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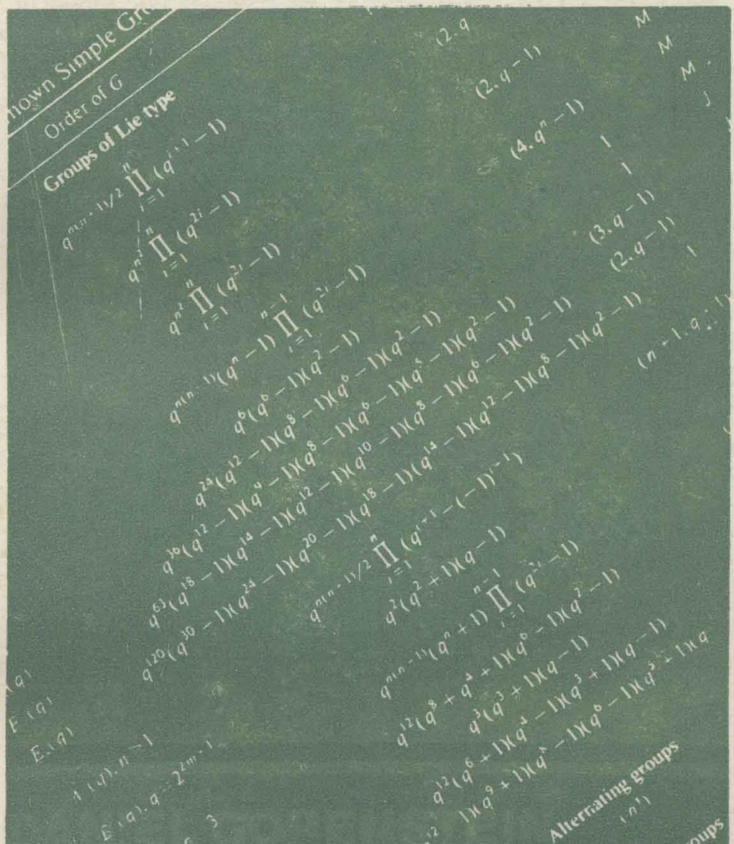
FINITE SIMPLE GROUPS

AN INTRODUCTION TO THEIR CLASSIFICATION



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Finite Simple Groups
An Introduction to Their Classification

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Introduction

In February 1981, the classification of the finite simple groups (D1)* was completed,[†, ‡] representing one of the most remarkable achievements in the history or mathematics. Involving the combined efforts of several hundred mathematicians from around the world over a period of 30 years, the full proof covered something between 5,000 and 10,000 journal pages, spread over 300 to 500 individual papers.

The single result that, more than any other, opened up the field and foreshadowed the vastness of the full classification proof was the celebrated theorem of Walter Feit and John Thompson in 1962, which stated that every finite group of odd order (D2) is solvable (D3)—a statement expressible in a single line, yet its proof required a full 255-page issue of the *Pacific Journal of Mathematics* [93].

Soon thereafter, in 1965, came the first new *sporadic* simple group in over 100 years, the Zvonimir Janko group J_1, to further stimulate the

*To make the book as self-contained as possible, we are including definitions of various terms as they occur in the text. However, in order not to disrupt the continuity of the discussion, we have placed them at the end of the Introduction. We denote these definitions by (D1), (D2), (D3), etc.

† The classification theorem asserts that an arbitrary finite simple group is necessarily isomorphic to one of the groups on a specified list of simple groups. (See Section 2.11 for the detailed list.)

‡ The final mathematical step of the classification was carried out by Simon Norton of the University of Cambridge, in England, establishing the "uniqueness" of the Fischer–Griess sporadic simple group F_1. Griess had earlier constructed F_1 in terms of complex matrices of degree 196,883. Moreover, Thompson had shown that there existed at most one simple group of "type F_1" which could be so represented by complex matrices of this degree. What Norton did was to prove that any group of type F_1 could, in fact, be represented by such matrices. (All these results are described more fully in Section 2.10.)

We also note that as of February 1981 several manuscripts (including Norton's) concerning the classification were still in preparation.

interest of the mathematical community in finite simple groups [187]. The sporadic groups acquired their name because they are not members of any infinite family of finite simple groups. Emile Mathieu, in 1861, had discovered five such groups [210–212], yet J_1 remained undetected for a full century, despite the fact that it has only 175,560 elements (a very small number by the standards of simple group theory). Then in rapid succession over the next ten years, 20 more sporadic groups were discovered, the largest of these the group F_1 of Bernd Fischer and Robert Griess (recently constructed by Griess [152]) of order 808, 017, 424, 794, 512, 875, 886, 459, 904, 961, 710, 757, 005, 754, 268, 000, 000, 000 (approximately 10^{54}), and because of its size, originally dubbed the "monster." An additional intriguing aspect of these new sporadic groups is the fact that several have depended upon computer calculations for their construction.

