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Edited by A. Dold and B. Eckmann

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Adviser: Roberto Conti

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E. Sernesi (Ed.)

Theory of Moduli

Montecatini Terme, 1985

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Lectures given at the 3rd 1985 Session of the
Centro Internazionale Matematico Estivo (C.I.M.E.)
held at Montecatini Terme, Italy, June 21–29, 1985

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continued on page 235

INTRODUCTION

This volume contains the texts of the three main lecture series given at the CIME session on "Theory of moduli" held in Montecatini during the period 21-29 June, 1985.

The lectures survey some important areas of current research in topology, complex analysis, algebraic geometry, which have as their common denominator the study of moduli spaces. Hopefully, this volume will be a useful reference text on the subject.

Other, more specialized, lectures were also given during the session but they are not reproduced here.

I am very grateful to the three authors, to the other lecturers and to all the participants to the conference for their interest and collaboration.

My thanks go also to the CIME for making the conference possible.

Edoardo Sernesi

C.I.M.E. Session on "Theory of Moduli"

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TABLE OF CONTENTS

MODULI OF ALGEBRAIC SURFACES

F. Catanese* - Università di Pisa**

Contents of the Paper

Introduction

This paper reproduces with few changes the lectures I actually delivered at the C. I. M. E. Session in Montecatini, with the exception of most part of one lecture where I talked at length about the geography of surfaces of general type: the reason for not including this material is that it is rather broadly covered in some survey papers which will be published shortly ([Pe], [Ca 3], [Ca 2]).

Concerning my original (too ambitious) intentions, conceived when I accepted Eduardo Sernesi's kind invitation to lecture about moduli of surfaces, one may notice some changes from the preliminary program: the topics "Existence of moduli spaces for algebraic varieties" and "Moduli via periods" were not treated. The first because of its broadness and complexity (I realized it might require a course on its own, while I mainly wanted to arrive to talk about surfaces of general type), the second too because of its vastity and also for fear of overlapping with the course by Donagi (which eventually did not treat period maps and variation of

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*A member of G. N. S. A. G. A. of C. N. R., and in the M. P. I. Research Project in Algebraic Geometry.

**The final version of the paper was completed during a visit of the author to the University of California, San Diego.

Hodge structures). Anyhow the first topic is exhaustively treated in Popp's lecture notes ([Po]) and in the appendices to the second edition of Mumford's book on Geometric Invariant Theory ([Mu 2]), whereas the nicest applications of the theory of variation of Hodge structures to moduli of surfaces are amply covered in the book by Barth-Peters-Van de Ven ([B-P-V]).

Also, I mainly treated moduli of surfaces of general type, and fortunately Seiler lectured on the results of his thesis ([Sei 1,2,3]) about the moduli of (polarized) elliptic surfaces: I hope his lecture notes are appearing in this volume.

Instead, the part on Kodaira-Spencer's theory of deformations and its connections with the classical theory of continuous systems started to gain a dominant role after I gave a series of lectures at the Institute for Scientific Interchange (I. S. I.) in Torino on this subject. In fact, after Zappa (cf. [Zp], [Mu 3]) discovered the first example of obstructed deformations, a smooth curve in an algebraic surface, it was hard to justify most of the classical statements about moduli (and in fact, cf. lecture four, some classical problems about completeness of the characteristic system have a negative answer).

Interest in moduli was revived only through the pioneering work of Kodaira-Spencer and later through Mumford's theory of geometric invariants. Mumford's theory is more algebraic and deals mostly with the problem of determining whether a moduli space exists as an algebraic or projective variety, whereas the transcendental theory of Kodaira and Spencer (in fact applied in an algebraic context by Grothendieck and Artin) applies to the more general category of com plex manifolds (or spaces), at the cost of producing only a local theory. In both issues, it is clear that it is not possible to have a good theory of moduli without imposing some restriction on complex manifolds or algebraic varieties.

Surfaces of general type are a case when things work out well, and one would like first to investigate properties and structure of this moduli spaces, then to draw from these results useful geometric consequences. It is my impression that for these purposes (e.g. to count number of moduli) the Kodaira-Spencer theory is by far more useful, and not difficult to apply in many concrete cases. In fact, it seems that in most applications only elementary deformation theory is needed, and that's one reason why these lecture notes cover very little of the more sophisticated theory (cf. §10 for more details). The other reason is that the author is not an expert in modern deformation theory and realized rather late about the existence or importance of some literature on the subject: in particular we would like to recommend the beautiful survey paper ([Pa]) by Palamodov on deformation of complex spaces, whose historical introduction contains rather complete information regarding the material treated in the first three lectures.

