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Problems in Optics

PROBLEMS IN OPTICS

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FOREWORD

COLLECTIONS of problems are useful both for faculty use in the evaluation of the state of a student's knowledge and for the student himself to use in self-evaluation. This collection of problems is at the level of the present state of knowledge expected of a student candidate for certification in optics and many of these problems are, in fact, drawn from certification examinations.

Physical optics is a traditional subject and a very large choice of problems is available in this area. An attempt has been made here to provide a broad selection of modern material using some of the newer experimental and theoretical results and, in addition, those areas of electromagnetic theory relevant to optics.

Quantum optics, which involves the elements of wave mechanics and its applications to atomic and molecular spectroscopy and, thus, to the propagation of electromagnetic radiation in material media, has only recently been introduced into optics courses. As a result of the relatively short experience in the presentation of these techniques, the problems in this area are generally presented at a somewhat lower level than the classical problems in spite of their significance in modern optical work.

An attempt has been made here to find a balance between extreme detail in solution and sufficient detail as to be of use. In general, whenever detail is not presented in the solution, reference is made to the general principle used. References are often given in the form § 8.3 (chapter 8, section 3) or § B.3 (Appendix B, section 3) and are keyed to the complementary volume *Optics*: Part 1, Electromagnetic Optics; Part 2, Quantum Optics, which forms part of this series. References to Appendices A and B of this volume are given in the form Appendix A (or B) and references to Problems (or parts thereof) as Problem 1 (or Problem 1, II. 1, etc.).

Many thanks are due our colleagues who provided us with a selection of problems, thus enhancing our coverage. To these individuals, MM. Boiteaux, Fert, Françon, Jacquinot, Kahane, Nikitine, Rouard, Rousset, Servant, Vienot, goes our gratitude. The solutions, however, are ours, and thus any error in detail or omission must remain with us.

We are also grateful to Professor J. W. Blaker for the accurate translation from the French.

M. R., J. P. M.

PRINCIPAL PHYSICAL CONSTANTS

(MKSA rationalized units)

Avogadro's number	$\mathcal{N}$	$= 6.025\times10^{26}$ molecules/kilomole
Volume of one kilomole of an ideal gas at standard conditions	V_m	$= 22{,}420$ m^3
Ideal gas constant	R	$= 8.3169\times10^{3}$ joules/kilomole-°K
Boltzmann constant	k	$= R/\mathcal{N} = 1.380\times10^{-23}$ joule/°K
Permittivity of free-space	ε_0	$= 8.834\times10^{-12}$ farads/m
Permeability of free-space	μ_0	$= 4\pi\times10^{-7} = 1.257\times10^{-6}$ henrys/m
Faraday's constant	$\mathcal{F}$	$= 96.522\times10^{6}$ coul/kilomole
Electron charge	e	$= 1.602\times10^{-19}$ coul
Rest mass of the electron	m_e	$= 9.1083\times10^{-31}$ kg
Mass of the proton	M_H	$= 1.6724\times10^{-27}$ kg
Specific charge of the electron	e/m_e	$= 1.759\times10^{11}$ coul/kg
Planck's constant	h	$= 6.6252\times10^{-34}$ joule-sec
Speed of light in vacuum	c	$= 2.997\,93\times10^{8}$ m/s
Rydberg constant for H	R_H	$= 10{,}967{,}758$ m^{-1}
Ground state radius of H	r_0	$= 0.5292\times10^{-10}$ m
Bohr magneton	μ_B	$= eh/4\pi m_e = 9.27\times10^{-24}$ A-m^2
Compton wavelength for the electron	λ_c	$= 2h/m_e c = 4.8524\times10^{-12}$ m

Energy conversion factors:

$$1 \text{ calorie} = 4.185 \text{ joules}$$

$$1 \text{ electron-volt} = 1.602\times10^{-19} \text{ joules} = 8068 \text{ cm}^{-1}\ (\times hc)$$

Unless otherwise indicated, these constants will be used for the calculations which follow.

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INTERFERENCE

PROBLEM 1

Visibility of Young's Fringes

In all of these problems assume that the source is monochromatic and radiates at a wavelength $\lambda = 0.55\ \mu$.

