage is designed to aid the understanding of the neophyte. In addition, I have not always used the same matrix structure to describe various Lie algebras, since I feel it is more useful to have several alternative descriptions of an algebra than one canonical one. I hope the cognoscenti will understand and appreciate these usages.

The physicists will be unhappy that so many important topics have been omitted. This work contains no systematic discussion of the representation theory of Lie groups and Lie algebras. Those interested in such material are urged to consult the books of H. Weyl [1, 2] and the works of É. Cartan [1, 24, 27, 28], the two classical giants in this field. Nor is there any systematic discussion of the theory of the special functions of mathematical physics. This material is treated in the books of N. Ja. Vilenkin [1], W. Miller [2], and J. D. Talman [1].

Finally, there is no systematic discussion of the applications of Lie group theory in modern physics. Such a systematic treatment, which would fill a volume in itself, could only be carried out after a treatment of the representation theory of Lie groups. In lieu of such treatment, numerous exercises indicating physical applications have been included at the end of each chapter. In addition, a number of physics papers dealing with every sort of application of Lie groups in physics have been placed in the bibliography. The interested reader has only to pick out some interesting-sounding titles from the bibliography and to follow them into the current literature. He is sure to be surrounded by Lie groups and unbelieveable applications in no time at all.

For the sins of omission and commission I am deeply sorry. I hope the former are somewhat compensated for by references in the bibliography. The latter I hope are few and far between.

I would like to thank my former students at M.I.T. and the students and faculty at U.S.F. for their many useful comments and suggestions. In addition, I would like to express my gratitute to Professors I. M. Singer and S. Helgason for useful discussions in the recent past, and for beautifully taught courses, which I had the privilege to attend, in the distant past. Thanks are also in order to Professor Peter Wolff for the key role he played in the preparation of the last half of the manuscript. An expression of gratitude is due to the programming policies of WCRB, which made the preparation of the manuscript reasonably pleasant, and to the staff of John Wiley & Sons for doing the same during the final stages of preparation. Finally, I would like to thank my wife for her patience during the preparation of this manuscript, as well as for typing large parts of it.

ROBERT GILMORE

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Contents