COMBINATORIAL ENUMERATION

I.P. Goulden
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COMBINATORIAL ENUMERATION

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Foreword

The progress of mathematics can be viewed as a movement from the infinite to the finite. At the start, the possibilities of a theory, for example, the theory of enumeration, appear to be boundless. Rules for the enumeration of sets subject to various conditions, or combinatorial objects as they are often called, appear to obey an indefinite variety of recursions, and seem to lead to a welter of generating functions. We are at first led to suspect that the class of objects with a common property that may be enumerated is indeed infinite and unclassifiable.

As cases file upon cases, however, patterns begin to emerge. Freakish instances are quietly discarded; impossible problems are recognized as such, and what is left organizes itself along a few general criteria.

We would like these criteria to eventually boil down to one, but by and large we must be content with a small finite number.

And so with the theory of enumeration, as Jackson and Goulden show in this book. There are two basic patterns, ordinary generating functions and exponential generating functions, the first counting unlabeled or linearly ordered objects, the second counting labeled objects. The various combinatorial interpretations of the Lagrange inversion formula give the deepest results in enumeration. The test case is the enumeration of permutations subject to various geometric conditions. The still largely mysterious q-analogs arise from adding an extra parameter to the enumeration of permutations.

Lastly, there is the connection between circular enumeration and exponential generating functions; this, as well as the other topics, is developed thoroughly and with a wealth of examples by Goulden and Jackson. Their book will be required reading from now on by any worker in combinatorics.

Gian-Carlo Rota

Cambridge, Massachusetts
April 1983

Preface

The theory of enumeration has developed rapidly during the past century, with the increasing awareness of the importance of discrete structures. Work on its mathematical foundations has been inspired by MacMahon's "Combinatory Analysis," published in 1915, and Rota's series entitled "On the Foundations of Combinatorial Theory," begun in 1964. Our objectives in writing this book are to give a unified account of a generating function approach to this area and to give a reasonably complete collection of representative results. We have illustrated the theory with a range of examples to reveal something of its generality and subtlety. The book is written not only for the combinatorial theorist but also for the mathematician, the physicist, and the computer scientist, in whose fields problems of this type occur. Hitherto, much of the material included here has been available only in the research journals.

The general principle behind our account is a very simple one. First, combinatorial arguments are used to derive bijections (decompositions) between sets of discrete structures, and these are then reduced to functional relationships between formal power series by associating generating functions with sets. The type of the generating function (whether **ordinary** (Chapter 2) or **exponential** (Chapter 3)) depends on the decomposition. In manipulating generating functions, we appeal to results from analysis and linear algebra that are developed from the ring of formal power series and Laurent series in Chapter 1.

Among the structures considered are permutations, sequences, integer partitions, trees, maps, plane partitions, and lattice paths. The examples following each decomposition have been selected to illustrate the variety of the enumerative results that can be obtained from a single decomposition. Many of these can be derived separately, and more quickly, by methods peculiar to the particular problem. However, such methods may be hard to discover without knowing the results in advance and tend to give less insight into the relationships between problems.

The exercises are organized as a compendium of supplementary results whose solutions are given in detail to encourage readers to probe further. They contain additional decompositions, further development and generalization of the ideas presented in the text, and a gradual evolution of the technical details, both combinatorial and algebraic.

We have not attempted to give a complete bibliography of the field; instead, we have confined ourselves to references that either relate closely to specific points in the material or that are more detailed accounts of particular topics. In the interest of brevity, these two types of references are not distinguished in the notes and references following each section.

Inevitably, it has been necessary to exclude a number of important enumerative areas. The main exclusions are incidence algebras, ring-theoretic methods, theory of chromatic polynomials, asymptotics, root systems, and graphical enumeration, each of which warrants separate treatment.