I

1. A point source S_0 illuminates two narrow, parallel slits, F_1 and F_2, ruled vertically in an opaque screen. The slits are separated by 2 mm. One observes the interference pattern in a plane π parallel to and at a distance of 1 m from the screen. A point M in the plane π is assigned the coordinates X and Y (Y parallel to the slits). Determine the expression governing the distribution of the illumination over the plane π.

2. How is the image modified when S_0 is replaced by a narrow slit F_0 parallel to F_1 and F_2? Calculate the interference pattern.

3. The observation of the fringes is made using a Fresnel eyepiece similar to a thin lens of focal length $f = 2$ cm. What are the advantages of observation with an eyepiece in comparison to observation with the naked eye? Indicate the positions of the eyepiece and the eye with respect to plane π for which the observation of the fringes is made under the best conditions.

II

Cover the slit F_1 with an absorbing screen (which introduces no phase-shift) of optical density $\Delta = 2$.

$$\left(\text{The optical density is defined by: } \Delta = \log_{10} \frac{\text{incident intensity}}{\text{transmitted intensity}}.\right)$$

Find the visibility, V, of the fringes defined by:

$$V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}},$$

where I_{max} and I_{min} represent the maximum and minimum intensities respectively.

III

Here a large incoherent source is used.

1. The source slit has a height h (fixed) and a width a (variable). This is situated at a distance $d = 1$ m behind the plane of the slits F_1 and F_2. Under these conditions, what is the expression for the illumination at a point M in the plane π? How does the visibility of the fringes, V, vary as a function of a? Use this expression to describe the phenomenon observed when one progressively opens the source slit F_0. Determine the maximum width of the slit so that the loss in contrast does not exceed 10%.

2. To increase the luminosity of the image an incoherently illuminated grating is used as a source (slits parallel to F_1 and F_2). Determine the width a of the transparent intervals and the grating step p so that the visibility retains its preceding value.

IV

1. Assume that the source slit F_0 is sufficiently narrow that it can be considered as a line and replace the Fresnel eyepiece observing apparatus by a photocell. Place the slit of the cell in the plane π parallel to the fringes. The height of the slit is fixed; its width is variable. Assume that the intensity of the photocurrent is proportional to the luminous flux falling on the cell. Give the law for the variation of the current as a function of the abscissa X of the slit. Describe what happens when the slit is opened.

2. What is the expression for the intensity of the current assuming that the source slit is not vanishingly fine but has width a? Determine the visibility factor.

V

1. Take the width of the source slit as $a = 0.01$ mm and the width of the slit of the detector as $b = 0.02$ mm. Find the visibility.

This theoretical visibility V_t is greater than the experimental visibility V_r which has a value $V_r = 0.5$. Show that this can be explained by taking into account the parasitic current $\mathfrak{I}_0$ (dark current) found in the absence of all luminous flux. Calculate the ratio, $\mathfrak{I}_0/\mathfrak{I}_{max}$, of the dark current to the maximum signal intensity.

2. The width of the slit of the detector is fixed by its construction at a value $b = 0.02$ mm, while, on the other hand, the width a of the source slit can be altered.

Calculate V_r and present graphically its variation as a function of a. For what value of a will V_r be maximum? What can be concluded from this investigation?

SOLUTION

I. *Coherent illumination*

1. *Point source*

Designate by x and y the coordinates in the plane of the pupil and by X and Y the coordinates of a point M in the image plane (Fig. 1.1). The infinitely thin slits diffract uniformly in the plane perpendicular to Oy.

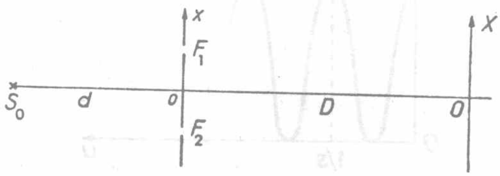

FIG. 1.1

Only the line Ox is illuminated with a light distribution

$$I = 4\cos^2(\pi us) \tag{1}$$

where

$$u = \frac{\sin i}{\lambda} \approx \frac{i}{\lambda} = \frac{X}{D}\frac{1}{\lambda}. \tag{2}$$

One gets this result from the fact that, for coherent illumination, the distribution of the amplitude in the image is equal to the Fourier transform of the amplitude distribution in the pupil (see Appendix A).