Since the style of the paper is already rather informal, we don't attempt any discussion of the main ideas here in the introduction, and, before describing with more detail the contents, we remark that the paper (according to the C. I. M. E. goals) is directed to and ought to be accessible to non specialists and to beginning graduate students. Of course, reasons of space have obliged us to assume some familiarity with the language of algebraic geometry, especially sheaves and linear systems.

Finally, in many points references are omitted for reasons of economy and the lack of a quotation of some author's name (or paper) should not be interpreted as any claim of originality on my side, or as an underestimation of some scientific work.

§1-5 summarizes the essentials of the Kodaira-Spencer-Kuranishi results needed in later sections, following existing treatments of the topic ([K-M], [Ku 3]), whereas §6 is devoted to a single but enlightening example. §7 deals with deformations of automorphisms, whereas §8-9 are devoted to Horikawa's theory of deformations of holomorphic maps, with more emphasis to applications, such as deformation of surfaces in 3-space, or of complete intersections, and include some examples of everywhere obstructed deformations, due to Mumford and Kodaira. §10 is a "mea culpa" of the author for the topics he did not treat, §11-13 try to compare Horikawa's and Schlessinger-Wahl's theory of embedded deformations, whereas §12 consists of a rewriting, with some simplifications of notation, of Kodaira's paper ([Ko 3]) treating embedded deformations of surfaces with ordinary singularities. §14-17 give a basic resumé on classification of surfaces and §18-19 are devoted to basic properties of surfaces of general type and a sketchy discussion of Gieseker's theorem on their moduli spaces. §20-23 include a rough outline of recent work of the author and a result of I. Reider: §20 deals with the number of moduli of surfaces of general type, §22 outlines the deformation theory of $(\mathbb{Z}/2)^2$ covers, §21 and 23 exhibit examples of moduli spaces with arbitrarily many connected components having different dimensions, and discuss also the problem whether the topological or the differentiable structure should be fixed.

Acknowledgments: It is a pleasure to thank the Centro Internazionale Matematico Estivo and the Institute for Scientific Interchange of Torino for their invitations to lecture on the topics of these notes, and for their hospitality and support. I'm also very grateful to the University of California at San Diego for hospitality and support, and especially to Ms. Annetta Whiteman for her excellent typing.

LECTURE ONE: ALMOST COMPLEX STRUCTURES and the KURANISHI FAMILY

In this lecture I will review the construction, due to Kuranishi, of the complex structures, on a compact complex manifold M, sufficiently close to the given one. To do this, one has to use the notion of almost complex structures, of integrable ones: in a sense one of the main theorems, due to Newlander and Nirenberg, is a direct extension of a basic theorem of differential geometry, the theorem of Frobenius.

§1. Almost complex structures

Let M be a differentiable (or C^ω, i.e. real analytic) manifold of dimension equal to $2n$, T_M its real tangent bundle.

Definition 1.1. An almost complex structure on M is the datum of a splitting $T_M \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}$, with $T^{1,0} = \overline{T^{0,1}}$.

Naturally, the splitting of $T_M \otimes \mathbb{C}$ induces a splitting for the complexified cotangent bundle $\overset{\vee}{T}_M \otimes \mathbb{C} = (T^{1,0})^\vee \oplus (T^{0,1})^\vee$ ($(T^{1,0})^\vee$ is the annihilator of $T^{0,1}$), and for all the other tensors. In particular for the r^{th} exterior power of the cotangent bundle, one has the decomposition $\Lambda^r(\overset{\vee}{T}_M \otimes \mathbb{C}) = \bigoplus_{p+q=r} \Lambda^p(T^{1,0})^\vee \otimes \Lambda^q(T^{0,1})^\vee$.

We shall denote by $\mathcal{E}^{p,q}$ the sheaf of C^∞ sections of $\Lambda^p(T^{1,0})^\vee \otimes \Lambda^q(T^{0,1})^\vee$ (resp. by $\mathcal{A}^{p,q}$ the sheaf of C^ω sections), by $\mathcal{E}^r$ the sheaf of C^∞ sections of $\Lambda^r(\overset{\vee}{T}_M \otimes \mathbb{C})$.

