Excercises in SET THEORY

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PREFACE

These exercises are intended to assist students in understanding the elements of set theory. The arrangement of topics follows the development found in Paul Halmos' *Naive Set Theory*, although a student studying some other textbook should find the problems useful. The exercises are, on the whole, routine explorations of the definitions and theorems of set theory and are not puzzle or contest problems. The problems vary somewhat in difficulty, but nearly all should yield to the average student, provided he is endowed also with persistence. It is the author's belief that maximal benefit will accrue to the student who does not look at the answers before he himself has some solution to check. A sense of pride in this matter on the student's part will be a productive attitude for him. There are, of course, multiple answers possible for many problems, while only one is offered in the answer section. Perhaps the student may happily find more elegant solutions than those given.

At the beginning of each chapter there is a brief compilation of results taken from *Naive Set Theory* (NST) and sometimes elsewhere that are relevant to the exercises of that chapter. There are exercises in algebra involving monoids, semigroups, groups, rings, fields, vector spaces, and algebras. All necessary definitions are included, but references to helpful books are also included. These algebraic exercises are included in expectation that the student will have previously completed an undergraduate course or two in algebra and will profit from applications of the abstract concepts of set theory to the familiar. In the chapter on ordinals the exercises lead to a proof of the compatibility of the recursive and set theoretic treatments of ordinal arithmetic. Although it ought to be obvious, it will be stated that no originality is claimed for the exercises.

L. E. S.

Table Showing Relationship of Exercises to Set Theory Text

Exercises in Set Theory	*Naive Set Theory*
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CHAPTER I

ELEMENTARY CONCEPTS

1.0 $x \in A$ reads x is a member (or element) of set A. (NST p2)
$x \notin A$ reads x is not a member of set A. (NST p2)

Axiom of Extension. $A = B$ means $x \in A$ iff (if and only if) $x \in B$. (NST p2)

Definition of Subsets. $A \subset B$ means if $x \in A$ then $x \in B$. (NST p3)
$A \subsetneq B$ means $A \subset B$ and $A \neq B$, A is a proper subset of B.

If $S(y)$ is a condition, then $x \in \{y|\ S(y)\}$ iff $S(x)$. (NST pp4, 10)

Axiom of Specification. If A is a set and $S(x)$ is a condition, then $\{x|\ x \in A$ and $S(x)\}$ is a set. (NST p60)

Axiom of Pairing. If A and B are sets, then there exists a set C so that $A \in C$ and $B \in C$. (NST p9)

Definition of Pair. $\{A, B\} = \{x|\ x = A$ or $x = B\}$. (NST p9)

Axiom of Unions. If $\mathcal{C}$ is a set, then there exists a set U such that if $x \in X$ for some $X \in \mathcal{C}$, then $x \in U$. (NST p12)

Definition of Union. $\bigcup \mathcal{C} = \{x|\ x \in X$ for some $X \in \mathcal{C}\}$. (NST p12)

Definition of Intersection. For $\mathcal{C} \neq \phi$, $\bigcap \mathcal{C} = \{x|\ x \in X$ for all $X \in \mathcal{C}\}$. (NST p15)

Definition of Relative Complement. $A - B = \{x|\ x \in A$ and $x \notin B\}$. (NST p17) If the set A is understood from context, denote $A - B$ by B'.

Axiom of Powers. If E is a set, then there exists a set P such that if $X \subset E$, then $X \in P$. (NST p19)

Definition of Power Set. $\mathcal{P}E = \{x \mid x \subset E\}$. (NST p19)

Definition of the Empty Set. $\emptyset = \{x \mid x \neq x\}$.

1.1 The following symbols are often used to study the algebra of sentences:

$\wedge$	representing	and
$\vee$	representing	or
$\neg$	representing	not
$\Longrightarrow$	representing	if ... then ..., implies
$\Longleftrightarrow$	representing	if and only if
$\exists$	representing	there exists, for some
$\forall$	representing	for all

These symbols are employed according to the following rules of construction for sentences. If p and q are sentences, then $p \wedge q$, $p \vee q$, $\neg p$, $p \Longrightarrow q$, $p \Longleftrightarrow q$ are sentences. If p is a sentence, then $\exists x: p$, $\forall x: p$ are sentences. For example, given that p, q, r are sentences, it is verified that $\neg [(p \wedge \neg q) \vee r]$ is a sentence in the following way. q is a sentence; therefore $\neg q$ is a sentence. p and $\neg q$ are sentences; therefore $p \wedge \neg q$ is a sentence. $p \wedge \neg q$ and r are sentences; therefore $(p \wedge \neg q) \vee r$ is a sentence. $(p \wedge \neg q) \vee r$ is a sentence; therefore $\neg [(p \wedge \neg q) \vee r]$ is a sentence.