The amplitude in the exit pupil is

$$f(x) = \delta\left(x+\frac{s}{2}\right)+\delta\left(x-\frac{s}{2}\right). \tag{3}$$

The amplitude in the image plane is

$$F(u) = \text{F.T.}[f(x)] \tag{4}$$

$$F(u) = \Delta(u)\,[e^{j\pi us}+e^{-j\pi us}] \tag{5}$$

with

$$\Delta(u) = \text{F.T.}[\delta(x)] = 1 \tag{6}$$

from which

$$F(u) = 2\cos\pi us \rightarrow \text{period } 2/s.$$

and

$$I(u) = |F(u)|^2 = 4\cos^2\pi us \rightarrow \text{period } 1/s. \tag{8}$$

2. *Linear source*

Here one observes no interference along the lines parallel to Oy. Each point on the source slit gives a light distribution centred on the geometric image and parallel to Ox. One then has fringes parallel to F_1 and F_2.

The period of these fringes is such that:

$$\Delta u = 1/s \tag{9}$$

giving a linear fringe spacing:

$$\Delta X = \lambda \frac{D}{s}. \tag{10}$$

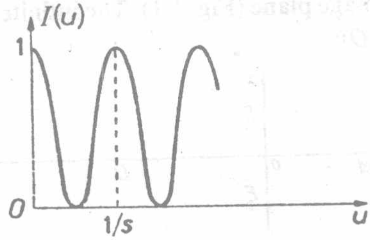

FIG. 1.2

Numerically:

$$\Delta X = 0.55 \times \frac{10^{-3}}{2} = 0.275 \text{ mm}.$$

3. *Observation of the fringes*

Naked eye. A normal eye working at its near point (25 cm) has difficulty in resolving the image. In effect the fringe spacing is seen at an angle:

$$\varepsilon = \frac{0.275}{250} \approx 10^{-3} \text{ rad}.$$

This value is only slightly larger than the angular limit of resolution of the eye which is of the order of 1 minute or 3×10^{-4} rad.

Eyepiece + eye. To avoid fatigue it is preferable that the eye does not accommodate. For this reason one uses an eyepiece whose focal plane coincides with the plane π; the image is then formed at infinity. This image is easily resolvable since the angular fringe spacing becomes

$$\varepsilon = \frac{0.275}{20} = 0.0135 \text{ rad}.$$

The magnification of the eyepiece is

$$G = \frac{\varepsilon'}{\varepsilon} = \frac{\text{angle at which the image is seen}}{\text{angle of the object when at the near point}}.$$

Note. In principle the slits diffract through an angle of 180° so that, even with large aperture, the eyepiece cannot collect all of the rays. The observer, in order to collect the maximum light, must place his pupil in the plane F_1', F_2' conjugate to the plane F_1, F_2 (Fig. 1.3).

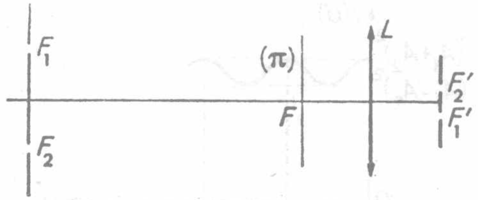

FIG. 1.3

The slits are at a distance ξ from the lens, their images are at a distance ξ', such that:

$$\frac{1}{\xi'}-\frac{1}{\xi}=\frac{1}{f}$$

$$\frac{1}{\xi'}=\frac{1}{100+f}+\frac{1}{f}=\frac{52}{102}$$

$$\xi' = 1.965 \text{ cm} \approx 2 \text{ cm}.$$

The magnification is equal to

$$\frac{\eta'}{\eta}=\frac{\xi'}{\xi}=\frac{1}{52}.$$

The image has dimension

$$\eta'=\frac{\eta}{52}=\frac{2}{52}\approx 0.04 \text{ mm}.$$

All of the rays which enter the eyepiece get to the eye since the value of η' is less than the minimum diameter of the pupil of the eye.