The De Rham algebra is the differential graded algebra $(\mathcal{E}^*, d)$, where $\mathcal{E}^* = \bigoplus_{r=0}^{2n} \mathcal{E}^r$, and d is the operator of exterior differentiation. For a function f, $df \in \mathcal{E}^{1,0} \oplus \mathcal{E}^{0,1}$ and one can write accordingly $df = \partial f + \bar\partial f$; the problem is whether for all forms φ one can write $d = \partial + \bar\partial$, with $\partial\colon \mathcal{E}^{p,q} \to \mathcal{E}^{p+1,q}$, $\bar\partial\colon \mathcal{E}^{p,q} \to \mathcal{E}^{p,q+1}$ (then one has $\partial^2 = \bar\partial^2 = \partial\bar\partial + \bar\partial\partial = 0$, since $d^2 = 0$). Hence one poses the following

Definition 1.2. The given almost complex structure is integrable if

$$d(\mathcal{E}^{p,q}) \subset \mathcal{E}^{p+1,q} \oplus \mathcal{E}^{p,q+1} .$$

As a matter of fact, it is enough to verify this condition only for $p = 1$, $q = 0$.

Lemma 1.3. The almost complex structure is integrable $\iff$ $d(\mathcal{E}^{1,0}) \subset \mathcal{E}^{2,0} \oplus \mathcal{E}^{1,1}$. [Hence another equivalent condition is: $\mathcal{E}^{1,0}$ generates a differential ideal.]

Proof. The question being local, we can take a local frame for $\mathcal{E}^{1,0}$, i.e. sections $\omega_1, \ldots, \omega_n$ of $\mathcal{E}^{1,0}$ whose values are linearly independent at each point (locally, $\mathcal{E}^{1,0}$ is a free module of rank n over $\mathcal{E}^0$, and $\{\omega_1, \ldots, \omega_n\}$ is a basis). Our weaker condition is thus that

$$(1.4) \qquad d\omega_\alpha = \sum_{\beta<\gamma} \varphi_{\alpha\beta\gamma}\, \omega_\beta \wedge \omega_\gamma + \sum_{\beta,\gamma} \psi_{\alpha\beta\bar{\gamma}}\, \omega_\beta \wedge \overline{\omega_\gamma}$$

(where $\varphi_{\alpha\gamma\beta}$ and $\psi_{\alpha\beta\bar{\gamma}}$ are functions) since every $\omega \in \mathcal{E}^{1,0}$ can be written as $\Sigma_{\alpha=1}^n f_\alpha \omega_\alpha$, and $\{\omega_\beta \wedge \omega_\gamma \mid 1 \le \beta < \gamma \le n\}$ is a local frame for $\mathcal{E}^{2,0}$, $\{\omega_\beta \wedge \bar{\omega}_\gamma \mid 1 \le \beta, \gamma \le n\}$ is a local frame for $\mathcal{E}^{1,1}$. Now $\mathcal{E}^{0,1} = \overline{\mathcal{E}^{1,0}}$ hence $d(\mathcal{E}^{0,1}) \subset \mathcal{E}^{1,1} \oplus \mathcal{E}^{0,2}$ and one verifies $d(\mathcal{E}^{p,q}) \subset \mathcal{E}^{p+1,q} \oplus \mathcal{E}^{p,q+1}$ by induction on p, q, since locally any $\eta \in \mathcal{E}^{p,q}$ can be written as $\Sigma_{\alpha=1}^n \eta_\alpha \wedge \omega_\alpha + \Sigma_{\alpha=1}^n \vartheta_\alpha \wedge \bar{\omega}_\alpha$, with $\eta_\alpha \in \mathcal{E}^{p-1,q}$, $\vartheta_\alpha \in \mathcal{E}^{p,q-1}$. Q.E.D.

At this stage, one has to observe that if M is a complex manifold, then $(T^\vee)^{1,0} = (T^{1,0})^\vee$ is generated (by definition !) by the differentials df of holomorphic functions (at least locally, if one has a chart $(z_1, \ldots, z_n)\colon U \to \mathbb{C}^n$, $dz_1, \ldots, dz_n$ give a frame for $(T^\vee)^{1,0}$). Conversely, one defines, given an almost complex structure, a function f to be holomorphic if $\bar{\partial} f = 0$ (i.e., $df \in \mathcal{E}^{1,0}$); one sees easily, by the local inversion theorem of U. Dini, that the almost complex structure comes from a complex structure on M if and only if for each p in M there do exist holomorphic functions $F_1, \ldots, F_n$ defined in a neighborhood U of p and giving a frame of $\mathcal{E}^{1,0}$ over U. This occurs exactly if and only if the almost complex structure is integrable: we have thus the following (cf. [N-N], [Hör] for a proof).