The pioneer in the field was Richard Brauer, who began to study simple groups in the late 1940s. He was the first to see the intimate and fundamental relationship between the structure of a group and the *centralizers* (D4) of its *involutions* (elements of order 2; D5), obtaining both quantitative and qualitative connections. As an example of the first, he showed that there are a finite number of simple groups with a specified centralizer of an involution [46]. As an example of the second, he proved that if the centralizer of an involution in a simple group G is isomorphic to the general linear group $GL(2, q)$ (D6) over the finite field with q elements, q odd, then either G is isomorphic to the three-dimensional projective special linear group $L_3(q)$ (D7), or else $q = 3$ and G is isomorphic to the smallest Mathieu group M_{11} of order $8 \cdot 9 \cdot 10 \cdot 11$ [40, 42]. This last result, which Brauer announced in his address at the International Congress of Mathematicians in Amsterdam in 1954, represented the starting point for the classification of simple groups in terms of the structure of the centralizers of involutions. Moreover, it foreshadowed the fascinating fact that conclusions of general classification theorems would necessarily include sporadic simple groups as exceptional cases.

In the early years, Brauer had been essentially a lone figure working on simple groups, although Claude Chevalley's seminal paper of 1955 on the finite groups of Lie type [66] had considerable impact on the field. By the late 1950s two disciples of Brauer, Michio Suzuki and Feit, had joined the battle. However, it was the Feit–Thompson theorem that provided the primary impetus for the great expansion of the study of simple groups. The field literally exploded in the 1960s, with a large number of talented young mathematicians attracted to the subject, primarily in the United States, England, Germany, and Japan. For the next fifteen years, the papers came pouring out—long, long papers: Thompson's classification of *minimal* simple groups (i.e., simple groups in which all proper subgroups are solvable), 410 pages in six parts, the first appearing in 1968 and the last in 1974 [289]; John Walter's classification of simple groups with abelian Sylow 2-subgroups, 109 pages of the *Annals of Mathematics* in 1969 [314]; the Alperin–Brauer–Gorenstein classification of simple groups with quasi-dihedral or wreathed Sylow 2-subgroups (see D7), 261 pages of the *Transactions of the American Mathematical Society* in 1970 [3]; the Gorenstein–Harada classification of simple groups whose 2-subgroups are generated by at most 4 elements, a 461-page *Memoir of the American Mathematical Society* in 1971 [136], to name but a few. Even near the end, we find Michael Aschbacher's fundamental "classical involution" theorem, 115 pages of the *Annals of Mathematics* in 1977 [13].

Furthermore, the search for new simple groups was keeping pace with this effort, with a discovery rate of roughly one per year. The phenomenon can be compared with elementary particle theory, in which one must scan a large horizon with the aid of one's intuition and theoretical knowledge in the hope of distinguishing a new particle. If Janko's group J_1 has been constructed from the centralizer of one of its involutions [isomorphic to $Z_2 \times L_2(4)$], then examine other likely candidates as potential centralizers of involutions in a new simple group. If Janko's second group J_2 turns out to be a *transitive rank* 3 permutation group (D8), conduct a more general investigation of such permutation groups. If the automorphism group of the remarkable 24-dimensional Euclidean lattice of John Leech yields John Conway's three simple groups .1, .2, .3, then look for other integral Euclidean lattices that may have a "large" automorphism group. Any plausible direction is worth considering; just keep in mind that the probability of success is very low. In the end, several sporadic groups were discovered by both the centralizer-of-involution and rank 3 permutation group approach, but unfortunately the study of integral lattices yielded no further new groups.

Another aspect of sporadic group theory makes the analogy with elementary particle theory even more apt. In a number of cases (primarily but not exclusively those in which computer calculations were ultimately required), "discovery" did not include the actual construction of a group—all that was established was strong evidence for the existence of a simple group G satisfying some specified set of conditions X. The operative metamathematical principle is this: if the investigation of an arbitrary group G having property X does not lead to a contradiction but rather to a "compatible" internal subgroup structure, then there exists an actual group with property X. In all cases, the principle has been vindicated; however, the interval between discovery and construction has varied from a few months to several years. Although the major credit has usually gone to the initial discoverer, existence and uniqueness were often established by others, or at least with someone else's assistance.