We have benefited both directly and indirectly from conversations with friends and colleagues. In particular we wish to thank D. Ž. Djoković, P. Flajolet, I. M. Gessel, M. Guy, J. Lawrence, A. Mandel, R. P. Stanley, and N. Wormald. One of us (D.M.J.) would like to express a debt of gratitude to the late Dr. J. C. P. Miller for his encouragement of this project, and the Department of Pure Mathematics and Mathematical Statistics (University of Cambridge), the Computer Laboratory (University of Cambridge), and the Institute National de Recherche en Informatique et en Automatique (Paris), for their hospitality during several summers of uninterrupted work. Finally, we are grateful to Mrs. Susan Embro and Mrs. Sandy Tamowski for so skillfully executing the long and, at times, trying task of typing our manuscript and to H. D. L. Night for preparing the illustrations.

I. P. Goulden
D. M. Jackson

Waterloo, Ontario
April 1983

Notation

For graph-theoretic definitions see Bondy and Murty (1976). In general, sans serif capitals denote sets and boldface letters denote matrices or vectors. In the text [a.b.c.] denotes exercise c in section b of chapter a, and a.b.c. denotes paragraph c in section b of chapter a.

N	$\{0, 1, 2, \dots\}$
N_+	$\{1, 2, \dots\}$
N_n	$\{1, \dots, n\}$
$\lvert\mathsf{S}\rvert$	cardinality of the set S
S^*	free monoid on S
S^+	$\mathsf{S}^* - \{\varepsilon\}$, where ε is the empty string
$\mathbb{Q}$	the set of rationals
$\mathbb{Z}$	the set of all integers
$\mathsf{M}_{m,n}(\mathsf{R})$	the set of all $m \times n$ matrices with elements in R
$\mathsf{M}_n(\mathsf{R})$	$\mathsf{M}_{n,n}(\mathsf{R})$
$\lfloor \lambda \rfloor$	integer part of λ
δ_{ij}	Kronecker delta: $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$
$\mapsto$	elementwise action of a mapping
$\mathbf{x}^{\mathbf{i}}$	$x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}$ where $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{i} = (i_1, \dots, i_n)$
$\mathbf{i}!$	$i_1! \cdots i_n!$
$\begin{bmatrix} m \\ \mathbf{i} \end{bmatrix}$	multinomial coefficient $m!/\mathbf{i}!$ where $i_1 + \cdots + i_n = m$
$f \circ \boldsymbol{\gamma}$ (umbral)	$\sum_{k \geq 0} f_k \gamma_k$ where $f = 1 + f_1 x + f_2 x^2 + \cdots$ and $\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \gamma_2, \dots)$
$[b_{ij}]_{m \times n}$	the $m \times n$ matrix whose (i, j)-element is b_{ij}
$\lVert a_{ij} \rVert_{m \times m}$	determinant of the $m \times m$ matrix whose (i, j)-element is a_{ij}
$\mathrm{cof}_{ij} \mathbf{A}$	cofactor of the (i, j)-element of $\mathbf{A}$
$\mathrm{diag}(\mathbf{x})$	the diagonal matrix with x_i in row i
$\mathbf{B}[\alpha \mid \beta]$	the submatrix of $\mathbf{B}$ with row and column labels in $\alpha \subseteq \mathsf{N}_m$, $\beta \subseteq \mathsf{N}_n$, respectively.

$\mathbf{B}(\alpha\mid\beta)$	$\mathbf{B}[\mathsf{N}_m - \alpha \mid \mathsf{N}_n - \beta]$
$[\mathbf{B}\mid\mathbf{b}]_i$	the matrix obtained by replacing column i of $\mathbf{B}$ by the column vector $\mathbf{b}$, where $\mathbf{b}$ has m elements
adj $\mathbf{A}$	the adjoint of $\mathbf{A} \in \mathsf{M}_m(\mathsf{R})$
$\mathbf{K}!$	$\prod_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}} k_{ij}!$ where $\mathbf{K} = [k_{ij}]m \times n$
$\mathbf{A}^{\mathbf{K}}$	$\prod_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}} a_{ij}^{k_{ij}}$ where $\mathbf{A} = [a_{ij}]m \times n$
$\mathbf{J}_{m,n}$	$[1]_{m,n}$
$\mathbf{J}_n$	$\mathbf{J}_{n,n}$