Theorem 1.4 (Newlander-Nirenberg). An almost complex structure on a C^∞ manifold comes from a (unique) complex structure if and only if it is integrable.

Following Weil ([We], p. 36-37) we shall give a proof in the case where everything is real-analytic, because then we see why this is an extension of the theorem of Frobenius that we now recall (see [Spiv I] for more details, or [Hi]).

Theorem 1.5. Let $\varphi_1, \ldots, \varphi_r$ be 1-forms defined in an open set Ω in $\mathbb{R}^n$ and linearly independent at any point of Ω. Then for each point p in Ω there do exist local coordinates $x_1, \ldots, x_n$ such that the span of $\varphi_1, \ldots, \varphi_r$ equals the span of $dx_1, \ldots, dx_r$, $\Longleftrightarrow$ $\varphi_1, \ldots, \varphi_r$ span a differential ideal (i.e., $\forall\, i = 1, \ldots, r$ $\exists$ forms ϑ_{ij} $(j = 1, \ldots, r)$, s.t. $d\varphi_i = \Sigma_{j=1}^r \varphi_j \wedge \vartheta_{ij}$).

<u>Proof</u>. The usual way to prove the theorem is to consider, $\forall$ p' in Ω the space $V_{p'}$ of tangent vectors killed by $\varphi_1, \ldots, \varphi_r$: then in a neighborhood U of p there exist vector fields $X_{r+1}, \ldots, X_n$ spanning $V_{p'}$ for any p' in U. Since

$$\varphi_i([X_j, X_k]) = X_j(\varphi_i(X_k)) - X_k(\varphi_i(X_j)) - d\varphi_i(X_j, X_k)$$

we see that the vector field $[X_j, X_k]$ at each p' in U lies in $V_{p'}$. One looks then for coordinates $x_1, \ldots, x_n$ s.t. $V_{p'}$ is spanned by $\partial/\partial x_{r+1}, \ldots, \partial/\partial x_n$, and these coordinates are obtained by induction on (n-r). In fact, by taking integral curves of the vector field X_n, one can assume $X_n = \partial/\partial x_n$, and replaces X_i by $Y_i = X_i - (X_i x_n) X_n$, which span the subspace $W_{p'}$ of vectors in $V_{p'}$ killing x_n, and so also the vector field $[Y_i, Y_j]$ at each point p' in U lies in $W_{p'}$ (if $X(x_n) = 0$, $Y(x_n) = 0 \Rightarrow [X,Y](x_n) = 0$!). By induction there are coordinates $(y_1, \ldots, y_n)$ with $W_{p'}$ spanned by $\partial/\partial y_{r+1}, \ldots, \partial/\partial y_{n-1}$. We can replace $X_n = \Sigma_{j=1}^n a_j(y)(\partial/\partial y_j)$ by $Y_n = \Sigma_{j=1}^r a_j(y)(\partial/\partial y_j) + a_n(y)(\partial/\partial y_n)$; since $[(\partial/\partial y_i), Y_n]$ $(i = r+1, \ldots, n-1)$ equals

$$\sum_{j \neq n+1, \ldots, n-1} \frac{\partial a_j(y)}{\partial y_i} \frac{\partial}{\partial y_j}$$

but on the other hand, this vector field is in V'_p, thus it is a multiple of Y_n by a function f. But then, on the one hand, $[(\partial/\partial y_i), Y_n](x_n) = 0$ (since $Y_n(x_n) = X_n(x_n) = 1$!), on the other hand this quantity must equal $fY_n(x_n) = f$. Hence the functions $a_j(y)$ $(j = 1, \ldots, r, n)$ depend only upon the variables $y_1, \ldots, y_r, y_n$, so, by taking integral curves of the vector field Y_n, we can assume $Y_n = \partial/\partial y_n$ also.