The truth values of the sentences constructed from the first five symbols are determined formally from defining truth tables. These are (using 1 as a symbol for true and 0 for false):

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

p	$\neg p$
1	0
0	1

p	q	$p \Longrightarrow q$
1	1	1
1	0	0
0	1	1
0	0	1

p	q	$p \Longleftrightarrow q$
1	1	1
1	0	0
0	1	0
0	0	1

For example, if p is a false sentence and q is a true sentence, then $p \wedge q$, $p \Longleftrightarrow q$ are false and $p \Longrightarrow q$, $p \vee q$, $\neg p$ are true.

Two sentences with the same truth values are equivalent ($\equiv$). Equivalences can be systematically verified using the truth tables iteratively as in these examples.

$$p \Longrightarrow q \equiv \neg p \vee q$$

p	q	$p \Longrightarrow q$	$\neg p$	$\neg p \vee q$
1	1	1	0	1
1	0	0	0	0
0	1	1	1	1
0	0	1	1	1

The column for $p \Longrightarrow q$ and the column for $\neg p \vee q$ contain the same truth values.

$$p \Longrightarrow q \equiv \neg q \Longrightarrow \neg p$$

p	q	$\neg q$	$\neg p$	$\neg q \Longrightarrow \neg p$	$p \Longrightarrow q$
1	1	0	0	1	1
1	0	1	0	0	0
0	1	0	1	1	1
0	0	1	1	1	1

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

Using truth tables, prove each of the following equivalences.

a) $p \Longrightarrow q \equiv \neg p \vee q$

b) $p \Longrightarrow q \equiv \neg (p \wedge \neg q)$

c) $p \vee q \equiv \neg (\neg p \wedge \neg q)$

d) $p \Longrightarrow q \equiv \neg q \Longrightarrow \neg p$

e) $p \Longrightarrow q \equiv (p \wedge \neg q) \Longrightarrow \neg p$

f) $p \Longrightarrow q \equiv (p \wedge \neg q) \Longrightarrow q$

g) $p \Longrightarrow q \equiv (p \wedge \neg q) \Longrightarrow (r \wedge \neg r)$

h) $p \Longleftrightarrow q \equiv (p \Longrightarrow q) \wedge (q \Longrightarrow p)$

i) $p \wedge q \equiv q \wedge p$

j) $p \vee q \equiv q \vee p$

k) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

l) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

m) $p \equiv \neg (\neg p)$

n) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

o) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

p) $p \wedge p \equiv p$

q) $p \vee p \equiv p$

This exercise suggests how logic is formalized. The student interested in pursuing the subject should consult one of the texts on logic suggested in the reference list.

1.2 **The quantifiers $\forall$ and $\exists$ are related as follows: $\forall x: p$ is equivalent to $\neg (\exists x: \neg p)$, or in words, (for all x, p is true) is equivalent to (it is not the case that there exists an x such that p is false).**

Prove this equivalence: $\neg (\forall x): (x \in X \Longrightarrow x \in Y)$ is equivalent to $(\exists x): (x \in X \wedge x \notin Y)$. Write out a statement of this theorem using the word *subset*.

1.3 Prove that equality for sets is reflexive, symmetric, and transitive; that is, prove these theorems:

a) $X = X$ for all sets X.

b) If $X = Y$, then $Y = X$ for all sets X, Y.

c) For all sets, X, Y, Z, if $X = Y$ and $Y = Z$, then $X = Z$.

1.4 Prove that inclusion for sets is reflexive, antisymmetric, and transitive; that is, prove these theorems:

a) $X \subset X$ for all sets X.

b) $X \subset Y$ and $Y \subset X$ imply $X = Y$ for all sets X, Y.

c) For all sets X, Y, Z if $X \subset Y$ and $Y \subset Z$ then $X \subset Z$.

1.5 Let $A \subsetneqq B$ and $B \subsetneqq C$. Prove that $A \subsetneqq C$.

1.6 Let $A \subset B$ and $B \subset C$ and $C \subset A$. Prove that $A = B = C$.

1.7 Let $P \subset U$ and $Q \subset U$. Define $P' = \{x | x \in U \text{ and } x \notin P\}$ and Q similarly. Prove that $P \subset Q$ iff $Q' \subset P'$. Prove furthermore that $P = (P')'$. Note: *iff* is an abbreviation for *if and only if*.

1.8 Prove that $\emptyset \subset P$. Prove that $P \subset \emptyset$ iff $P = \emptyset$.

1.9 By definition
$$P \cup Q = \{x | x \in P \text{ or } x \in Q\} = \bigcup\{P, Q\}$$
$$P \cap Q = \{x | x \in P \text{ and } x \in Q\} = \bigcap\{P, Q\}.$$
If P and Q are sets, then prove that $P \cup Q$ and $P \cap Q$ are sets.