Contents

Notation xxiii

1 MATHEMATICAL PRELIMINARIES 1

1.1 The Ring of Formal Power Series 1

1.1.1 Formal Power Series, 1
1.1.2 The Coefficient Operator, 1
1.1.3 Infinite Sums and Products, 2
1.1.4 Compositional and Multiplicative Inverses, 2
1.1.5 The Formal Derivative and Integral, 3
1.1.6 The Logarithmic, Exponential, and Binomial Power Series, 4
1.1.7 Circular and Hyperbolic Power Series, 7
1.1.8 Formal Differential Equations, 8
1.1.9 Roots of a Power Series, 9
1.1.10 Matrices Over the Ring of Formal Power Series, 10
1.1.11 Formal Laurent Series, 11

Notes and References 12
Exercises 12

1.2 The Lagrange Theorem for Implicit Functions 15

1.2.1 Proposition, 15
1.2.2 **Theorem (Residue Composition), 15**
1.2.3 An Identity (by Residue Composition), 16
1.2.4 **Theorem (Lagrange), 17**
1.2.5 A Functional Equation, 18
1.2.6 The Central Trinomial Numbers, 19
1.2.7 Abel's Extension of the Binomial Theorem, 19
1.2.8 **Theorem (Multivariate Residue Composition), 19**
1.2.9 **Theorem (Multivariate Lagrange), 21**
1.2.10 A Functional Equation in Two Variables, 22
1.2.11 **Theorem (MacMahon Master Theorem), 23**
1.2.12 Dixon's Identity, 23
1.2.13 **Corollary (Lagrange Theorem for Monomials), 25**

Notes and References 26
Exercises 26

2 THE COMBINATORICS OF THE ORDINARY GENERATING FUNCTION 29

2.1 Introduction 29

2.1.1 The Elementary Counting Lemmas, 29
2.1.2 Decompositions and Weight Functions, 30
2.1.3 Direct and Indirect Decompositions, Combinatorial Marking, and Multivariate Generating Functions, 30
2.1.4 A Classical Application of Enumerative Arguments, 31
2.1.5 Recursive and "At-Least" Decompositions, 31

2.2 The Elementary Counting Lemmas 31

2.2.1 Definition (Distinguishability), 32
2.2.2 Definition (Weight Function, Weight), 32
2.2.3 Remark (The General Enumerative Problem), 32
2.2.4 Example (Enumerative Problems), 32
2.2.5 Definition (Ordinary Generating Function), 33
2.2.6 Remark (s-Objects), 33
2.2.7 Example (a Generating Function), 33
2.2.8 Proposition, 34
2.2.9 Definition (Decomposition, ω-Preserving), 34
2.2.10 Proposition, 34
2.2.11 Example (Terquem Problem: Decomposition), 35
2.2.12 **Lemma (Sum), 36**
2.2.13 Example (Sum), 36
2.2.14 **Lemma (Product), 36**
2.2.15 Example (Terquem Problem: Generating Function), 37
2.2.16 Definition (Additively Weight-Preserving Decomposition), 38
2.2.17 Remark (Direct, Indirect, Recursive Decompositions), 39
2.2.18 Example (an Indirect Decomposition), 39
2.2.19 Example (a Recursive Decomposition), 40
2.2.20 Definition (Composition), 41
2.2.21 Example (Composition), 41
2.2.22 **Lemma (Composition), 42**
2.2.23 Example (Composition), 43
2.2.24 Definition (s-Derivative), 44
2.2.25 Example (s-Derivative of a Set of Permutations), 44
2.2.26 **Lemma (Differentiation), 45**
2.2.27 Example (s-Derivative), 45
2.2.28 Definition ("Exact," "At-Least" Generating Functions), 46
2.2.29 **Lemma (the Principle of Inclusion and Exclusion), 47**
2.2.30 Example (Derangements), 48