<u>Q.E.D.</u>

We have given a proof of the well known theorem of Frobenius just to notice that the only fact that is repeatedly used is the following: if X is a non zero vector field, then there exist coordinates $(x_1, \ldots, x_n)$ s.t. $X = \partial/\partial x_n$. This follows from the theorem of existence and unicity for ordinary differential equations and from Dini's theorem. Both these results hold for holomorphic functions (they are even simpler, then), therefore, given a non zero holomorphic vector field $Z = \Sigma_{i=1}^n a_i(w)\, \partial/\partial w_i$ on an open set in $\mathbb{C}^n$ (i.e., the a_i's are holomorphic functions), there exist local holomorphic coordinates $z_1, \ldots, z_n$ around each point such that $Z = \partial/\partial z_n$.

The conclusion is that the theorem of Frobenius holds verbatim if we replace $\mathbb{R}^n$ by $\mathbb{C}^n$, we consider holomorphic (1,0) forms $\varphi_1, \ldots, \varphi_r$, and we require local holomorphic coordinates $z_1, \ldots, z_n$ s.t. the $\mathbb{C}$-span of $\varphi_1, \ldots, \varphi_r$ be the $\mathbb{C}$-span of $dz_1, \ldots, dz_r$. The proof of the Newlander-Nirenberg theorem in the real analytic case follows then from the following.

Lemma 1.6. Let Ω be an open set in $\mathbb{R}^{2n}$, let $\omega_1, \ldots, \omega_n$ be real analytic complex valued 1-forms defining an integrable almost complex structure (i.e., 1.4 holds). Then, around each point $p \in \Omega$, there are complex valued functions $F_1, \ldots, F_n$ s.t. the span of $dF_1, \ldots, dF_n$ equals the span of $\omega_1, \ldots, \omega_n$.

Proof. Take local coordinates $x_1, \ldots, x_n$ around p s.t. each ω_α is expressed by a power series $\sum_{j=1}^{2n} \sum_K f_{\alpha j, K} x^K dx_j$, where $K = (k_1, \ldots, k_{2n})$ denotes a multi-index. Then $\overline{\omega_\alpha} = \sum_{j,K} \overline{f_{\alpha j, K}} x^K dx_j$ and, if we consider $\mathbb{R}^{2n}$ as contained in $\mathbb{C}^{2n}$, upon replacing the monomial x^K by the monomial z^K and x_j by dz_j (here x_j is the real part of z_j!), ω_α and $\overline{\omega_\alpha}$ extend to holomorphic 1-forms ω_α, η_α in a neighborhood of p in $\mathbb{C}^{2n}$. Since $\omega_1, \ldots, \omega_n, \overline{\omega}_1, \ldots, \overline{\omega}_n$ are a local frame for $\mathcal{E}^1$, the ω_α's, η_α's give a basis for the module of holomorphic 1-forms, therefore one can write

$$d\omega_\alpha = \sum_{\beta<\gamma} \varphi_{\alpha\beta\gamma}\, \omega_\beta \wedge \omega_\gamma + \sum_{\beta,\gamma} \psi_{\alpha\beta\gamma}\, \omega_\beta \wedge \eta_\gamma + \sum_{\beta<\gamma} \xi_{\alpha\beta\gamma}\, \eta_\beta \wedge \eta_\gamma .$$

By restriction to $\mathbb{R}^{2n}$, using (1.4) we see that $\xi_{\alpha\beta\gamma} \equiv 0$, hence $\omega_1, \ldots, \omega_n$ span a differential ideal, hence Frobenius applies and there exist new holomorphic coordinates in $\mathbb{C}^{2n}$, $w_1, \ldots, w_{2n}$ s.t. the span of $dw_1, \ldots, dw_n$ equals the span of $\omega_1, \ldots, \omega_n$. We simply take F_i to be the restriction of w_i to $\mathbb{R}^{2n}$. Q.E.D.

Remark 1.7. Assume that for $t = (t_1, \ldots, t_m)$ in a neighborhood of the origin in $\mathbb{C}^m$ one is given real analytic 1-forms $\omega_{t,1}, \ldots, \omega_{t,n}$ as in lemma 1.6 which are expressed by convergent power series in $t_1, \ldots, t_m$, and define an integrable almost complex structure when t belongs to a complex analytic subspace B containing the origin. Then, for t in B, the conclusions of lemma 1.6 hold with $F_{t,1}, \ldots, F_{t,n}$ expressed as convergent power series in $(t_1, \ldots, t_m)$. In fact, if a vector field X_t is given by a convergent power series in $t_1, \ldots, t_m$ also the solutions of the associated differential equation are power series in $t_1, \ldots, t_m$: moreover, by the local inversion theorem for holomorphic functions, if $f(x,t)\colon \Omega \to \Omega$ is locally invertible, real analytic in x and complex analytic in t, then the local inverse is also complex analytic in t.