II. *The vibrations passing through F_1 and F_2 are in phase but have different amplitudes*

When the vibrations are out of phase by ϕ, the intensity at point M is given by

$$I(M)=A_1^2+A_2^2+2A_1A_2\cos\phi=I_1+I_2+2\sqrt{I_1I_2}\cos\phi. \tag{11}$$

The maximum and minimum intensities are respectively equal to

$$I_{\max}=(\sqrt{I_1}+\sqrt{I_2})^2$$
$$I_{\min}=(\sqrt{I_1}-\sqrt{I_2})^2,$$

the visibility is

$$V=\frac{2\sqrt{I_1I_2}}{I_1+I_2}. \tag{12}$$

Assuming that the optical density filter is placed in front of F_1 one has:

$$\log_{10}\frac{I_2}{I_1}=2 \quad \text{where} \quad \frac{I_2}{I_1}=100,$$

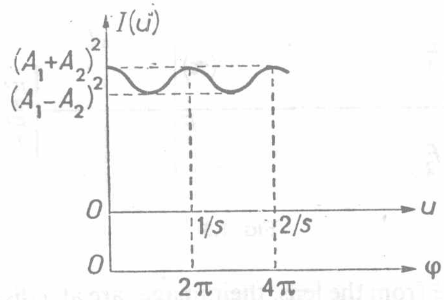

FIG. 1.4

from which $$V = 0.2 \quad \text{(Fig. 1.4)}.$$

The positions of the maxima and minima are the same with or without the filter. On the other hand the visibility, V, is not unity unless the amplitudes passing through the slits are equal.

III. *Large source. Incoherent illumination* (§ 6.7)

1. *The source is a large slit*

All the points lying on a line parallel to Oy give fringes parallel to Oy with period $\Delta u = 1/s$.
Break the slit (width a) into an infinite number of vanishingly thin slits.
Let v be the reduced coordinate of a point in the source plane. The width of the slit can be characterized by

$$v_0 = a/\lambda d. \tag{13}$$

The intensity produced on M by an element of width $\mathrm{d}v$ is

$$\mathrm{d}I = A \times h\{1+\cos 2\pi[(u+v)s]\}\,\mathrm{d}v. \tag{14}$$

A = constant, λvs = path difference between the disturbances arriving from F_1 and F_2.
Each elementary slit of infinitesimal width gives a system of fringes with period $\Delta u = 1/s$ and centred on the geometric image of the elementary slit.
Thus, the intensity transmitted to M by the slit source is

$$I = Ah \int_{-v_0/2}^{+v_0/2} [1+\cos 2\pi(u+v)s]\,\mathrm{d}v \tag{15}$$

$$I = I_0\left[1+\frac{\sin \pi v_0 s}{\pi v_0 s}\cos 2\pi us\right]. \tag{16}$$

One finds:

$$V = \frac{\sin \pi v_0 s}{\pi v_0 s}.$$

The graph of V is given in Fig. 1.5.

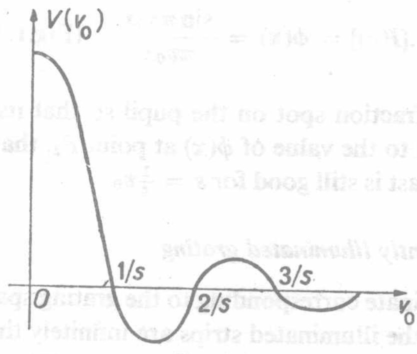

FIG. 1.5

Numerical application. One wants $V \geqslant 0.9$ so that

$$\frac{\sin \pi v_0 s}{\pi v_0 s} = 0.9 \rightarrow \pi v_0 s = \frac{\pi}{4} \rightarrow v_0 = \frac{1}{4s}.$$

From the definition of v_0 one gets

$$\frac{1}{4s} = \frac{a}{\lambda d} \quad \text{so that} \quad a = d\frac{\lambda}{4s} = 10^6 \times \frac{0.55 \times 10^{-3}}{4 \times 2}$$

$$V = 0.9 \quad \text{for} \quad a \approx 70\ \mu.$$

The fringes vanish for $a = 275\ \mu$.