Notes and References 48

2.3 Preliminary Examples 48

2.3.1 **Decomposition (Subsets), 48**
2.3.2 Subsets, 49
2.3.3 Recurrence Equation for Binomial Coefficients, 49
2.3.4 **Decomposition (Multisets), 50**
2.3.5 Multisets, 50

2.3.6 Definition (Composition of an Integer), 51
2.3.7 Decomposition (Compositions of an Integer), 51
2.3.8 Compositions of an Integer, 51
2.3.9 Correspondence (Multisets–Compositions), 52
2.3.10 Compositions and Parts, 53
2.3.11 Recurrence Equation for Compositions and Parts, 53
2.3.12 An Identity from Compositions, 54
2.3.13 Definition (Succession in a Set), 54
2.3.14 Decomposition (Subsets), 55
2.3.15 Subsets with Successions, 55
2.3.16 Definition (Skolem Subsets), 55
2.3.17 Decomposition (Skolem Subsets), 56
2.3.18 The Skolem Problem, 56
2.3.19 Definition (Circular Succession), 56
2.3.20 Decomposition (Subsets), 57
2.3.21 Example (Circular Successions), 57
2.3.22 Subsets and Circular Successions, 58

Notes and References 59
Exercises 59

2.4 Sequences 62

2.4.1 Definition (Substring, Subsequence, Block), 63
2.4.2 Definition (Maximal Block), 63
2.4.3 Decomposition ({0, 1}-Sequences), 63
2.4.4 {0, 1}-Sequences and Maximal Blocks of 1's, 63
2.4.5 Decomposition ({0, 1}-Sequences), 64
2.4.6 {0, 1}-Sequences and Maximal Blocks of 0's and 1's, 64
2.4.7 Definition (Type of a Sequence), 65
2.4.8 Sequences and Type, 65
2.4.9 Sequences and Type Restrictions, 65
2.4.10 Compositions and Part Restrictions, 66
2.4.11 Remark (Sequences and Compositions), 67
2.4.12 Ordered Factorizations, 67
2.4.13 Definition (Rise, Level, Fall), 68
2.4.14 Definition (Smirnov Sequence), 68
2.4.15 Decomposition (Smirnov Sequences), 68
2.4.16 The Smirnov Problem, 69
2.4.17 Sequences and Levels, 69
2.4.18 Sequences and Maximal Blocks, 70
2.4.19 Decomposition (Sequences and Falls), 71
2.4.20 The Simon Newcomb Problem, 71
2.4.21 Permutations and Falls, 72
2.4.22 Definition (Sequence with Strictly Increasing Support), 73
2.4.23 Decomposition (Sequences with Strictly Increasing Support), 73
2.4.24 Sequences of Type (2, ..., 2) with Strictly Increasing Support, 73
2.4.25 Definition (Dirichlet Generating Function), 74
2.4.26 Lemma (Product), 74
2.4.27 Definition (Multiplicatively Weight-Preserving Decomposition), 75
2.4.28 Ordered Factorizations and Dirichlet Generating Functions, 75
2.4.29 Remark (Sequences and Ordered Factorizations), 76