1.10 Prove that $\emptyset$ is a neutral element for the union of sets, i.e., that $X \cup \emptyset = \emptyset \cup X = X$ for all sets X.

1.11 Let $X \cup Y = X$ for all sets X. Prove that $Y = \emptyset$.

1.12 Prove that $X \cap \emptyset = \emptyset$ for all sets X.

1.13 Prove the following theorems:

a) The commutativity of union, $P \cup Q = Q \cup P$.

b) The commutativity of intersection, $P \cap Q = Q \cap P$

c) The associativity of union, $P \cup (Q \cup R) = (P \cup Q) \cup R$.

d) The associativity of intersection, $P \cap (Q \cap R) = (P \cap Q) \cap R$

e) The distributivity of intersection with respect to union, $P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R)$.

f) The distributivity of union with respect to intersection, $P \cup (Q \cap R) = (P \cup Q) \cap (P \cup R)$.

g) The idempotency of union, $P \cup P = P$.

h) The idempotency of intersection, $P \cap P = P$.

1.14 Define $0 = \emptyset$, $1 = 0 \cup \{0\}$, $2 = 1 \cup \{1\}$, $3 = 2 \cup \{2\}$, $4 = 3 \cup \{3\}$. Prove that 0, 1, 2, 3, 4 are sets. These sets which are denoted by the symbols used for the nonnegative integers will show up frequently in these exercises.

1.15 Express the set 4 using only the symbols $\{$, $\}$, $\emptyset$, , .

1.16 Prove each of these statements true or false:

a) $1 \in 2$ c) $1 \cap 2 = 0$ e) $(0 \cap 2) \in 1$.

b) $1 \subset 2$ d) $1 \cup 2 = 2$

1.17 In order to generalize the construction of 1.14, assume an intuitive knowledge of the natural numbers (nonnegative whole numbers) and proof by mathematical induction. For each natural number n define a set S_n as follows: $S_0 = \emptyset$, $S_{n+1} = S_n \cup \{S_n\}$. Prove that S_0 is a set. Prove that if S_n is a set, then S_{n+1} is a set. Conclude that S_n is a set for each natural number n. Finally, identify n and S_n; that is, use the symbol n to denote the natural number as intuitively understood and the constructed set. These sets, 0, 1, 2, 3,... will be used in succeeding exercises.

1.18 Of which of the following sets is x a member, x a subset, x neither a member nor a subset?

A) $\{\{x\}, y\}$ D) $\{x\} - \{\{x\}\}$
B) x E) $\{x\} \cup x$
C) $\emptyset \cap x$ F) $\{x\} \cup \{\emptyset\}$

1.19 Show that
a) $\bigcup \{\{a, b, c\}, \{a, d, e\}, \{a, f\}\} = \{a, b, c, d, e, f\}$
b) $\bigcap \{\{a, b, c\}, \{a, d, e\}, \{a, f\}\} = \{a\}$
c) $\bigcup \{1\} = 1$
d) $\bigcap \{1\} = 1$
e) $\bigcup \{A\} = A$ for all sets A
f) $\bigcap \{A\} = A$ for all sets A.

1.20 Express the following sets using the sets 0, 1, 2, etc.
$\emptyset$, $\bigcup\emptyset$, $\mathcal{P}\emptyset$, $\bigcup\bigcup\emptyset$, $\mathcal{P}\mathcal{P}\emptyset$, $\bigcup\bigcup\bigcup\emptyset$, $\mathcal{P}\mathcal{P}\mathcal{P}\emptyset$.

1.21 Let $X = \{\{2, 5\}, 4, \{4\}\}$. Find $\bigcap(\bigcup X - 4)$.

1.22 Construct $(\bigcup\mathcal{P} 1)'$ where $A' = 3 - A$ for any set A.

1.23 Construct $\bigcap \bigcup(\mathcal{P} 2 - 2)$.

1.24 Let $X = \{\{\{1, 2\}, \{1\}\}, \{\{1, 0\}\}\}$ Construct $\bigcup X$, $\bigcap X$, $\bigcup\bigcup X$, $\bigcap\bigcap X$, $\bigcup\bigcap X$, $\bigcap\bigcup X$.

1.25 Let $X = \{\{1, 2\}, \{2, 0\}, \{1, 3\}\}$. Construct $\bigcup X$, $\bigcap X$, $\bigcup\bigcup X$, $\bigcap\bigcap X$, $\bigcup\bigcap X$, $\bigcap\bigcup X$.

1.26 Let P, Q, R be subsets of U and let the relative complements ($'$) be with respect to U. Then prove the following:
a) $P \subset Q$ iff $P \cap Q' = \emptyset$
b) $P \subset Q$ iff $P' \cup Q = U$
c) $P \subset Q$ iff $(P \cap Q') \subset P'$
d) $P \subset Q$ iff $(P \cap Q') \subset Q$
e) $P \subset Q$ iff $(P \cap Q') \subset (R \cap R')$.