The Van Cittert–Zernike theorem (Appendix B) gives this result immediately. The degree of coherence between the slits F_1 and F_2 is given by the Fourier transform of the intensity distribution in the source plane. Since the problem is one-dimensional, it is sufficient to assume that the source is a slit parallel to OY with a width a and that the pupil is formed by two points, P_1 and P_2, set in an opaque screen (P_1 and P_2 corresponding to the intersection of the slits F_1 and F_2 with the line Ox are separated by a distance s). The intensity distribution in the source can be represented by a rectangular function (Fig. 1.6).

$$I(v) = 0 \quad \text{for} \quad v < -v_0/2 \quad \text{and} \quad v > +v_0/2, \tag{17}$$
$$I(v) = 1 \quad \text{for} \quad -v_0/2 < v < +v_0/2.$$

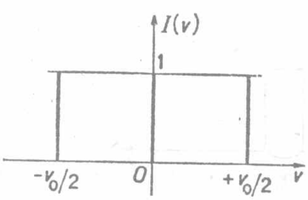

FIG. 1.6

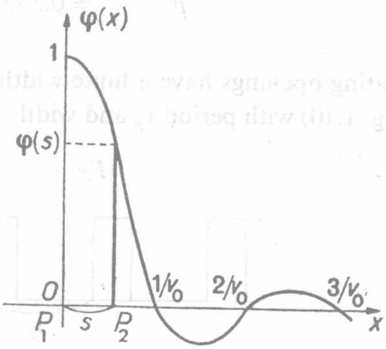

FIG. 1.7

One finds

$$F.T.[I(v)] = \phi(x) = \frac{\sin \pi v_0 x}{\pi v_0 x} \quad \text{(Fig.1.7)}. \tag{18}$$

Place each imaginary diffraction spot on the pupil so that its centre coincides with P_1. The fringe visibility is equal to the value of $\phi(x)$ at point P_2, that is, at $\phi(s)$ (Fig. 1.7). One can see that the fringe contrast is still good for $s = \frac{1}{4}v_0$.

2. *The source is an incoherently illuminated grating*

Call v_p the reduced coordinate corresponding to the grating spacing p.

(a) Assume initially that the illuminated strips are infinitely thin.

The intensity distribution in the source is a Dirac series (Fig. 1.8). Its Fourier transform is a Dirac series of period $1/v_p$ (Fig. 1.9).

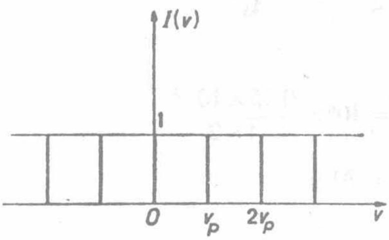

FIG. 1.8

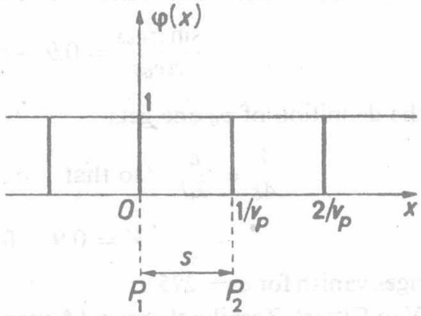

FIG. 1.9

As before, place the imaginary diffraction spot $\phi(x)$ on the pupil so that $\phi(0)$ coincides with P_1. The fringe visibility will be unity if

$$1/v_p = s \quad \text{(Fig. 1.9)}$$

that is, if

$$s = \lambda d/p,$$

so that

$$p = \frac{\lambda d}{s} = 0.55 \times \frac{10^3}{2} = 275\ \mu.$$

(b) The grating openings have a finite width a. $I(v)$ is a unbounded series of rectangular functions (Fig. 1.10) with period v_p and width v_0.

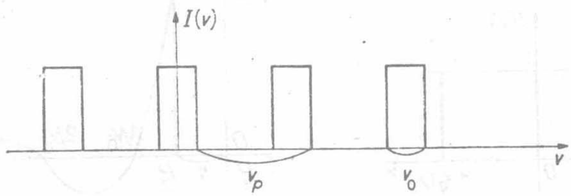

FIG. 1.10

The Fourier transform is shown in Fig. 1.11.

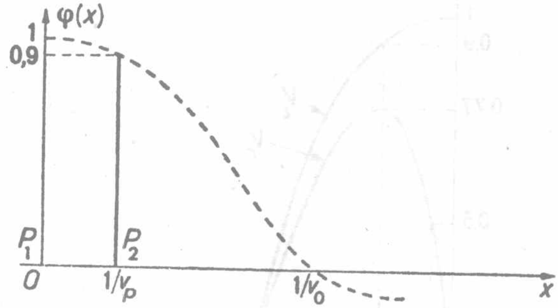

FIG. 1.11

To have an image well contrasted one needs

$$s = \frac{1}{\nu_p} = \frac{1}{4\nu_0}.$$

Numerical results

Grating spacing $p = 275\ \mu$.