Notes and References 76
Exercises 76

2.5 Partitions of an Integer 80

2.5.1 Definition (Partition), 80
2.5.2 **Decomposition (Partitions), 80**
2.5.3 The Number of Partitions, 81
2.5.4 Example (Calculation of $p(6)$), 81
2.5.5 Distinct Parts, 81
2.5.6 Largest Part Exactly m, 82
2.5.7 Definition (Conjugate), 82
2.5.8 **Decomposition (Partitions with Given Largest Part), 82**
2.5.9 All Partitions—Euler's Theorem, 82
2.5.10 Definition (Ferrers Graph, Durfee Square), 83
2.5.11 Definition (Abutment), 84
2.5.12 **Decomposition (All Partitions), 84**
2.5.13 All Partitions—q-Analogue of Kummer's Theorem, 85
2.5.14 All Partitions—Euler's Theorem, 85
2.5.15 Definition—Maximal Triangle, 86
2.5.16 **Decomposition (Partitions with Distinct Parts), 86**
2.5.17 Partitions with Distinct Parts—An Identity, 86
2.5.18 Partitions with Distinct Parts—Euler's Theorem, 87
2.5.19 **Decomposition (Self-conjugate Partitions), 87**
2.5.20 **Decomposition (Self-conjugate Partitions), 88**
2.5.21 Self-conjugate Partitions—An Identity, 88
2.5.22 Self-conjugate Partitions—Euler's Theorem, 89
2.5.23 **Decomposition (Sylvester: All Partitions), 89**
2.5.24 The Jacobi Triple Product Identity, 91
2.5.25 Euler's Theorem for Pentagonal Numbers, 92

Notes and References 93
Exercises 93

2.6 Inversions in Permutations and q-Identities 96

2.6.1 Definition (Inversion), 96
2.6.2 **Algorithm (Inversion), 96**
2.6.3 Proposition, 97
2.6.4 **Lemma (Inversions), 97**
2.6.5 Example (Between-Set and Within-Set Inversions), 98
2.6.6 **Lemma (Between-Set and Within-Set Generating Functions), 98**
2.6.7 Recurrence Equation for q-Binomial Coefficients, 100
2.6.8 Definition (Increasing, Decreasing, Cup-, Cap-permutations), 100
2.6.9 **Lemma (Cup- and Cap-permutations), 101**
2.6.10 **Theorem (Bimodal Permutation), 102**
2.6.11 Corollary (q-Analogue of the Binomial Theorem), 103
2.6.12 Three Finite Product Identities, 104
2.6.13 Proposition, 104
2.6.14 Four Infinite Product Identities, 105

Notes and References 106
Exercises 106

2.7 Planted Plane Trees 109

2.7.1 Definition (Branch, Branch List), 110
2.7.2 **Decomposition (Planted Plane Cubic Trees), 110**

2.7.3 Planted Plane Cubic Trees and Nonroot Monovalent Vertices, 111
2.7.4 **Decomposition (Branch), 111**
2.7.5 Planted Plane Trees and Nonroot Vertices, 112
2.7.6 Definition (Degree Sequence), 112
2.7.7 Planted Plane Trees and Degree Sequence, 112
2.7.8 Planted Plane Trees and Bivalent Vertices, 113
2.7.9 **Decomposition (Planted Plane Trees), 114**
2.7.10 Planted Plane Trees and Bivalent Vertices, by Composition, 114
2.7.11 Planted Plane Trees with No Isolated Bivalent Vertices, 115
2.7.12 Definition (2-Chromatic Tree), 116
2.7.13 **Decomposition (Planted Plane 2-Chromatic Trees), 116**
2.7.14 Planted Plane 2-Chromatic Trees, 116
2.7.15 Definition (Height of a Vertex), 117
2.7.16 Vertices of Given Degree and Height in Planted Plane Trees (First Method), 118
2.7.17 **Decomposition (Planted Plane Trees with a Single Distinguished Vertex of Degree d and Height h), 120**
2.7.18 Vertices of Given Degree and Height in Planted Plane Trees (Second Method), 121
2.7.19 Remark (Finding Decompositions), 121
2.7.20 Definition (Left-most Path), 122
2.7.21 **Decomposition (Left-most Path), 122**
2.7.22 Planted Plane Trees, Left-most Paths, and Degree Sequence, 122