The excitement generated by the discovery (and construction) of these new simple groups was intense. Moreover, there was a long period in which it was felt that there might well exist infinitely many sporadic groups (from the point of view of classification at least, this was a disturbing thought, since this possibility would very likely have precluded the achievement of a complete classification of simple groups. A certain haphazard, almost random quality accompanied the search; some of the groups seemed literally plucked from thin air. I have always felt great admiration for the remarkable intuition of these indefatigable explorers.

It is essential to distinguish between the notion of *discovery* (including construction) and *classification*. One can search for a new simple group in any direction, and discovery is its own reward, requiring no further theoretical justification. However, in contrast, the solution of a general classification problem must be systematic and all-inclusive—*every* simple group with the specified property must be determined. In particular, the analysis must uncover every sporadic group satisfying the given conditions, previously discovered or not. For example, Fischer's first three sporadic groups, $M(22)$, $M(23)$, and $M(24)$, were discovered and constructed in the process of proving just such a classification theorem [97, 98]. Likewise, some years earlier, Suzuki's exceptional family of groups of Lie type of characteristic 2 was discovered in the process of classifying groups in which the centralizer of every involution has order a power of 2 [276, 278].

Whatever one's attitude toward the possible number of sporadic groups, it was certainly true that a complete classification of the finite simple groups was regarded at that time as very remote, for the steady stream of developments was producing as much turmoil as light. The chaotic state of affairs was well expressed in the verses of a song entitled "A Simple Song" (to be sung to the tune of "Sweet Betsy from Pike")), published in the *American Mathematical Monthly* (1973, p. 1028), and summed up in its final stanza:

> No doubt you noted the last lines don't rhyme.
> Well, that is, quite simply, a sign of the time.
> There's chaos, not order, among simple groups;
> And maybe we'd better go back to the loops.

I believe I have the distinction of being the original optimist regarding a possible classification of the simple groups. Even as early as 1968, in the final section of my book *Finite Groups* [130], I had placed great emphasis on

Thompson's classification of minimal simple groups, a magnificent result in which Thompson showed the fundamental significance of the "local" methods of the odd-order paper for the study of simple groups. In my comments, I suggested that his techniques might well be applicable to much broader classification problems. Over the next few years my thoughts in this direction continued to evolve, and gradually I developed a global picture of how it might be possible to carry through a complete classification. At J. L. Alperin's urging, I presented these ideas at a group theory conference at the University of Chicago in 1972, and in four lectures I laid out a *sixteen*-step program for classifying the finite simple groups [132; Appendix]. The program was met with considerable skepticism. I doubt that I made any converts at that time—the pessimists were still strongly in the ascendancy.

However, in the next few years substantial progress was made on some of the individual portions of the program: the complete classification of "nonconnected" simple groups, the first inroads into the "*B*-conjecture," and a deeper understanding of the structure of centralizers of involutions in groups of "component" type. Aschbacher, who had entered the field during this period, came on now like a whirlwind, moving directly to a leadership position and sweeping aside all obstacles, as he proved one astonishing result after another. Within five years, the program, which at its formulation had been a far-off dream, began to take on a sense of immediacy, with genuine prospects for fulfillment.

Hardly surprisingly, individual steps of the program had to be modified along the way—in 1972, the key notions of "tightly embedded subgroup," "Aschbacher block," and George Glauberman's entire theory of "pushing up" had not even existed. In addition, the program overemphasized the role of the prime 3 in the analysis of groups of "characteristic 2 type." But more significantly, in 1972 I had not appreciated the far-reaching impact that Fischer's "internal geometric" approach would have for the study of simple groups. Despite these shortcomings, the overall program remained largely intact, so that at all times we had a way of measuring the extent of our achievements and could describe quite accurately the steps remaining to complete the classification.

The turning point undoubtedly occurred at the 1976 summer conference in Duluth, Minnesota. The theorems presented there were so strong that the audience was unable to avoid the conclusion that the full classification could not be far off. From that point on, the practicing finite group theorists became increasingly convinced that the "end was near"—at first within five years, then within two years, and finally momentarily. Residual skepticism was confined largely to the general mathematical community, which quite reasonably would not accept at face value the assertion that the classification theorem was "almost proved."