§2. Small deformations of a complex structure

If U is a vector subspace of a vector space V, and W is a supplementary subspace of U in V (thus we identify V with $U \oplus W$), then all the subspaces U', of the same dimension, sufficiently close to U, can be viewed as graphs of a linear map from U to W: we apply this principle pointwise to define a small variation of an almost complex structure (hence also of a complex structure).

Definition 2.1. A small variation of an almost complex structure is a section φ of $T^{1,0} \otimes (T^{0,1})^\vee$ (the variation is said to be of class C^r if φ is of class C^r).

Remark 2.2. To a small variation φ we associate the new almost complex structure s.t. $T^{0,1}_\varphi = \{(u,v) \in T^{1,0} \oplus T^{0,1} \mid u = \varphi(v)\}$, since there is a canonical isomorphism of $T^{1,0} \otimes (T^{0,1})^\vee$ with $\mathrm{Hom}(T^{0,1}, T^{1,0})$.

We assume from now on that M is a complex manifold: then, in terms of local holomorphic coordinates $(z_1, \ldots, z_n)$ one can write φ as

(2.3) $$\varphi = \sum_{\alpha,\beta} \varphi^{\bar\beta}_\alpha(z)\, d\bar z_\beta \otimes \frac{\partial}{\partial z_\alpha}$$

so that

$$T^{0,1}_\varphi = \left\{ \left(\sum_\alpha u_\alpha \frac{\partial}{\partial z_\alpha}, \sum_\beta v_\beta \frac{\partial}{\partial \bar z_\beta} \right) \,\middle|\, u_\alpha = \sum_\beta \varphi^{\bar\beta}_\alpha v_\beta \right\}$$

and is annihilated by $(T^{1,0}_\varphi)^\vee$, the span of $\{\omega_\alpha = dz_\alpha - \Sigma_\beta \varphi^{\bar\beta}_\alpha d\bar z_\beta\}$. On the other hand, by what we've seen $T^{0,1}_\varphi$ is spanned by the ξ_γ's, where

$$\xi_\gamma = \frac{\partial}{\partial \bar z_\gamma} + \sum_\alpha \varphi^{\bar\gamma}_\alpha \frac{\partial}{\partial z_\alpha} .$$

Since $d\omega_\alpha = -\Sigma_\beta d\varphi^{\bar\beta}_\alpha \wedge d\bar z_\beta$, we are going to write down the integrability condition (1.4), which can be interpreted as

(2.4) $$d\omega_\alpha(\xi_\gamma, \xi_\delta) = 0 \quad \forall \ \alpha, \gamma, \delta \ (\gamma < \delta) .$$

We have

$$-d\omega_\alpha = \sum_{\beta,\epsilon} \left(\frac{\partial \varphi^{\bar\beta}_\alpha}{\partial z_\epsilon} dz_\epsilon \wedge d\bar z_\beta + \frac{\partial \varphi^{\bar\beta}_\alpha}{\partial \bar z_\epsilon} d\bar z_\epsilon \wedge d\bar z_\beta \right) ,$$

which belongs to $\mathcal{E}^{1,1} \oplus \mathcal{E}^{0,2}$, hence kills pairs of vectors of type (1,0). We get thus the condition

$$d\omega_\alpha \left(\frac{\partial}{\partial \bar z_\gamma}, \frac{\partial}{\partial \bar z_\delta} \right) + d\omega_\alpha \left(\sum_{\alpha'} \varphi^{\bar\gamma}_{\alpha'} \frac{\partial}{\partial z_{\alpha'}}, \frac{\partial}{\partial \bar z_\delta} \right)$$
$$+ d\omega_\alpha \left(\frac{\partial}{\partial \bar z_\gamma}, \sum_{\alpha''} \varphi^{\bar\delta}_{\alpha''} \frac{\partial}{\partial z_{\alpha''}} \right) = 0 ,$$

boiling down to

$$(2.5') \qquad \frac{\partial \varphi_\alpha^{\bar\delta}}{\partial \bar z_\gamma} - \frac{\partial \varphi_\alpha^{\bar\gamma}}{\partial \bar z_\delta} + \sum_\epsilon \frac{\partial \varphi_\alpha^{\bar\delta}}{\partial z_\epsilon} \varphi_\epsilon^{\bar\gamma} - \frac{\partial \varphi_\alpha^{\bar\gamma}}{\partial z_\epsilon} \varphi_\epsilon^{\bar\delta} = 0 .$$