Notes and References 124

Exercises 125

2.8 Sequences with Distinguished Substrings 128

2.8.1 Definition (A-Type of a Sequence), 128
2.8.2 Definition (k-Cluster), 128
2.8.3 Definition (Cluster Generating Function), 129
2.8.4 Definition, 130
2.8.5 Proposition, 131
2.8.6 **Theorem (Distinguished Substring), 131**
2.8.7 Sequences with No pth Powers of Strings of Length k, 132
2.8.8 Sequences and Strictly Increasing Substrings, 134
2.8.9 Definition (Connector Matrix), 135
2.8.10 **Lemma (Cluster Generating Function for an Arbitrary Set), 135**
2.8.11 Example, 136

Notes and References 137

Exercises 137

2.9 Rooted Planar Maps and the Quadratic Method 138

2.9.1 The Quadratic Method, 138
2.9.2 Definition (Planar Map), 139
2.9.3 Proposition (Euler's Polyhedral Formula), 140
2.9.4 Definition (Rooted Planar Map, Root Edge, Root Face, Root Vertex), 140
2.9.5 **Decomposition (Rooted Near-triangulation; Root Edge), 141**
2.9.6 Rooted Near-triangulations and Inner Faces, 142
2.9.7 Rooted Near-triangulations, Inner Faces, and Degree of Outer Face, 143

2.9.8 **Decomposition (Rooted Planar Map; Root Edge), 145**
2.9.9 Rooted Planar Maps, 146
2.9.10 **Decomposition (2-Edge-Connected Rooted Planar Map), 149**
2.9.11 2-Edge-Connected Rooted Planar Maps, 150
2.9.12 **Decomposition (Nonseparable Rooted Planar Map), 151**
2.9.13 Nonseparable Rooted Planar Maps, 152

Notes and References 154

Exercises 154

3 THE COMBINATORICS OF THE EXPONENTIAL GENERATING FUNCTION 158

3.1 Introduction 158

3.1.1 The Elementary Counting Lemmas, 158
3.1.2 The $*$-Product and $*$-Composition, 159
3.1.3 The $*$-Derivative, 159
3.1.4 The Γ-Series, 160

3.2 The Elementary Counting Lemmas 160

3.2.1 Definition (s-Tagged Configuration, Tag Set, Tag Weight), 160
3.2.2 Example (s-Tagged Configurations), 161
3.2.3 Definition (Exponential Generating Function), 161
3.2.4 Proposition, 161
3.2.5 Proposition, 161
3.2.6 **Lemma (Sum), 162**
3.2.7 Example (Exponential Generating Functions), 162
3.2.8 Example (Derangements), 162
3.2.9 Definition ($*$-Product), 163
3.2.10 Example ($*$-Product), 163
3.2.11 **Lemma ($*$-Product), 163**
3.2.12 Example (a Permutation Problem), 164
3.2.13 Example (a Sequence Problem), 165
3.2.14 Definition ($*$-Composition with Respect to s-Objects), 165
3.2.15 Example ($*$-Composition), 166
3.2.16 **Lemma ($*$-Composition), 166**
3.2.17 Partitions of a Set, 167
3.2.18 Definition ($*$-Differentiation), 167
3.2.19 **Lemma ($*$-Differentiation), 168**
3.2.20 Remark (a Construction for the $*$-Derivative), 168
3.2.21 Definition (Alternating Permutation), 168
3.2.22 André's Problem, 169

Notes and References 169

3.3 Trees and Cycles in Permutations and Functions 170

3.3.1 **Decomposition (Derangements), 170**
3.3.2 Derangements (Indirect Decomposition), 170
3.3.3 A Recurrence Equation for the Derangement Number, 171