But the classification of the finite simple groups is not an ordinary theorem: it makes more sense to think of it as an entire field of mathematics, in which by some accident the central questions become incorporated into the statement of a single theorem. Indeed, as in any other broad field of mathematics, later results depend upon earlier theorems. Thus, all the major results after 1963 depend crucially upon the solvability of groups of odd order, and none provides an alternate proof. As one can well imagine, the logical interconnections between the several hundred papers making up the classification proof are often very subtle, and it is no easy task to present a completely detailed flow diagram.

Because of the excessive lengths of the papers and the specialized techniques developed for the study of simple groups, the classification proof has remained quite inaccessible to non-finite group theorists. Not because of lack of interest: indeed, many mathematicians have followed the developments rather closely, especially those related to the sporadic groups and the groups of Lie type. But very few have managed to penetrate beneath these "boundary" aspects of the subject to the core of the classification proof. Even within the field, many finite group theorists, working in a specialized area of the subject, have had similar difficulties developing a global picture of the full classification proof.

I hope that this detailed outline of the classification theorem will represent a first step toward correcting this situation by illuminating the broad features of the subject of simple groups—the known simple groups themselves, the techniques underlying the classification, and the major components of the classification proof.

The concept of a group is so central to mathematics, it is difficult to imagine that ideas that have been so fruitful for the study of simple groups will find no further mathematical applicability. The "signalizer functor" theorem, the "classical involution" theorem, the "B-property" of finite groups, the "root involution" theorem, the "$C(G; T)$" theorem—to name but a few of the basic results—have such conceptually natural statements, one would certainly hope that analogues of at least some of these theorems exist for some families of rings or algebras. Thus, I have had in mind the subsidiary objective of enabling (primarily) algebraists, number theorists, and geometers to consider the central ideas of finite simple group theory in relation to their own fields.

Although this book is clearly aimed at a mathematical audience, portions of it should be of interest to physicists, crystallographers, and perhaps other theoretical scientists as well—especially the descriptions of the known finite simple groups: the construction of the groups of Lie type from their associated Lie algebras, the origins and definitions of each of the sporadic groups, the list of orders of the known simple groups, many of their basic properties, etc.

For expository purposes it has seemed best to split the total endeavor into two parts. In this book we present a global picture of the classification proof (Chapter 1), a detailed description of the known simple groups, including the sporadic groups (Chapters 2, 3), and a long discussion of the major techniques underlying the proof of the classification theorem (Chapter 4). Its contents represent an updated and considerably expanded version of an article written for the Brauer memorial issue of the *Bulletin of the American Mathematical Society*, published in January 1979 [132].

We hope that the material covered here, which can be regarded as preparatory to the classification proof itself, will stimulate the reader to pursue the more detailed outline, which will form the contents of two further books. However, by itself, this book should provide the reader with considerable insight into simple group theory: in particular, an overall picture of the fundamental four-part subdivision of the classification proof, the group-theoretic origins and definitions of each of the known simple groups, and a good feeling for the methods that have been developed for the study of simple groups.

I have attempted to make the book as self-contained as reasonably possible, so that it will be accessible to anyone with a sound mathematical background and a modest knowledge of abstract algebra. In particular, I have included the definition of essentially every term used in the text (even such basic notions as simple group and Sylow's theorem). Furthermore, even though the organizational structure follows a precise logical pattern, I have tried to make the individual sections more or less independent, so that a selective reading of the text is possible. Moreover, I have included only a few proofs (and outlines of proofs); these have been selected either because of the intrinsic importance of the result or as illustrations of a group-theoretic technique in action. However, I do not mean to imply that the book will necessarily make easy reading, for finite simple group theory involves many deep and difficult concepts whose applicability is often very elaborate.

There is one aspect of the classification "proof" that we must mention before concluding this introduction. Indeed, many of the papers on simple groups are known to contain a considerable number of "local" errors. The fact that it seems beyond human capacity to present a closely reasoned argument of several hundred pages with absolute accuracy may provide the explanation, but this explanation does not eliminate doubt concerning the validity of the proof. Most of these errors, when uncovered, can be fixed up "on the spot." But many of the arguments are ad hoc, so how can one be certain that the "sieve" has not let slip a configuration leading to yet another simple group? The prevailing opinion among finite group theorists is that the overall proof is accurate and that with so many individuals working on simple groups during the past fifteen years, often from such differing perspectives, every significant configuration has loomed into view sufficiently often and could not have been overlooked.