The condition that (2.5′) holds for each α, and $\gamma < \delta$, can be written more simply as

$$(2.5) \qquad \bar\partial\varphi = \frac{1}{2}[\varphi,\varphi] ,$$

where

$$\bar\partial\varphi = \sum_\alpha \left(\sum_{\gamma<\delta} \left(\frac{\partial \varphi_\alpha^{\bar\gamma}}{\partial \bar z_\delta} - \frac{\partial \varphi_\alpha^{\bar\delta}}{\partial \bar z_\gamma} \right) d\bar z_\gamma \wedge d\bar z_\delta \right) \otimes \frac{\partial}{\partial z_\alpha} ,$$

$$[\varphi,\varphi] = 2 \sum_\alpha \sum_{\gamma<\delta} \left(\sum_\epsilon \frac{\partial \varphi_\alpha^{\bar\delta}}{\partial z_\epsilon} \varphi_\epsilon^{\bar\gamma} - \frac{\partial \varphi_\alpha^{\bar\gamma}}{\partial z_\epsilon} \varphi_\epsilon^{\bar\delta} \right) d\bar z_\gamma \wedge d\bar z_\delta \otimes \frac{\partial}{\partial z_\alpha}$$

$$= \sum_{\alpha,\epsilon,\gamma,\delta} d\bar z_\gamma \varphi_\epsilon^{\bar\gamma} \left(\frac{\partial \varphi_\alpha^{\bar\delta}}{\partial z_\epsilon} \right) \wedge d\bar z_\delta \otimes \frac{\partial}{\partial z_\alpha} - d\bar z_\delta \varphi_\alpha^{\bar\delta} \left(\frac{\partial \varphi_\epsilon^{\bar\gamma}}{\partial z_\alpha} \right) \wedge d\bar z_\gamma \otimes \frac{\partial}{\partial z_\epsilon} .$$

We shall explain these definitions while recalling some standard facts on Dolbeault cohomology and Hodge theory (harmonic forms).

So, let V be a holomorphic vector bundle, and let (U_α) be a cover of M by open sets where one has a trivialization $V_{|U_\alpha} \simeq U_\alpha \times \mathbb{C}^r$, hence fibre vector coordinates v_α, related by $v_\alpha = g_{\alpha\beta} v_\beta$ where $g_{\alpha\beta}$ is an invertible $r \times r$ matrix of holomorphic functions. We let $\mathcal{E}^{0,p}(V)$ be the space of (C^∞) sections of $V \otimes \Lambda^p(T^{0,1})^\vee$: since $\bar\partial g_{\alpha\beta} = 0$, it makes sense to take $\bar\partial$ of (0, p) forms with values in V (i.e., elements of $\mathcal{E}^{0,p}(V)$), and we have the Dolbeault exact sequence of sheaves

$$0 \to \mathcal{O}(V) \to \mathcal{E}(V) \xrightarrow{\bar\partial_1} \mathcal{E}^{0,1}(V) \xrightarrow{\bar\partial_2} \cdots \xrightarrow{\bar\partial_n} \mathcal{E}^{0,n}(V) \to 0 ,$$

where $\mathcal{O}(V)$ is the sheaf of holomorphic sections of V. We have the theorem of Dolbeault (the $\mathcal{E}^{0,k}(V)$ are soft sheaves).

<u>Theorem 2.6.</u>

$$H^i(M, \mathcal{O}(V)) \simeq \frac{\ker H^0(\bar\partial_{i+1})}{\operatorname{Im} H^0(\bar\partial_i)} .$$

So $\bar\partial$ is well defined for our $\varphi \in \mathcal{E}^{0,1}(T^{1,0})$. For further use, we shall use the notation $\Theta = \mathcal{O}(T^{1,0})$. To explain the bracket operation, we notice that this is a bilinear operation