3.3.4 Definition (Circular Permutation), 171
3.3.5 **Decomposition (Cycle; for Permutations), 171**
3.3.6 Derangements (Direct Decomposition), 172
3.3.7 Involutions, 172
3.3.8 Definition (Labeled Tree), 173
3.3.9 **Decomposition (Labeled Branch), 174**
3.3.10 Rooted Labeled Trees, 174
3.3.11 Proposition, 175
3.3.12 **Decomposition (Cycle; for Functions), 176**
3.3.13 Functions and Cycle Type, 176
3.3.14 Expectation of the Number of Cycles of Given Length in Functions, 176
3.3.15 Idempotent Functions, 177
3.3.16 **Decomposition (Recursive Cycle; for Permutations), 178**
3.3.17 Derangements and Involutions (Recursive Decomposition), 178
3.3.18 Remark (Distinguished Tagged s-Objects), 179
3.3.19 Correspondence (Functions from N_n to N_n-Rooted Labeled Trees), 179
3.3.20 Rooted Labeled Trees (Direct Decomposition), 181
3.3.21 Definition (Spanning Tree), 181
3.3.22 Lemma (Edge-Weighted Tree), 181
3.3.23 **Theorem (Matrix-Tree), 182**
3.3.24 In-Directed and Out-Directed Spanning Arborescences, 183
3.3.25 Matrix-Tree Theorem for Undirected Graphs, 185
3.3.26 Labeled Trees, 185

Notes and References 185

Exercises 186

3.4 2-Covers of a Set and Homeomorphically Irreducible Labeled Graphs 197

3.4.1 **Decomposition ((0, 1)-Matrices), 197**
3.4.2 Example, 198
3.4.3 (0, 1)-Matrices with No Rows or Columns of 0's, 198
3.4.4 Recurrence Equation, 199
3.4.5 A Differential Decomposition for Matrices with No 0 Rows or Columns, 200
3.4.6 Definition (Proper k-Cover, Restricted Proper k-Cover), 201
3.4.7 **Decomposition (Proper 2-Covers), 201**
3.4.8 Proper 2-Covers, 202
3.4.9 Restricted Proper 2-Covers, 203
3.4.10 A Differential Equation for Restricted Proper 2-Covers, 204
3.4.11 A Differential Decomposition for Restricted Proper 2-Covers, 204
3.4.12 **Decomposition (Simple Labeled h-Graphs), 206**
3.4.13 Proposition, 207
3.4.14 Simple Labeled h-Graphs, 207
3.4.15 A Recurrence Equation for Simple Labeled h-Graphs, 209

Notes and References 209

Exercises 209

3.5 Coefficient Extraction for Symmetric Functions 213

3.5.1 Definition (p-Regular Graph), 213
3.5.2 The Ordinary Generating Function for Simple Labeled Graphs, 213
3.5.3 Definition (Monomial Symmetric Function), 214

3.5.4 Proposition, 214
3.5.5 A Differential Equation for Simple Labeled Graphs in Terms of Power Sum Symmetric Functions, 215
3.5.6 Proposition, 216
3.5.7 **Theorem (Γ-Series), 217**
3.5.8 The Γ-Series for Simple 3-Regular Labeled Graphs, 218
3.5.9 A Recurrence Equation for Simple 3-Regular Labeled Graphs, 219
3.5.10 Non-negative Integer Matrices with Line Sum Equal to 2, 221
3.5.11 Differential Decomposition for Simple 3-Regular Labeled Graphs, 224