Width of the grating openings $a = 70\ \mu$.

Note. One can also get these results by another simple process (Fig. 1.12).

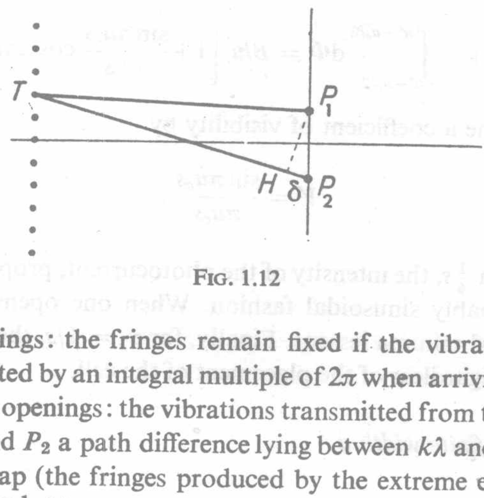

FIG. 1.12

(a) Fine grating openings: the fringes remain fixed if the vibrations transmitted by an opening T are phase-shifted by an integral multiple of 2π when arriving at P_1 and P_2.

(b) Grating with large openings: the vibrations transmitted from the edges of any window should produce at P_1 and P_2 a path difference lying between $k\lambda$ and $(k+\frac{1}{8})\lambda$ in which case the fringes do not overlap (the fringes produced by the extreme edges of an opening are shifted by a maximum of $\frac{1}{4}$ fringe).

IV. *The opening of the detector has finite width b*

1. *The source slit is infinitely thin*

The fringes on plane π have unit contrast (see question I). On the other hand, because of the finite width of the detector slit, the flux recorded by the receiver is never zero (Fig. 1.13). The illumination is the same at all points along a single vertical in the observing plane. Break the window of the receiving cell down into elements of width du and height l.

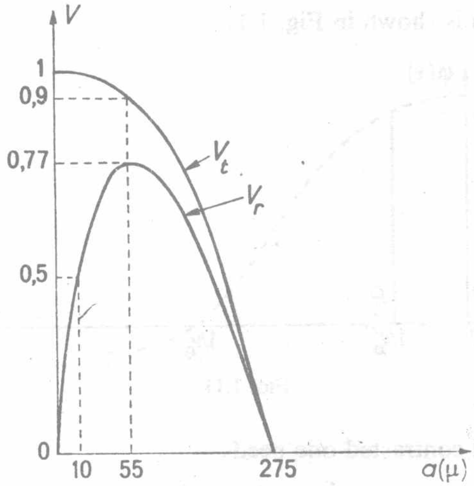

FIG. 1.13

Call u_c the reduced coordinate which corresponds to the linear width b of the slit. The flux which penetrates through the surface element at abscissa u' is

$$d\Phi = Bl(1+\cos 2\pi us)\,du, \tag{19}$$

from which

$$\Phi(u') = \int_{u'-u_c/2}^{u'+u_c/2} d\Phi = Blu_c\left[1+\frac{\sin \pi u_c s}{\pi u_c s}\cos 2\pi u' s\right]. \tag{20}$$

As before, one can define a coefficient of visibility by

$$V = \frac{\sin \pi u_c s}{\pi u_c s}. \tag{21}$$

As long as u_c is less than $\frac{1}{4}s$, the intensity of the photocurrent, proportional to the luminous flux, varies in a reasonably sinusoidal fashion. When one opens the slit, the difference between the maxima and minima lessens. Finally, for $u_c = 1/s$, the intensity of the photocurrent does not vary regardless of the placement of the cell.

2. *The source slit has a finite width a*

One has

$$I(u) = I_0\left[1+\frac{\sin \pi v_0 s}{\pi v_0 s}\cos 2\pi us\right], \tag{22}$$

from which

$$\Phi(u') = Blv_0\int_{u'-u_c/2}^{u'+u_c/2}\left[1+\frac{\sin \pi v_0 s}{\pi v_0 s}\cos 2\pi us\right]du, \tag{23}$$

$$\Phi(u') = Blu_c v_0\left[1+\frac{\sin \pi u_c s}{\pi u_c s}\times\frac{\sin \pi v_0 s}{\pi v_0 s}\cos 2\pi u' s\right]. \tag{24}$$