However, clearly the first task of the post-classification era must involve a reexamination of the proof to eliminate these local errors. That reexamination will have two other objectives as well. First, because the full proof evolved over a thirty-year period, some of the early papers were written without the benefit of subsequent developments. Second, because of the lengths of most of the major papers, prior results were usually quoted wherever possible, even when a slight additional argument might have avoided a particular reference.

Thus the local errors will most likely be corrected as part of the broader task of "revising" the existing classification proof in an attempt to discover its "essential" core. Helmut Bender actually began this effort ten years ago, making significant simplifications in the local group-theoretic portions of the odd-order paper [28]. The "Bender method," as his approach came to be called, subsequently developed into a standard technique, finding application in several classification problems (see Sections 4.3, 4.8). But only very recently, with the full classification proof nearly in view, have finite group theorists begun to systematically consider a global reexamination.

The outline I shall present is meant to be a historical summary of the original classification proof. Thus, apart from Bender's work, which has already become an integral part of the proof, I shall avoid discussion in the body of the text of any recently achieved revisions. In the final chapter of the sequel, I shall briefly describe some of the revisionist game plans for improving the classification proof.

Finally, except for a few passing remarks, I shall not discuss the remarkable, recently discovered connections between the Fischer–Griess group F_1 and classical elliptic function theory. These connections have their origin in the serendipitous observation of John McKay that the coefficient of q in the expansion at infinity of the elliptic modular function $J(q)$ is 196,884, while the minimal degree of a faithful irreducible complex representation of F_1 is 196,883. Although considerable "numerological" interconnections have since been uncovered [70, 195, 298], the deeper explanation of this relationship remains a mystery. However, because approximately 20 of the 26 sporadic groups are embedded one way or the other in F_1, there is a distinct possibility that there may ultimately exist a unified, coherent description of most of the sporadic groups. Although these developments are not needed for the classification theorem per se, they do indicate that an interest in the finite simple groups will persist long after the classification.

Definitions for the Introduction

D1. A group G is *simple* if its only normal subgroups are the entire group G and the trivial subgroup consisting of the identity element 1 of G. In general, a subgroup X of a group G is *normal* if $g^{-1}xg \in X$ for every $x \in X$ and $g \in G$. When this is the case, the set of (right) cosets of X in G form a group, called the *factor* or *quotient* group of G by X and denoted by G/X. Multiplication in G/X is given by the rule $(Xg)\cdot(Xg') = X(gg')$ for $g, g' \in G$. Here, for a given $g \in G$, the *coset* Xg denotes the set of elements xg with x ranging over X. The mapping ϕ: $g \mapsto Xg$ is a homomorphism of G onto G/X. Every homomorphic image of G is easily shown to arise in this fashion. Thus a group G is simple if and only if its only homomorphic images are itself and the trivial group.

D2. The *order* of a group is the number of its elements.

D3. A group is *solvable* if it possesses a normal series with abelian (i.e., commutative) factors. A *normal series* of a group G is a chain of subgroups $G = G_1, G_2, \ldots, G_n = 1$ with each G_i normal in G_{i-1}, $2 \leq i \leq n$. The quotient groups G_{i-1}/G_i are called the *factors* of the series.

D4. The *centralizer* of a subset X of a group is the set of elements of G that commute elementwise with X, i.e., the set of $g \in G$ such that $g^{-1}xg = x$ for all $x \in X$. The *center* of G is the set of elements of G whose centralizer is G itself—i.e., the set of $x \in G$ that commute with every element of G.

D5. The *order* of an element x of a group G is the order of the *cyclic* subgroup that it generates, i.e., the subgroup $\{x^i | i \in \mathbf{Z}\}$.

D6. $GL(n, q)$ is the group of all nonsingular $n \times n$ matrices with entries in the finite field $GF(q)$ with q elements. It has a normal subgroup $SL(n, q)$, the *special linear* group, consisting of those matrices of determinant 1. The factor group of $SL(n, q)$ by its subgroup of scalar matrices (of determinant 1) is the *projective special linear* group and is denoted by both $PSL(n, q)$ and $L_n(q)$. It is known to be simple when $n \geq 3$ or $n = 2$ and $q \geq 4$.

D7. A 2-group S is *quasi-dihedral* if S is generated by elements x, y subject to the relations $x^2 = y^{2^n} = 1$, $x^{-1}yx = y^{-1+2^{n-1}}$, $n \geq 3$; and S is *wreathed* if S is generated by elements x, y, z, subject to the relations $x^{2^n} = y^{2^n} = z^2 = 1$, $xy = yx$, $z^{-1}xz = y$, $z^{-1}yz = x$, $n \geq 2$.