Notes and References 227

Exercises 227

4 THE COMBINATORICS OF SEQUENCES 230

4.1 Introduction 230

4.2 The Maximal String Decomposition Theorem 231

4.2.1 Definition (π_1-String, Maximal π_1-String Type), 231
4.2.2 Definition (π_1-String Enumerator, Maximal π_1-String Length Enumerator), 232
4.2.3 **Theorem (Maximal String Decomposition), 232**
4.2.4 Definition (Rises, Successions, c-Successions), 233
4.2.5 **Lemma ($<$-Transformation, $+$-Transformation, $\oplus$-Transformation), 233**
4.2.6 Definition (π_1-Alternating Sequence), 234
4.2.7 **Corollary (π_1-Alternating Sequences of Even Length), 234**
4.2.8 $<$-Alternating Permutations of Even Length, 234
4.2.9 $+$-Alternating Permutations of Even Length, 234
4.2.10 $\oplus$-Alternating Permutations of Even Length, 235
4.2.11 $<$-Alternating Permutations of Even Length, Inversions, 235
4.2.12 **Corollary (Sequences, Type, Occurrences of π_1), 235**
4.2.13 Sequences and Rises (Simon Newcomb Problem), 236
4.2.14 Sequences and Levels (Smirnov Problem), 236
4.2.15 **Corollary (Sequences, π_1-Strings of Length p), 237**
4.2.16 Sequences with π_1-Strings of Length 3, 237
4.2.17 Permutations with $<$-Strings of Length 3, Inversions, 237
4.2.18 Permutations with Prescribed Product of Maximal $<$-String Lengths, 237
4.2.19 **Theorem (Maximal String Decomposition; Distinguished Final String), 238**
4.2.20 **Corollary (π_1-Alternating Sequences of Odd Length), 238**
4.2.21 $<$-Alternating Permutations of Odd Length, with Inversions, 239
4.2.22 $<$-Alternating Permutations of Odd Length, 239

Notes and References 239

Exercises 240

4.3 The Pattern Algebra 243

4.3.1 Definition (Pattern), 243
4.3.2 Definition (Incidence Matrix, Set of Encodings), 244

4.3.3 Definition (Fundamental Generating Functions), 245
4.3.4 Proposition (Incidence Matrices for Union and Product), 245
4.3.5 **Lemma (Sum, Product for the Fundamental Generating Functions), 245**
4.3.6 Proposition (π_1-String Enumerator), 246
4.3.7 π_1-Alternating Sequences of Odd Length, 247
4.3.8 **Theorem (Elimination), 248**
4.3.9 Definition (Left-, Right-Expansions), 249
4.3.10 **Algorithm (Factored Expansion), 250**
4.3.11 Remark (General Strategy), 250
4.3.12 Proof of the Maximal String Decomposition Theorem, 250
4.3.13 Definition (Sequence with Repeated Pattern), 251
4.3.14 Sequences with Repeated Pattern $\pi_1^2\pi_2^2$, 252
4.3.15 Permutations with Repeated Pattern $\pi_1^2\pi_2^2$, for Rises, 253
4.3.16 Sequences with Fixed Pattern, 254
4.3.17 Permutations with Fixed Pattern and Inversions, 255
4.3.18 A Tripartite Problem, 256
4.3.19 A q-Identity for the Tangent Function, 257
4.3.20 A q-Identity from Permutations with Repeated Pattern, 258

Notes and References 259
Exercises 259

4.4 The Logarithmic Connection for Circular Permutations 268

4.4.1 Definition (Circular Sequence), 268
4.4.2 **Theorem (Logarithmic Connection), 270**
4.4.3 **Theorem (Maximal String Decomposition for Circular Permutations), 271**
4.4.4 Circular Permutations and Maxima, 272
4.4.5 Directed Hamiltonian Cycles of the Complete Directed Graph with Distinguished Hamiltonian Cycles, 273
4.4.6 Hamiltonian Cycles of the Complete Graph with a Distinguished Hamiltonian Cycle, 275
4.4.7 The Ménage Problem, 275

Notes and References 277
Exercises 277

4.5 Permanents and Absolute Problems 281

4.5.1 Proposition (Permanent), 281
4.5.2 Definition (Absolute Partition for Permutations), 281
4.5.3 **Lemma (Absolute Partition), 281**
4.5.4 Derangements, 282
4.5.5 The Ménage Problem (Absolute), 282
4.5.6 Definition (k-Discordant Permutation), 283
4.5.7 3-Discordant Permutations, 283
4.5.8 Definition (Latin Rectangle), 284
4.5.9 **Decomposition ((3 × n)-Latin Rectangles), 284**
4.5.10 (3 × n)-Latin Rectangles, 285
4.5.11 A Correspondence for Sequences and Absolute Sequences, 286
4.5.12 Permutations Whose Cycles Have Repeated Pattern $\pi_1^2\pi_2^2$, 287

Notes and References 287
Exercises 287