1.27 Prove that $A \cup B = A$ iff $B \subset A$.

1.28 Prove that $A \cap B = A$ iff $A \subset B$.

1.29 Give an example of two sets A, B such that $(\bigcap A) \cap (\bigcap B) \neq \bigcap(A \cap B)$.

1.30 Prove that $(\bigcap A) \cap (\bigcap B) \subset \bigcap(A \cap B)$.

1.31 Give an example of two sets A, B such that $\mathcal{P}(A \cup B) \neq (\mathcal{P} A) \cup (\mathcal{P} B)$.

1.32 Assuming a knowledge of **R**, the real numbers, what subset is described here? $\{x \mid x^2 > 2\} \cap \{x \mid |x-2| < |x+3|\}$.

1.33 Define $A + B = (A-B) \cup (B-A)$ to be the symmetric difference of the sets A and B. Prove the following:

a) $A + \emptyset = A$

b) $A + B = B + A$

c) $A + (B+C) = (A+B) + C$

d) $A \cap (B+C) = (A \cap B) + (A \cap C)$

e) $A - B \subset A + B$

f) $A = B$ iff $A + B = \emptyset$

g) $A + C = B + C$ implies $A = B$.

1.34 Prove that $\{A\}$ is a set without using the axiom of pairing, given A is a set.

CHAPTER 2

THE ORDERED PAIR AND THE CARTESIAN PRODUCT

2.0 Definition of Ordered Pair. $(a, b) = \{\{a\}, \{a, b\}\}$. (NST p23)

Definition of Cartesian Product. $A \times B = \{x \mid x = (a, b)$ for some $a \in A$ and some $b \in B\}$. (NST p24)

2.1 Construct these sets:

a) $2 \cup 3$ d) 2×1 g) 1×1

b) $2 \cap 3$ e) 1×2

c) 2×3 f) 0×1

2.2 Show that $\{x, y\}$ cannot serve as the definition for an ordered pair; show that it does not have the property $(x, y) = (a, b)$ iff $x = a$ and $y = b$.

2.3 An ordered pair is by definition a set. Show by example that not every ordered pair has two members.

2.4 Prove this proposition false: $X \times Y = Y \times X$ for all sets X, Y. The Cartesian product is not commutative.

2.5 Prove that the Cartesian product is nonassociative.

2.6 Give an example of two sets X, Y such that $X \times Y = Y \times X$.

2.7 Prove that the Cartesian product is distributive with respect to union: $X \times (Y \cup Z) = (X \times Y) \cup (X \times Z)$ for all X, Y, Z.

2.8 Give examples of sets such that $X \cup (Y \times Z) \neq (X \cup Y) \times (X \cup Z)$

2.9 Prove that $\bigcap\bigcap(x, y) = x$.

2.10 Prove $[\bigcap\bigcup(x, y)] \cup [\bigcup\bigcup(x, y) - \bigcup\bigcap(x, y)] = y$.

2.11 Prove that $X \times X = Y \times Y$ implies $X = Y$.

2.12 Prove that $X \times Y = X \times Z$ and $X \neq \emptyset$ imply $Y = Z$.

CHAPTER 3

RELATIONS

3.0 **Definition of a Relation from X to Y.** R is a relation from X to Y iff $R \subset X \times Y$. One writes $x R y$ if $(x, y) \in R$.

Definition of the Image of a Relation. Image $R = \{y \mid (x, y) \in R$ for some $x \in X\}$.

Definition of the preimage of a Relation. Preimage $R = \{x \mid (x, y) \in R$ for some $y \in Y\}$.

Definition. A relation R on X (from X to X) is

a) reflexive iff for all $x \in X$, $(x, x) \in R$
b) irreflexive iff for all $x \in X$, $(x, x) \notin R$
c) transitive iff $(x, y) \in R$ and $(y, z) \in R$ imply $(x, z) \in R$
d) atransitive iff $(x, y) \in R$ and $(y, z) \in R$ imply $(x, z) \notin R$.
e) symmetric iff $(x, y) \in R$ implies $(y, x) \in R$
f) antisymmetric iff $(x, y) \in R$ and $(y, x) \in R$ imply $x = y$. (NST p54)

Definition of Composition. If R is a relation from X to Y and S is a relation from Y to Z then $S \circ R = \{(x, z) \mid (x, y) \in R$ and $(y, z) \in S$ for some $y \in Y\}$. (NST p41)

Definition of Inverse Relation. If R is a relation from X to Y then $R^{-1} = \{(y, x) \mid (x, y) \in R\}$. (NST p40)