D8. The group of all permutations of a (finite) set Ω (i.e., one-to-one transformations of Ω onto itself) under the natural operation of composition is called the *symmetric* group on Ω. Any subgroup X of the symmetric group is called a *permutation* group (on Ω). The cardinality of Ω is called the *degree* of X. X is *k-fold transitive* on Ω if any two ordered k-tuples of distinct elements of Ω can be transformed into each other by an element of X. One writes *transitive*, *doubly transitive*, *triply transitive*, etc., for 1-fold, 2-fold, 3-fold transitive, etc. X has (permutation) *rank r* if X is transitive on Ω and the subgroup of X fixing a point of Ω has exactly r orbits on Ω. Thus double transitivity is equivalent to permutation rank 2.

1

Local Analysis and the Four Phases of the Classification

1.1. From Character Theory to Local Analysis

It seems best to begin with an explanation of the historical origins and general meaning of the primary underlying method of the classification proof: *local group-theoretic analysis*. This was not always the principal approach to the study of simple groups, for Brauer's methods were almost entirely *representation-* and *character-theoretic* (D1).* In the middle 1930s he had introduced and developed the concept of *modular* characters (see D1) of a finite group. He soon realized the power of these ideas, which played an instrumental role in his proof of a conjecture of Artin on L-series in algebraic number fields, and he saw how they could be applied to obtain deep results concerning the structure of simple groups.† From the middle 1940s until his death, Brauer systematically developed the general theory of modular characters and *blocks* of irreducible characters (D2), with increasingly significant applications to simple group theory.

These methods were especially suited for investigating "small" simple groups: complex linear groups of transformations of low dimension, alternating groups of low degree (D3), groups with very restricted Sylow 2-subgroups [e.g., quaternion, dihedral, quasi-dihedral, wreathed, abelian, etc. (D4)]. This was certainly fortunate, since at the outset of the study of simple groups, it was obviously most natural to focus on the smallest ones.

*Again we include some definitions, denoted (D1), (D2), etc., which we have placed at the end of the section (see the footnote on page 1). We follow the same procedure in Sections 1.2 and 1.3.

†Throughout the text the term *simple* group will always refer to a *nonabelian* simple group.

Indeed, the methods were so effective that in the early years there was a strong conviction that character theory would remain a principal tool—perhaps even *the* most essential—for investigating simple groups (even though it was also recognized that larger-rank situations would involve considerable computational difficulties).

However, even in treating small groups, the method had drawbacks, for the way it worked was this: if one had rather precise information about the structure of some subgroup H of G (such as the centralizer of an involution), one could relate the characters of H to those of G and use this connection to obtain conclusions about the structure of G. This was the thrust of the Brauer method.

The difficulties arise if one asks a broad enough question, for then one cannot assert *a priori* that any critical subgroup H of G has a restricted shape. I should like to illustrate this point by considering a specific classification problem, namely, the determination of all simple groups G of order $p^a q^b r^c$, p, q, r primes with $p < q < r$. In view of the classical Burnside theorem that all groups of order $p^a q^b$ are solvable [see Theorem 4.130], this is a problem of natural interest. Among the known simple groups, there are exactly *eight* whose orders have this form (see D5):

$$A_5, \quad A_6, \quad L_2(7), \quad L_2(8), \quad L_2(17), \quad L_3(3), \quad U_3(3), \quad \text{and} \quad U_4(2).$$

Each of these can certainly be considered to be a "small" group (the largest order is, in fact, 25,920). (Note that in these groups $p=2$, $q=3$, and $r=5, 7, 13$, or 17.)

The obvious conjecture is that an arbitrary simple group of order $p^a q^b r^c$ is necessarily isomorphic to one of these eight groups.

For brevity, let us call these eight groups *K_3-groups* and an arbitrary group of order $p^a q^b r^c$ a (p, q, r)-group, so that our conjecture assumes the form:

A simple (p, q, r)-group is necessarily a K_3-group.

Of course, until shown otherwise, our conjecture may be false. As in every general classification problem, it is therefore natural to focus attention on a *minimal* counterexample G to the conjecture, i.e., a simple (p, q, r)-group G of *least* order which is not a K_3-group. Clearly, to establish the conjecture, we must show that no such group G exists—equivalently, that G must, in fact, be a K_3-group. The advantage of considering a minimal counterexample is that if H is any proper subgroup of G, then the (nonsolvable) composition factors of H (D6) are simple (p, q, r)-groups of